Robust Synchronization in Networks of Cyclic Feedback Systems

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***Abstract*— This paper presents a result on the robust synchronization of outputs of statically interconnected non-identical cyclic feedback systems that are used to model, among other processes, gene expression. The result uses incremental versions of the small gain theorem and dissipativity theory to arrive at an upper bound on the norm of the synchronization error between corresponding states, giving a measure of the degree of convergence of the solutions. This error bound is shown to be a function of the difference between the parameters of the interconnected systems, and disappears in the case where the systems are identical, thus retrieving an earlier synchronization result.**

I. Introduction

Networks of interconnected oscillating dynamical systems often exhibit some convergence property in terms of the values of their states or frequencies, leading to state synchronization or phase-locking. Whether or not this phenomenon takes place depends on the coupling structure, the strength of this coupling and the similarity of the interconnected systems in terms of structure and parameters.

This paper investigates the relationship between the degree of output synchronization in networks of parametrized *cyclic feedback systems* (CFSs) and the variation of their parameters under the assumption that the CFSs are structurally the same.

CFSs represent a class of dynamical systems that has been widely studied in the literature and has been used to model gene expression [1], [2], [3], [4], [5]. CFSs have a common overall structure composed of a unity-gain negative feedback around a cascade of subsystems. The last subsystem in the cascade is generally a bounded, monotonically increasing nonlinearity.

In [5] it was shown that the solutions of the above defined class of systems are limited to a number of scenarios: a single equilibrium, a single (non-constant) periodic solution or a combination of equilibria with homoclinic and heteroclinic orbits. The more recent work [6] presents a necessary and sufficient condition for the existence of a diagonal Lyapunov function proving the global asymptotic stability of the system. This result was obtained using the passivity properties of the subsystems making up the cascade and their interconnection structure. The condition obtained placed an upper bound on the product of the individual subsystems' 'secant gains', see [7]. As is seen in [1], limit cycles are typically observed when stability is lost due to the breaking of the above described secant gain condition.

The results herein extend those in [8] where incremental dissipativity tools were employed to show that the differences between the outputs of interconnected, identical CFSs tend to zero under strong, linear, static coupling. Such incremental stability analysis of signals was covered in [9], [10], [11].

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In this paper, we prove that under the same coupling assumption as in [8], the differences between the outputs of interconnected, non-identical CFSs will asymptotically tend to finite, generally non-zero limits. To do this, we compare each CFS subsystem to a corresponding nominal subsystem representing an 'average' of the corresponding subsystems. Based on the monotonicity property of the defining function of each CFS subsystem, we show that each subsystem is incrementally output semipassive with an associated positive scalar quantity, henceforth referred to as the 'incremental secant gain'. From here we relate the output synchronization error to the coupling strength. We do this in two ways:

We first present a small-gain theorem result that shows that in a network of interconnected, nonidentical CFSs, if the product of the incremental secant gains is less than unity, which can always be achieved by strengthening the coupling, then the differences between corresponding states of the different CFSs of the network will be upper-bounded by a constant.

The second method directly employs the incremental dissipation inequalities defining the incremental output semipassivity of each subsystem. Using the tools in [6] and [8], we show that if the coupling is larger than a certain threshold (determined by the incremental secant gains), the entire network becomes incrementally output semipassive, with the result that the corresponding outputs of the interconnected CFSs converge to within a finite distance of each other.

Each method leads to different sufficient conditions (the latter being less conservative) allowing to prove that, under strong coupling, the differences between corresponding states of the different CFSs of the network are asymptotically upper-bounded by a constant. Furthermore we show that in both cases the value of this constant tends towards zero as the difference between the CFSs reduces, thereby recovering the result presented in [8].

II. Notation

This section introduces the notation that will be used in later sections. We shall be considering networks of N cyclic feedback systems (CFSs). Each individual CFS is composed of a cascade of n scalar subsystems H_i, $i = 1, \cdots, n$, in negative feedback with a unity gain, as illustrated in Figure 1.

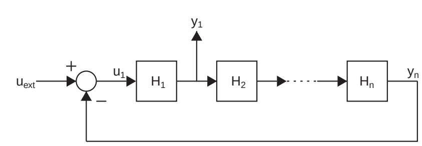

Fig. 1. Isolated cyclic feedback system.

The i^{th} state of the j^{th} CFS is represented by $x_{i_j} \in \mathbb{R}$, which is also the output y_{i_j} of the i^{th} subsystem of the j^{th} CFS. The vector $\mathbf{x}_i \in \mathbb{R}^N$ is the vector of the i^{th} states from each CFS, i.e. $\mathbf{x}_i = [\ x_{i_1} \ \ \cdots \ \ x_{i_N}\ ]^T$. The input to the i^{th} subsystem of the j^{th} CFS is given by $u_{i_j} \in \mathbb{R}$, and the vector of inputs to the i^{th} subsystems is $\mathbf{u}_i = [\ u_{i_1} \ \ \cdots \ \ u_{i_N}\ ]^T$. The vector of outputs from the i^{th} subsystems is $\mathbf{y}_i = [\ y_{i_1} \ \ \cdots \ \ y_{i_N}\ ]^T$, and the vector of all the outputs is $Y = [\ y_{1_1} \ \ \cdots \ \ y_{n_1} \ \ \cdots \ \ y_{1_N} \ \ \cdots \ \ y_{n_N}\ ]^T$.

We define the operator Π as $\Pi = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$, $N \in \mathbb{Z}_+$, I_N being the $N \times N$ identity matrix and $\mathbf{1}$ the N-dimensional vector of ones. As described in [12], [8], the operator Π measures the lack of consensus between the elements of a vector $\mathbf{z} \in \mathbb{R}^N$ in the following sense: the j^{th} element of the vector $\Pi\mathbf{z}$ is the difference between the j^{th} element of z and the average of all the elements of z. Note that $\Pi^T\Pi = \Pi$.

For any real matrix Γ, we denote by Γ_s its symmetric part, i.e. $\Gamma_s = \frac{1}{2}\left(\Gamma + \Gamma^T\right)$. Finally, we use the notation $|\cdot|$ to denote the euclidian norm of a vector.

III. Cyclic Feedback Systems

As illustrated in Figure 1, a CFS is defined as a cascade of subsystems with a unity negative feedback.

