Dissipativity characterization of a class of oscillators and networks of oscillators

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Nonlinear oscillations in phys. syst. : the Van der Pol oscillator

Sustained energy exchange between two or several storage elements.

\[ \phi_R(v) = -Rv + \frac{1}{3}v^3 \]

\[ LC \frac{d^2 v}{dt^2} + L \left( -R + v^2 \right) \frac{dv}{dt} + v = 0 \]

The active element is a static resistance with a negative characteristic at low energy and with a positive characteristic at high energy.
Goals and motivation

• **Main theme:** Global analysis and control of sustained oscillations (limit cycles) in networks of oscillators.

• **Based upon:**
  – high-dimensional generalizations of the Van der Pol and Rayleigh equations,
  – passivity and related concepts,
  – application of input-output stability theory to the analysis of limit cycles.

• **Goal:** Definition of a class of oscillators whose orbital stability properties extends directly to networks of oscillators.

• **Motivation:**
  – Bridge the gap between equilibrium points and limit cycle analysis tools in nonlinear systems,
  – Control of amplitude, frequency and phase of oscillations in networks of oscillators.

• **Applications:**
  – robotics: rhythmic tasks robots (i.e. walking robots, juggling robots, dexterous robots),
  – biology: fundamental oscillation mechanism.
• Class of system studied

• Main result: the “dissipative” oscillator theorem

• Class of networks studied

• Extension of the main result to this class of networks
Class of systems studied

- Storage elements: a passive system and an integrator.

- Static "energy regulating" element: \( \phi_k(\sigma)y = -ky + \phi(\sigma)y \), with \( \sigma = \text{output} (y \text{ or } x) \).

- \( Q_k \) passive \( \Rightarrow \dot{S} \leq \left((k - k^*) - \phi(\sigma)\right)y^2 \). The influence of \( k - k^* > 0 \) is counteracted by \( \phi \) at high energy and not counteracted at low energy.

Basic mechanism in electro-mechanical systems.
Example: Van der Pol oscillator

\[
\dot{x} + (-k + x^2)\dot{x} + x = 0, \quad y = \dot{x}
\]

Properties:

- \( k \leq 0 \): GAS equilibrium.
- \( k = 0 \): Supercritical Hopf bifurcation.
- \( k > 0 \): GAS limit cycle (in \( \mathbb{R}^2 \setminus \{0\} \)).
- Robust to linear output interconnections.
“Dissipative oscillator” Theorem

Assumptions :

1. $P(s) = \text{transfer function of a LTI, passive system } H \text{ which is controllable, detectable, and relative degree one,}$

2. $P(0) \neq 0,$

3. $\phi(0) = \phi'(0) = 0, \phi''(0) > 0, \lim_{\sigma \to \pm \infty} \phi(\sigma) = +\infty, \phi(\sigma) > 0, \forall \sigma \neq 0.$

Results :

1. A Hopf bifurcation occurs at a critical value $k^*.$

2. If $\Pi$ is passive at $k = k^*$, the Hopf bifurcation is supercritical and the limit cycle is GAS for $k \gtrsim k^*.$
Some insights about the “Dissipative oscillator” Theorem

1. Local :

   • **linearized system stability analysis**: root locus argument ⇒ generically 2 complex conjugate eigenvalues cross the imaginary axis at $k^*$,

   • **at $k = k^*$, stability analysis in the center manifold**: $\Pi$ is passive ⇒ the system is 3-asymptotically stable ⇒ the Hopf bifurcation is supercritical,

2. Global :

   • **global boundedness of solutions for** $k \gtrsim k^*$ (Arcak, Kokotovic),

   • **at $k = k^*$, $\Pi$ is absolutely stable** ⇒ $\Pi$ is practically stable/semi-globally stable for $k \gtrsim k^*$ ⇒ the limit cycle is GAS for $k \gtrsim k^*$. 
Multiplier theory (IQC) broadens the application of the result.

Remark: As $k$ increases, the linearized system typically loses passivity before losing stability. The use of multipliers is important!
Example: Sustained oscillations for a damped pendulum

General 2\textsuperscript{nd} order transf. fun. of rel. degree 1 (e.g., pendulum) \( P(s) = \frac{\tau s + 1}{s^2 + 2\zeta s + 1} \) \((\frac{1}{\tau} \leq 2\zeta)\).

Using Zames-Falb multipliers, the origin could be proven absolutely stable for \( k \leq k^* \).

State-space for a single oscillator

State-space of a SINGLE oscillator for \( k = 1 \) and \( k_p = 1.900000e+00 \)

State-space of a SINGLE oscillator for \( k = 1 \) and \( k_p = 2.100000e+00 \)

(a) \( k = 1.9 \)  
(b) \( k = 2.1 \)
Network of $N$ identical dissipative oscillators.

Each oscillator is linearly coupled to the others: $u_i = -y_{ci} - \sum \gamma_{ij} y_j$.

$$\Gamma = (\gamma_{ij}), \ i, j = 1, \ldots, N$$

is the interconnection matrix.

- **Question**: Is the Hopf bifurcation characterization in the SISO case relevant for the study of the interconnection of oscillators?

- **Central idea**: Include the interconnection matrix $\Gamma$ in the nonlinear block.

- **Static nonlinearity property to be preserved**: $-kY +$ passive operator.

- **Observation**: This property is preserved if the interconnection matrix $\Gamma$ is symmetric, i.e. $\Gamma = \Gamma^T \Rightarrow \exists k_0 \geq 0$ such that $\Gamma' = \Gamma + k_0 I_N$ is a positive semidefinite matrix of rank $q < N$.

- **Consequence**: The bifurcation analysis in a symmetric network reduces to the bifurcation analysis of a single oscillator!
MIMO representation of the network

\[ \begin{align*}
\Phi_k & \quad \text{STATIC NONLINEARITY} \\
\text{diag}\{P(s)\} & \quad \text{Y} \\
\text{diag}\{\frac{1}{s}\} & \\
\text{diag}\{\phi_k'(\Sigma)\}Y + \Gamma'Y & \\
\text{diag}\{Q_{k'}(s)\} & \quad \text{Y}
\end{align*} \]

- \( \Gamma'^T = \Gamma' = \Gamma + k_0 I_N \geq 0 \)
- \( k' = k + k_0 \)

\[ \begin{align*}
\Gamma'^T = \Gamma : \\
-k'Y + \text{diag}\{\phi(\Sigma)\}Y + \Gamma'Y \quad \text{passive!}
\end{align*} \]

- Consequence:
  - Supercr. equiv. bif.,
  - Global limit cycle

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Example: Interconnection of “dissipative oscillators”

\[ P(s) = \frac{1}{s + 1} \]

\[ \Gamma = \begin{pmatrix} 0 & K_c & \cdots & K_c \\ K_c & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & K_c \\ K_c & \cdots & K_c & 0 \end{pmatrix}, \quad K_c > 0 \]

Time evolution of the outputs (3 oscillators in a TSI structure)

(a) \( k' = 1.9 \)

(b) \( k' = 2.1 \)
Dissipativity theory proves useful in (global) analysis of limit cycles. The dissipativity framework is well-suited to

- high dimensional results \((n > 2)\),
- interconnections (networks of oscillators),
- strong coupling analysis