Each constitutive subsystem H_{i_j} can be either dynamic or static. Dynamic subsystems are assumed to have the structure

$$H_{i_j} \begin{cases} \dot{x}_{i_j} = -f_{i_j}(x_{i_j}) + u_{i_j} \\ y_{i_j} = x_{i_j} \end{cases} \tag{1}$$

where $f_{i_j}(\cdot)$ is in $\mathcal{C}^1(\mathbb{R}_{\geq 0})$, monotonically increasing and is such that $f_{i_j}(0) = 0 \ \forall i, j$. On the other hand, static subsystems are assumed to have the structure

$$H_{i_j} \begin{cases} x_{i_j} = g_{i_j}(u_{i_j}) \\ y_{i_j} = x_{i_j} \end{cases} \tag{2}$$

where $g_{i_j}(\cdot)$ is in $\mathcal{C}^1(\mathbb{R}_{\geq 0})$, monotonically increasing $\forall i, j$, and, for $i = 1, \cdots, n-1$, $g_{i_j}(0) = 0 \ \forall j$. The last subsystem of the cascade is assumed to be static and represented by the monotonically increasing, $\mathcal{C}^1(\mathbb{R}_{\geq 0})$ map $g_{n_j}(\cdot) : [0, \infty) \to [-\hat{g}_{n_j}, 0)$, $\hat{g}_{n_j} > 0$. Therefore $g_{n_j}(\cdot)$ is bounded $\forall j$ and $g_{i_j}^{-1}(\cdot)$ exists $\forall i, j$. Note that with the above properties, the positive orthant is invariant (see [6]), and so, assuming all initial conditions lie in the positive orthant, we need only consider signals $x_{i_j} \geq 0$, $\forall i, j$.

The inputs u_{i_j} to each of these subsystems satisfy the following cyclic feedback interconnection rules:

$$\begin{aligned} u_{1_j} &= u_{\text{ext}_j} - y_{n_j} \\ u_{i_j} &= y_{i-1_j}, i = 2, \cdots, n \end{aligned} \tag{3}$$

where u_{ext_j} is the external input to the j^{th} CFS, which can be used to interconnect several CFSs.

In the rest of the paper, we will use the following ultimate boundedness assumption:

Definition 1 (Ultimate boundedness): The solution of a cyclic feedback system j is said to be *ultimately bounded* if, $\forall i \ \exists T_{i_j}, \hat{x}_{i_j}$ such that $x_{i_j}(t) \leq \hat{x}_{i_j} \ \forall t > T_{i_j}$.

The bound $\hat{x}_{i_j}$ can be found explicitly for several classes of CFSs. For example the Goodwin model given in [3] is a CFS which can be proven to be semipassive (see [13] for a definition of semipassivity), and therefore its solutions are bounded in the sense of Definition 1. However, even for some CFS models that are not semipassive, such as those discussed in [1], the bounds $\hat{x}_{i_j}$ can be found for both the isolated and the interconnected CFSs cases. Since the states x_{i_j} form the arguments of the functions $f_{i_j}(\cdot)$ and $g_{i_j}(\cdot)$, the invariance of the positive orthant, the boundedness properties of x_{i_j} and the monotonicity of these functions limit their domains and hence images to

$$\begin{aligned} &f_{i_j}(\cdot) : [0, \hat{x}_{i_j}] \to [0, f_{i_j}(\hat{x}_{i_j})] && i = 1, \cdots, n \\ &g_{i_j}(\cdot) : [0, \hat{x}_{i-1_j}] \to [0, g_{i_j}(\hat{x}_{i-1_j})] && i = 1, \cdots, n-1 \\ &g_{n_j}(\cdot) : [0, \hat{x}_{n-1_j}] \to [-\hat{g}_{n_j}, g_{n_j}(\hat{x}_{n-1_j})] && i = 1, \cdots, n-1 \end{aligned}$$

Henceforth we shall refer to the domain and image of $f_{i_j}(\cdot)$ or $g_{i_j}(\cdot)$ as $\mathcal{D}_{i_j}$ and $\mathcal{I}_{i_j}$ respectively.

A. Coupling

We assume that network coupling is static and linear. This type of coupling is conveniently defined using a coupling matrix $\Gamma \in \mathbb{R}^{N \times N}$. The network coupling topology is thus defined by the following relation between the inputs and outputs of the CFSs:

$$\mathbf{u}_{\text{ext}} = -\Gamma \mathbf{y}_1 \tag{4}$$

We further restrict the topology by assuming that:

(A1) $\text{rank}(\Gamma) = N - 1$
(A2) $\Gamma + \Gamma^T \geq 0$
(A3) $\Gamma\mathbf{1} = \Gamma^T\mathbf{1} = \mathbf{0}$, where $\mathbf{1}, \mathbf{0} \in \mathbb{R}^N$ are vectors whose elements are all 1 and 0 respectively.

Note that from (A3), $\Gamma\Pi = \Pi\Gamma = \Gamma$. We refer to [8] Section IV.A for a graph interpretation of these assumptions.

B. Incremental Systems & Incremental Storage Functions

We assume that all corresponding subsystems have the same parametric structure and only differ by the parameters of these structures, i.e. H_{i_k} and H_{i_l} have the same parametric structure, $\forall k, l \in 1, \cdots, N$. With each subsystem H_{i_j}, $j = 1, \cdots, N$, we associate the incremental dynamic subsystem

$$H_{i_\Delta} \begin{cases} \Pi\dot{\mathbf{x}}_i = -\Pi F_i(\mathbf{x}_i) + \Pi\mathbf{u}_i \\ \Pi\mathbf{y}_i = \Pi\mathbf{x}_i \end{cases} \tag{5}$$

or the incremental static subsystem

$$H_{i_\Delta} \begin{cases} \Pi\mathbf{x}_i = \Pi G_i(\mathbf{u}_i) \\ \Pi\mathbf{y}_i = \Pi\mathbf{x}_i \end{cases} \tag{6}$$

where $F_i(\mathbf{x}_i) = [\ f_{i_1}(x_{i_1}) \ \ \cdots \ \ f_{i_N}(x_{i_N})\ ]^T$, $G_i(\mathbf{u}_i) = [\ g_{i_1}(u_{i_1}) \ \ \cdots \ \ g_{i_N}(u_{i_N})\ ]^T$, and where $f_{i_j}(\cdot)$ and $g_{i_j}(\cdot)$ satisfy the previous assumptions (see (1) and (2)). The incremental inputs $\Pi\mathbf{u}_i$ are given by

$$\begin{aligned} \Pi\mathbf{u}_1 &= \Pi\mathbf{u}_{\text{ext}} - \Pi\mathbf{y}_n \\ \Pi\mathbf{u}_i &= \Pi\mathbf{y}_{i-1}, i = 2, \cdots, n \end{aligned} \tag{7}$$

where $\mathbf{u}_{\text{ext}} = [\ u_{\text{ext}_1} \ \ \cdots \ \ u_{\text{ext}_N}\ ]^T$.

Proposition 1: For the incremental subsystem H_{i_Δ} defined in (5) or (6), we can write an incremental storage function V_i which obeys a dissipation inequality of the form

$$\dot{V}_i = -\gamma_i \mathbf{x}_i^T \Pi^T \Pi M_i(\mathbf{x}_i) + \gamma_i \mathbf{x}_i^T \Pi^T \Pi \mathbf{u}_i \tag{8}$$

where γ_i is a positive constant, $M_i(\mathbf{x}_i) = [\ m_{i_1}(x_{i_1}) \ \ \cdots \ \ m_{i_N}(x_{i_N})\ ]^T$ and $m_{i_j}(\cdot) : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing function which equals $f_{i_j}(\cdot)$ when H_{i_j} is defined as in (1) and $g_{i_j}^{-1}(\cdot)$ when H_{i_j} is defined as (2).

Proof: Suppose the subsystem is of the dynamic type (1). Then using $V_i = \frac{\gamma_i}{2}\mathbf{x}_i^T\Pi^T\Pi\mathbf{x}_i$ we obtain (8) with $M_i(\mathbf{x}_i) = F_i(\mathbf{x}_i)$. For static subsystems, we use $V_i = 0$ as the storage function. This gives the dissipation inequality $\dot{V}_i = 0 = -\gamma_i\mathbf{x}_i^T\Pi^T\Pi M_i(\mathbf{x}_i) + \gamma_i\mathbf{x}_i^T\Pi^T\Pi\mathbf{u}_i$. with $M_i(\mathbf{x}_i) = G_i^{-1}(\mathbf{x}_i) = \begin{bmatrix} g_{i_1}^{-1}(x_{i_1}) & \cdots & g_{i_N}^{-1}(x_{i_N}) \end{bmatrix}^T$. ■

We now define a *nominal* CFS to which each CFS in the network is compared. As notation, the nominal counterpart of a set of functions $m_{i_j}(\cdot)$, $j = 1, \cdots, N$ is $\tilde{m}_i(\cdot)$. Let $\tilde{m}_i(\cdot) : \mathcal{D}_i \to \mathcal{I}_i$, where $\mathcal{D}_i, \mathcal{I}_i \subset \mathbb{R}$, be a monotonically increasing function that is a convex combination of the functions $m_{i_j}(\cdot)$, $j = 1, \cdots, N$:

$$\tilde{m}_i(x) = \sum_{j=1}^{N} \alpha_{i_j} m_{i_j}(x), \quad \sum_{j=1}^{N} \alpha_{i_j} = 1, \alpha_{i_j} \in [0,1], \forall j \tag{9}$$

where $m_{i_j}(\cdot)$ is defined in Proposition 1 and the positive scalars α_{i_j} are to be chosen. The domain $\mathcal{D}_i$ and image $\mathcal{I}_i$ of the functions $\tilde{m}_i(\cdot)$ are then respectively the unions of the domains and the images of the constituent functions $m_{i_j}(\cdot)$ for $j = 1, \cdots, N$.

We also define the function $\phi_{i_j}(\cdot) : [0, \hat{x}_{m_i}] \to \mathbb{R}$ to be $\phi_{i_j}(x_{i_j}) = m_{i_j}(x_{i_j}) - \tilde{m}_i(x_{i_j})$. In vector form, we have

$$\Phi_i(\mathbf{x}_i) = \begin{bmatrix} \phi_{i_1}(x_{i_1}) & \cdots & \phi_{i_N}(x_{i_N}) \end{bmatrix}^T$$

with

$$\Phi_i(\mathbf{x}_i) = M_i(\mathbf{x}_i) - \tilde{M}_i(\mathbf{x}_i) \tag{10}$$

where $\tilde{M}_i(\mathbf{x}_i) = \begin{bmatrix} \tilde{m}_i(x_{i_1}) & \cdots & \tilde{m}_i(x_{i_N}) \end{bmatrix}^T$. Finally we define metrics on the differences between a signal $m_{i_j}(\cdot)$ and its nominal counterpart $\tilde{m}_i(\cdot)$

$$\hat{\phi}_{i_j} = \sup_{x_{i_j} \in \mathcal{D}_i} |m_{i_j}(x_{i_j}) - \tilde{m}_i(x_{i_j})| \tag{11}$$

$$\hat{\Phi}_i = \begin{bmatrix} \hat{\phi}_{i_1}(x_{i_1}) & \cdots & \hat{\phi}_{i_N}(x_{i_N}) \end{bmatrix}^T \tag{12}$$

Lemma 1: For $\tilde{M}_i(\cdot)$ defined as above, $\mathbf{x}_i^T\Pi^T\Pi\tilde{M}_i(\mathbf{x}_i) > 0$. Furthermore, if the CFSs, coupled as in (4), are ultimately bounded in the sense of Definition 1, $\exists\gamma_i$: $\gamma_i\mathbf{x}_i^T\Pi^T\Pi\tilde{M}_i(\mathbf{x}_i) \geq \mathbf{x}_i^T\Pi\mathbf{x}_i$.

Proof: By expanding $\mathbf{x}_i^T\Pi\tilde{M}_i(\mathbf{x}_i)$ we obtain

$$\mathbf{x}_i^T\Pi\tilde{M}_i(\mathbf{x}_i) = \frac{1}{2N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left[\left(x_{i_j} - x_{i_k}\right)\left(\tilde{m}_i(x_{i_j}) - \tilde{m}_i(x_{i_k})\right)\right]$$

Since $\tilde{m}_i(\cdot)$ is monotonically increasing, we have $\left(x_{i_j} - x_{i_k}\right)\left(\tilde{m}_i(x_{i_j}) - \tilde{m}_i(x_{i_k})\right) \geq 0, \forall j, k$. Therefore $\mathbf{x}_i^T\Pi\tilde{M}_i(\mathbf{x}_i) \geq 0$.

For the second part simply let $\frac{1}{\gamma_i} = \inf_{x \in \mathcal{D}_i}\frac{\mathrm{d}\tilde{m}_i(x)}{\mathrm{d}x}$, where the infimum is over the domain of $\tilde{m}_i(\cdot)$ (which is closed and bounded because of the ultimate boundedness of the state x_{i_j}). Then, by the mean value theorem we have $\gamma_i(x_{i_j} - x_{i_k})(\tilde{m}_i(x_{i_j}) - \tilde{m}_i(x_{i_k})) > (x_{i_j} - x_{i_k})^2, \forall i, j$ and the result follows since

$$\mathbf{x}_i^T\Pi\mathbf{x}_i = \frac{1}{2N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left(x_{i_j} - x_{i_k}\right)^2$$

■

We now aim to write incremental dissipation inequalities satisfied by the incremental subsystems of the CFSs and refer the reader to [14], [15] for comprehensive treatments of passivity and the more general concept of dissipativity. We will show that in general these inequalities will be such that these subsystems are incrementally output semipassive: that is, they are incrementally output passive outside a ball centered on the origin $|\Pi\mathbf{y}_i| = 0$. To show this we make use of the nominal systems defined above. Before proceeding, however, we give a formal definition of incremental output semipassivity:

Definition 2 (Incremental Output Semipassivity): System H_{i_Δ} is incrementally output semipassive with respect to input $\mathbf{u}_i$ and output $\mathbf{y}_i$ if there exists a positive definite incremental storage function $V_i : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$, a constant $\varrho > 0$, and a function $\mathcal{H}_i(\cdot)$ such that

$$\dot{V}_i \leq -\mathcal{H}_i(\Pi\mathbf{y}_i) + (\Pi\mathbf{u}_i)^T(\Pi\mathbf{y}_i)$$

with $\mathcal{H}_i(\Pi\mathbf{y}_i) > 0, \quad \forall|\Pi\mathbf{y}_i| > \varrho$.
As a result, if there is no incremental input ($\Pi\mathbf{u}_i = 0$), the incremental outputs $\Pi\mathbf{y}_i$ will decrease to an absolute value of at least ϱ.

Lemma 2: Assume that the CFSs as defined in (1)-(3), and coupled as in (4), are ultimately bounded in the sense of Definition 1. Then the incremental subsystem H_{i_Δ} as defined in (5) or (6) is incrementally output semipassive in the sense of Definition 2 with $\varrho = \gamma_i|\hat{\Phi}_i|$ if

$$\frac{1}{\gamma_i} = \inf_{x_{i_j} \in \mathcal{D}_i}\frac{\mathrm{d}\tilde{m}_i(x_{i_j})}{\mathrm{d}x_{i_j}} > 0$$

where γ_i is the *incremental secant gain* of the i^{th} nominal subsystem and $\tilde{m}_i(\cdot)$ is defined in (9).

Proof: Using the incremental storage functions suggested in Proposition 1 with $\frac{1}{\gamma_i} = \inf_{x_{i_j} \in \mathcal{D}_i}\frac{\mathrm{d}\tilde{m}_i(x_{i_j})}{\mathrm{d}x_{i_j}}$, the incremental dissipation equality then becomes:

$$\dot{V}_i = -\gamma_i\mathbf{x}_i^T\Pi^T\Pi M_i(\mathbf{x}_i) + \gamma_i\mathbf{u}_i^T\Pi^T\Pi\mathbf{x}_i \tag{13}$$

Using $\Pi^T\Pi = \Pi$ and (10) we obtain

$$\begin{aligned} \mathbf{x}_i^T\Pi^T\Pi M_i(\mathbf{x}_i) &= \mathbf{x}_i^T\Pi M_i(\mathbf{x}_i) \\ &= \mathbf{x}_i^T\Pi\tilde{M}_i(\mathbf{x}_i) + \mathbf{x}_i^T\Pi\Phi_i(\mathbf{x}_i) \end{aligned} \tag{14}$$

Using (14) and Lemma 1,

$$\begin{aligned} \dot{V}_i &\leq -\mathbf{x}_i^T\Pi\mathbf{x}_i - \gamma_i\mathbf{x}_i^T\Pi\Phi_i(\mathbf{x}_i) + \gamma_i\mathbf{u}_i^T\Pi\mathbf{x}_i \\ &\leq -\mathbf{y}_i^T\Pi\mathbf{y}_i - \gamma_i\mathbf{y}_i^T\Pi\Phi_i(\mathbf{y}_i) + \gamma_i\mathbf{u}_i^T\Pi\mathbf{y}_i \\ &\leq -|\Pi\mathbf{y}_i|\left(|\Pi\mathbf{y}_i| - \gamma_i|\hat{\Phi}_i|\right) + \gamma_i\mathbf{u}_i^T\Pi\mathbf{y}_i \end{aligned} \tag{15}$$

The subsystem is therefore incrementally output semipassive in the sense of Definition 2 with $\varrho = \gamma_i|\hat{\Phi}_i|$.

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C. Synchronization and Incremental Stability

Definition 3 (Output synchronization): The i^{th} output of a collection of N CFSs is said to be synchronized when,

$$y_{i_j} = y_{i_k}, \forall j, k \in 1, \cdots, N$$

Definition 4 (Output synchronization error): The synchronization error σ_i of the i^{th} output $\mathbf{y}_i$, is defined as

$$\sigma_i = \lim_{t\to\infty} \sup \left(\sqrt{(\Pi \mathbf{y}_i(t))^T (\Pi \mathbf{y}_i(t))} \right)$$

The total output synchronization error is then given by

$$\sigma = \lim_{t\to\infty} \sup \left[((\Pi \otimes I_n)Y(t))^T ((\Pi \otimes I_n)Y(t)) \right]^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \sigma_i^2 \right)^{\frac{1}{2}}$$

We denote by $\hat{\sigma}_i$ and $\hat{\sigma}$ an upper bound on σ_i and σ respectively.

Obviously, output synchronization implies zero output synchronization error ($\sigma_i = 0,\ \forall i$). Therefore to prove asymptotic output synchronization, it is enough to prove that the signal $\Pi\mathbf{y}_i$ is asymptotically stable. Furthermore, if this signal is not asymptotically stable, we shall nevertheless prove that under certain conditions its magnitude will asymptotically decrease below a fixed value that will be referred to as the upper bound on the asymptotic synchronization error.

IV. Main Results

In this section, we obtain bounds on the synchronization error using two approaches: an incremental small-gain theorem approach and an approach based on incremental output semipassivity.

A. A incremental small-gain theorem result

Theorem 1: Consider a network of N non-identical CFSs as defined in (1)-(3), satisfying the assumptions of Lemma 2, coupled as in (4) via a coupling matrix Γ which satisfies the properties $(A1)-(A3)$, and such that each subsystem is incrementally output semipassive in the sense of Lemma 2. Defining γ_i ($i = 1, \cdots, n$) as in Lemma 2, let $\tilde{\gamma}_1 = \frac{\gamma_1}{1+\gamma_1\lambda_2(\Gamma_s)}$, where $\lambda_2(\Gamma_s) > 0$ is the second smallest eigenvalue of the symmetric part of Γ, and let $\tilde{\gamma}_i = \gamma_i$ for $i = 2, \cdots, n$. If the coupling is such that $\lambda_2(\Gamma_s) > \frac{-1+\gamma_1\gamma_2\cdots\gamma_n}{\gamma_1}$ then $\tilde{\gamma}_1 \cdots \tilde{\gamma}_n < 1$ and an upper bound $\hat{\sigma}_i$ on the synchronization error σ_i for the i^{th} output of the CFSs is

$$\hat{\sigma}_i = \Upsilon \prod_{j=1}^{i} \tilde{\gamma}_j + \Theta_i$$

where we define $\Theta_i = \sum_{j=1}^{i} |\hat{\Phi}_j| \prod_{k=j}^{i} \tilde{\gamma}_k$ and $\Upsilon = \min\left(|\hat{\mathbf{y}}_n|, \frac{1}{1-\tilde{\gamma}_1\cdots\tilde{\gamma}_n}\Theta_n\right)$ and $\hat{\mathbf{y}}_n = [\ \hat{g}_{n_1} \ \cdots \ \hat{g}_{n_N}\ ]^T$.

Proof: Using the dissipation inequality (15), the interconnection rules (7) and the coupling (4) we have, for the first incremental dissipation inequality

$$\dot{V}_1 \leq -|\Pi\mathbf{y}_1| \left((1+\gamma_1\lambda_2(\Gamma_s))|\Pi\mathbf{y}_1| - \gamma_1|\hat{\Phi}_1| - \gamma_1|\Pi\mathbf{y}_n| \right) \tag{16}$$

From this, observe that $V_1 = \frac{1}{2}\mathbf{x}_1^T\Pi\mathbf{x}_1 = \frac{1}{2}\mathbf{y}_1^T\Pi\mathbf{y}_1$ is monotonically decreasing with time for all $|\Pi\mathbf{y}_1| > \frac{\gamma_1}{1+\gamma_1\lambda_2(\Gamma_s)}\left(|\hat{\Phi}_1| + |\Pi\mathbf{y}_n|\right) = \tilde{\gamma}_1\left(|\hat{\Phi}_1| + |\Pi\mathbf{y}_n|\right)$.

Since $|\Pi\mathbf{y}_n| < |\hat{\mathbf{y}}_n|$, an initial bound $\hat{\sigma}_1(1)$ is given by

$$\hat{\sigma}_1(1) = \tilde{\gamma}_1\left(|\hat{\Phi}_1| + |\hat{\mathbf{y}}_n|\right)$$

Therefore after finite time, $|\Pi\mathbf{y}_1|$ will decrease below $\tilde{\gamma}_1\left(|\hat{\Phi}_1| + |\hat{\mathbf{y}}_n|\right)$. Now consider the dissipation inequality

$$\dot{V}_2 \leq -|\Pi\mathbf{y}_2| \left(|\Pi\mathbf{y}_2| - \tilde{\gamma}_2|\Phi_2| - \tilde{\gamma}_2|\Pi\mathbf{y}_1|\right)$$

The incremental storage function V_2 is monotonically decreasing for all $|\Pi\mathbf{y}_2| > \tilde{\gamma}_2(|\hat{\Phi}_2| + |\Pi\mathbf{y}_1|)$. Since after finite time $|\Pi\mathbf{y}_1| < \tilde{\gamma}_1\left(|\hat{\Phi}_1| + |\hat{\mathbf{y}}_n|\right)$, we know that $|\Pi\mathbf{y}_2|$ will, also after finite time, decrease below $\tilde{\gamma}_2\tilde{\gamma}_1\left(|\hat{\Phi}_1| + |\hat{\mathbf{y}}_n|\right) + \tilde{\gamma}_2|\hat{\Phi}_2|$, and so an initial upper bound on the synchronization error for the second output is

$$\hat{\sigma}_2(1) = \tilde{\gamma}_2\tilde{\gamma}_1\left(|\hat{\Phi}_1| + |\hat{\mathbf{y}}_n|\right) + \tilde{\gamma}_2|\hat{\Phi}_2|$$

This bound will place a limit on the synchronization error of the subsequent outputs. By repeating this method, the incremental state $\Pi\mathbf{y}_i$ obeys, after finite time, the initial upper bound $\hat{\sigma}_i(1)$ on the synchronization error σ_i

$$|\Pi\mathbf{y}_i| < \hat{\sigma}_i(1) = (\tilde{\gamma}_i\cdots\tilde{\gamma}_1)|\hat{\mathbf{y}}_n| + \Theta_i \tag{17}$$

We then have an upper bound on the synchronization error of each incremental output.

Specifically, if $\tilde{\gamma}_1\cdots\tilde{\gamma}_n < 1$, we have two upper bounds on $|\Pi\mathbf{y}_n|$, which are $|\hat{\mathbf{y}}_n|$ and $\hat{\sigma}_n(1)$. The smaller upper bound is determined by the magnitude of the differences in the CFS parameters and the strength of the coupling: if $\Theta_n < |\hat{\mathbf{y}}_n|$ then by making the coupling strength $\lambda_2(\Gamma_s)$ large enough $\tilde{\gamma}_1$ can be made sufficiently small so that $\hat{\sigma}_n(1) = (\tilde{\gamma}_n\cdots\tilde{\gamma}_1)|\hat{\mathbf{y}}_n| + \Theta_n < |\hat{\mathbf{y}}_n|$. As a result, for the n^{th} output, the upper bound is such that $|\Pi\mathbf{y}_n| < \min(|\hat{\mathbf{y}}_n|, \hat{\sigma}_n(1))$. If $|\hat{\mathbf{y}}_n| \leq \hat{\sigma}_n(1)$, then this upper bound cannot be reduced and the upper bound on σ_i is given by (17). However if $\hat{\sigma}_n(1) < |\hat{\mathbf{y}}_n|$ then substituting the bound on $|\Pi\mathbf{y}_n|$ into (16) yields

$$\dot{V}_1 \leq -|\Pi\mathbf{y}_1| \left((1+\gamma_1\lambda_2(\Gamma_s))|\Pi\mathbf{y}_1| - \gamma_1|\hat{\Phi}_1| - \gamma_1\hat{\sigma}_n(1) \right)$$

This iteration gives a new, reduced upper bound $\hat{\sigma}_1(2)$ on σ_1

$$|\Pi\mathbf{y}_1| < \hat{\sigma}_1(2) = \tilde{\gamma}_1\left(\hat{\Phi}_1 + \hat{\sigma}_n(1)\right)$$

As a consequence, the upper bound on σ_i is also reduced to

$$\hat{\sigma}_i(2) = (\tilde{\gamma}_i\cdots\tilde{\gamma}_1)\hat{\sigma}_n(1) + \Theta_i$$

This iterative procedure yields a difference equation for the κ^{th} iteration of the synchronization error of the i^{th} output

$$\hat{\sigma}_i(\kappa+1) = (\tilde{\gamma}_i\cdots\tilde{\gamma}_1)\hat{\sigma}_n(\kappa) + \Theta_i \tag{18}$$

Letting $\hat{\sigma}_n(0) = |\hat{\mathbf{y}}_n|$, it is easy to show that

$$\hat{\sigma}_n(\kappa) = (\tilde{\gamma}_1\cdots\tilde{\gamma}_n)^\kappa\hat{\sigma}_n(0) + \Theta_n\sum_{m=1}^{\kappa}(\tilde{\gamma}_1\cdots\tilde{\gamma}_n)^{m-1}$$

and so clearly, if $\tilde{\gamma}_1\cdots\tilde{\gamma}_n < 1$

$$\lim_{\kappa\to\infty}\hat{\sigma}_n(\kappa) = \frac{1}{1-\tilde{\gamma}_1\cdots\tilde{\gamma}_n}\Theta_n \tag{19}$$

Therefore if $\hat{\sigma}_n(1) < \hat{\sigma}_n(0) = |\hat{\mathbf{y}}_n|$ then we can see from (18) that $\hat{\sigma}_n(\kappa+1) - \hat{\sigma}_n(\kappa) = (\tilde{\gamma}_1\cdots\tilde{\gamma}_n)^\kappa(\hat{\sigma}_n(1) - \hat{\sigma}_n(0)) < 0$ and hence $\hat{\sigma}_n(\kappa)$ *decreases* to (19) as $\kappa\to\infty$.

From this we infer that the minimum value of $\hat{\sigma}_n$ is either $|\hat{\mathbf{y}}_n|$ (in the case $\hat{\sigma}_n(1) \geq |\hat{\mathbf{y}}_n|$) or (19) otherwise. Letting

$$\Upsilon = \min\left(|\hat{\mathbf{y}}_n|, \frac{1}{1-\tilde{\gamma}_1\cdots\tilde{\gamma}_n}\Theta_i\right)$$

then (18) implies that, after finite time

$$|\Pi\mathbf{y}_i| < \hat{\sigma}_i = \Upsilon\prod_{j=1}^{i}\tilde{\gamma}_j + \Theta_i, \quad \forall i$$

which gives the result. ■

Remark 1: Note that if all the CFSs are identical, that is, if $\hat{\phi}_{i_j} = 0, \forall i, j$ (from (11)), we obtain zero synchronization error and hence full output synchronization.
Another feature to note is that strengthening the coupling (by increasing the coupling gain and therefore $\lambda_2(\Gamma_s)$) can make the upper bound on the synchronization error for the first output $\hat{\sigma}_1$, arbitrarily small and reduce the upper bound on the synchronization error for the i^{th} output, $\hat{\sigma}_i$, $\forall i = 2, \cdots, N$, arbitrarily close to $\hat{\sigma}_i = \sum_{j=2}^{i}|\hat{\Phi}_j|\prod_{k=j}^{i}\tilde{\gamma}_k$.

B. An incremental output semipassivity result

Theorem 2: Consider a network of N non-identical CFSs as defined in (1)-(3), satisfying the assumptions of Lemma 2, coupled as in (4) via a coupling matrix Γ which satisfies the properties $(A1)-(A3)$, and such that each subsystem is incrementally output semipassive in the sense of Lemma 2. Let $\lambda_2(\Gamma_s)$ be the second smallest eigenvalue of the symmetric part of Γ. If $\lambda_2(\Gamma_s) > k^* = \frac{-1+\gamma_1\cdots\gamma_n\left(\cos^n\left(\frac{\pi}{n}\right)\right)}{\gamma_1}$ then the network of interconnected CFSs is incrementally output semipassive with a finite total output synchronisation error upper bound $\hat{\sigma}$ given by the right hand side of (25).

Proof: From (15), an incremental dissipation inequality of the form

$$\dot{V}_i \leq -\mathbf{y}_i^T\Pi\mathbf{y}_i + \gamma_i\mathbf{u}_i^T\Pi\mathbf{y}_i - \gamma_i\mathbf{y}_i^T\Pi\Phi_i(\mathbf{y}_i)$$

can be written for the i^{th} subsystem of the CFSs. Scaling $\dot{V}_1$ by $\frac{1}{1+k\gamma_1}$, $k > 0$, we then add and subtract $\frac{k\gamma_1}{1+k\gamma_1}\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1$ to it and using $\Pi^T\Pi = \Pi$ we obtain

$$\begin{aligned}\dot{V}_1 \leq &-\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1 + \tilde{\gamma}_1\mathbf{u}_1^T\Pi^T\Pi\mathbf{y}_1 + k\tilde{\gamma}_1\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1\\ &-\tilde{\gamma}_1\mathbf{y}_1^T\Pi^T\Pi\Phi_1(\mathbf{y}_1)\end{aligned}$$

where $\tilde{\gamma}_1 = \frac{\gamma_1}{1+k\gamma_1}$. Notice that we can make $\tilde{\gamma}_1$ arbitrarily small by increasing k. We then take as an incremental storage function V for the entire network of CFSs the linear combination of the incremental storage functions V_i of the individual subsystems i: $V = \sum_{i=1}^{n} d_iV_i$ where $d_i > 0$. The values of d_i are the same as those in [6], with $r = (\tilde{\gamma}_1\gamma_2\cdots\gamma_n)^{\frac{1}{n}}$:

$$d_1 = 1, d_2 = \frac{r^2}{\gamma_2^2}, d_3 = \frac{r^4}{(\gamma_2\gamma_3)^2}, \cdots, d_n = \frac{r^{2n-2}}{(\gamma_2\cdots\gamma_n)^2} \tag{20}$$

Let $D = \operatorname{diag}\{\ d_1, \ \cdots, \ d_n\ \}$ and

$$A = \begin{bmatrix} -1 & 0 & \cdots & 0 & -\tilde{\gamma}_1\\ \gamma_2 & -1 & 0 & \ddots & 0\\ 0 & \gamma_3 & -1 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & \gamma_n & -1\end{bmatrix}$$

Using the interconnection rules (7) we obtain:

$$\begin{aligned}\dot{V} \leq &\frac{1}{2}\left((\Pi\otimes I_n)Y\right)^T\left(I_N\otimes(DA+A^TD)\right)\left((\Pi\otimes I_n)Y\right)+\\ &d_1\tilde{\gamma}_1(\mathbf{u}_{\text{ext}}^T\Pi^T\Pi\mathbf{y}_1 + k\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1) + \sum_{i=1}^{n} d_i\gamma_i\mathbf{y}_i^T\Pi\Phi_i(\mathbf{y}_i)\end{aligned} \tag{21}$$

Now we include the coupling, $\mathbf{u}_{\text{ext}} = -\Gamma\mathbf{y}_1$. Note that $\Gamma\Pi = \Pi\Gamma$. We thus obtain

$$-\mathbf{y}_1^T\Gamma^T\Pi^T\Pi\mathbf{y}_1 = -\mathbf{y}_1^T\Pi^T\Gamma^T\Pi\mathbf{y}_1 \leq -\lambda_2(\Gamma_s)\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1 \tag{22}$$

Substituting (22) into (21)

$$\begin{aligned}\dot{V} \leq &\frac{1}{2}\left((\Pi\otimes I_n)Y\right)^T\left(I_N\otimes(DA+A^TD)\right)\left((\Pi\otimes I_n)Y\right)+\\ &d_1\tilde{\gamma}_1(k-\lambda_2(\Gamma_s))\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1 + \sum_{i=1}^{n} d_i\gamma_i\mathbf{y}_i^T\Pi\Phi_i(\mathbf{y}_i)\end{aligned}$$

Note (from [6]) that if the secant gain condition

$$\tilde{\gamma}_1\gamma_2\cdots\gamma_n \leq \sec^n\left(\frac{\pi}{n}\right) \tag{23}$$

holds then $DA + A^TD \leq -\epsilon I_n \leq 0$ (see [6]). By increasing k sufficiently, $\tilde{\gamma}_1$ can be made arbitrarilly small, and so the secant gain condition is satisfied provided that $k \geq k^* = \frac{-1+\gamma_1\cdots\gamma_n\cos^n\left(\frac{\pi}{n}\right)}{\gamma_1}$. For $k > k^*$, we thus can write the incremental dissipation inequality in the following way:

$$\begin{aligned}\dot{V} \leq &-\epsilon\left((\Pi\otimes I_n)Y\right)^T\left((\Pi\otimes I_n)Y\right)+\\ &d_1\tilde{\gamma}_1(k-\lambda_2(\Gamma_s))\mathbf{y}_1^T\Pi^T\Pi\mathbf{y}_1 + \sum_{i=1}^{n} d_i\gamma_i\mathbf{y}_i^T\Pi\Phi_i(\mathbf{y}_i)\end{aligned}$$

Finally, if the coupling strength $\lambda_2(\Gamma_s) \geq k (> k^*)$, we have:

$$\dot{V} \leq -\epsilon\left((\Pi\otimes I_n)Y\right)^T\left((\Pi\otimes I_n)Y\right) + \sum_{i=1}^{n} d_i\gamma_i\mathbf{y}_i^T\Pi\Phi_i(\mathbf{y}_i) \tag{24}$$

Define $\hat{\mathbf{\Phi}} = [\ \hat{\phi}_{1_1} \ \cdots \ \hat{\phi}_{n_1} \ \cdots \ \hat{\phi}_{1_N} \ \cdots \ \hat{\phi}_{n_N}\ ]^T$ and the diagonal matrix $\Lambda = \operatorname{diag}\{\ \tilde{\gamma}_1, \ \gamma_2, \ \cdots, \ \gamma_n\ \}$. We can equivalently rewrite (24) as

$$\begin{aligned}\dot{V} \leq &-\epsilon\left((\Pi\otimes I_n)Y\right)^T\left((\Pi\otimes I_n)Y\right)+\\ &\left((\Pi\otimes I_n)Y\right)^T\left((I_N\otimes D\Lambda)\hat{\mathbf{\Phi}}\right)\end{aligned}$$

Therefore $\dot{V} < 0$ for all

$$|(\Pi\otimes I_n)Y| > \frac{|((I_N\otimes D\Lambda)\hat{\mathbf{\Phi}})|}{\epsilon} \tag{25}$$

showing that the network of CFSs is incrementally output semipassive and that it achieves an upper bound on the total output synchronisation error given by $\hat{\sigma} = \frac{|((I_N\otimes D\Lambda)\hat{\mathbf{\Phi}})|}{\epsilon}$. ■

Remark 2: Note that if all the CFSs are identical, that is, if $\hat{\phi}_{i_j} = 0$, $\forall i, j$ (from (11)), the earlier result in [8] is retrieved, as strong coupling ensures that the network then becomes incrementally output strictly passive, resulting in zero synchronization error and hence full output synchronization.

V. EXAMPLE

The example we will consider here consists of $N = 2$ interconnected Goodwin model oscillators [3]. The j^{th} oscillator is given by the following model:

$$\begin{aligned}
\dot{x}_{1_j} &= -x_{4_j} - b_j x_{1_j} + u_{\text{ext}_j} \\
\dot{x}_{2_j} &= b_j x_{1_j} - b_j x_{2_j} \\
\dot{x}_{3_j} &= b_j x_{2_j} - b_j x_{3_j} \\
x_{4_j} &= -\frac{1}{1 + x_{3_j}^{p_j}}
\end{aligned}$$

where $u_{\text{ext}_j} = -k\left(N x_{1_j} - \sum_{l=1}^{N} x_{1_l}\right)$. Since the system is semipassive, its solutions are bounded with or without the coupling, as required by Theorems 1 and 2. The oscillator parameters are as follows $b_1 = b_2 = \frac{1}{2}$, $p_1 = 17$ and $p_2 = 20$. We take as the nominal system the first oscillator (where $j = 1$) and so the second oscillator's deviation from the first is such that $\hat{\phi}_{1_2} = \hat{\phi}_{2_2} = \hat{\phi}_{3_2} = 0$ and $\hat{\phi}_{4_2} = 0.0358$. The incremental secant gains of the nominal system are then $\tilde{\gamma}_1 = \frac{b_1}{b_1 + kN}$, $\tilde{\gamma}_2 = 1$, $\tilde{\gamma}_3 = \frac{1}{b_1}$ and $\tilde{\gamma}_4 = \sup_{x_3}\left(\frac{\mathrm{d}}{\mathrm{d}x_3}\left(-\frac{1}{1+x_3^{p_1}}\right)\right) = 4.6385$. With a coupling gain $k = 10$ we have $\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\gamma}_4 = 0.2263 < 1$, thus satisfying the conditions of both Theorems 1 and 2. With these parameters, and applying Theorem 1 we see that the upper bounds on the synchronization errors are as follows:

$$\begin{aligned}
\hat{\sigma}_1 &= 0.0052 & \hat{\sigma}_2 &= 0.0052 \\
\hat{\sigma}_3 &= 0.0105 & \hat{\sigma}_4 &= 0.2144
\end{aligned}$$

The time evolutions and the error bounds for the states x_{3_j} and x_{4_j}, for $j = 1, 2$ are shown in Figures 2(a) and 2(b). The upper bound on the total synchronization is then given by

$$\hat{\sigma} = \left| \begin{bmatrix} \hat{\sigma}_1 & \hat{\sigma}_2 & \hat{\sigma}_3 & \hat{\sigma}_4 \end{bmatrix}^T \right| = 0.2148$$

Using the passivity approach of Theorem 2 however, we find that the upper bound on the total output synchronization error is given by (25): $\hat{\sigma} = 0.2182$.

VI. DISCUSSION & FUTURE WORK

We have presented two methods for deriving upper bounds on the synchronization error of cyclic feedback systems that are interconnected via static, linear coupling. In the case of the example given in the previous section we saw that using either method gives a similar upper bound on the total output synchronization error $\hat{\sigma}$. The advantage of the method given in Theorem 1 is that bounds on the synchronization error of individual outputs $\mathbf{y}_i$ may be found. The disadvantage of that approach is that it requires the secant gain product to be less than unity whereas Theorem 2 requires that quantity to satisfy (23), a less conservative condition as highlighted in [6].

VII. ACKNOWLEDGMENT

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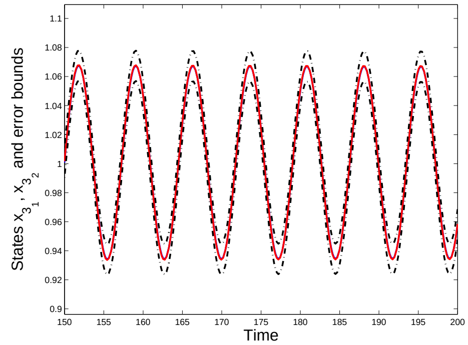
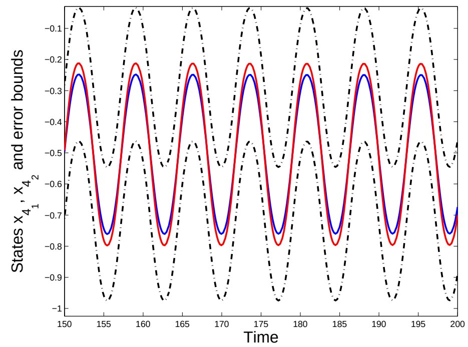


(a)



(b)

Fig. 2. Synchronization error bounds (dashed) and limit cycles (solid).

REFERENCES

[1] C. D. Thron, "The secant condition for instability in biochemical feedback control: Parts I and II," *Bulletin of Mathematical Biology*, vol. 53, pp. 383–401, 1991.

[2] J. Tyson and H. Othmer, "The dynamics of feedback control circuits in biochemical pathways," *Progress in Theoretical Biology*, vol. 5, pp. 1–62, 1978.

[3] B. Goodwin, "Oscillatory behavior in enzymatic control processes," *Advances in Enzyme Regulation*, vol. 3, pp. 425–428, 1965.

[4] D. Gonze, S. Bernard, C. Waltermann, A. Kramer, and H. Herzel, "Spontaneous synchronization of coupled circadian oscillators," *Biophysical Journal*, vol. 89, pp. 120–189, July 2005.

[5] J. Mallet-Paret and H. Smith, "The Poincare-Bendixson theorem for monotone cyclic feedback systems," *Journal of Dynamics and Differential Equations*, vol. 2, pp. 367–421, 1990.

[6] M. Arcak and E. Sontag, "Diagonal stability for a class of cyclic systems and applications," *Automatica*, vol. 42, pp. 1531–1537, 2006.

[7] E. Sontag, "Passivity gains and the "secant condition" for stability," *Automatica*, vol. 55, pp. 177–183, 2005.

[8] G.-B. Stan, A. Hamadeh, R. Sepulchre, and J. Goncalves, "Output synchronization in networks of cyclic biochemical oscillators," in *In proceedings of the 26th IEEE American Control Conference (IEEE-ACC 2007)*, July 2007.

[9] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Trans. on Automatic Control*, vol. 47, pp. 410–422, 2002.

[10] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34(6), pp. 683–696, 1998.

[11] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer, "Convergent dynamics, a tribute to Boris Pavlovich Demidovich," *Systems & Control Letters*, vol. 52, pp. 257–261, 2004.

[12] G.-B. Stan and R. Sepulchre, "Analysis of interconnected oscillators by dissipativity theory," *IEEE Transactions on Automatic Control*, vol. 52, pp. 256–270, 2007.

[13] A. Y. Pogromsky, "Passivity based design of synchronizing systems," *International Journal of Bifurcation and Chaos*, vol. 8(2), pp. 295–319, 1998.

[14] J. Willems, "Dissipative dynamical systems: Parts I and II," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–393, 1972.

[15] R. Sepulchre, M. Jankovic, and P. Kokotovic, *Constructive Nonlinear Control*. Springer, 1997.