Oscillators as systems

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Oscillators play an increasing role in systems and control
Rhythmic robotics, collective motion, life sciences, quantum control, resonant MEMS, . . .

No system theory for oscillators
external description, stability and robustness analysis, interconnections . . .

Oscillators as interconnections of open systems
What is an oscillator?

Traditional view:
- *nonlinear*: isolated periodic solution of (closed) dynamical system;
- *complicated*: analysis is local and restricted to low-dimensions;
- *fragile*: easily bifurcates to complex attractors.

Needed view: *system building block*
- interacts with its environment
- relaxes to a limit cycle when isolated.
Oscillators in biology

[Biochemical oscillations and cellular rhythms, Goldbeter, ’96]

<table>
<thead>
<tr>
<th>Rhythm</th>
<th>period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neural rhythms</td>
<td>0.01 to 10 s</td>
</tr>
<tr>
<td>Cardiac rhythm</td>
<td>1 s</td>
</tr>
<tr>
<td>Calcium oscillations</td>
<td>1 s to several minutes</td>
</tr>
<tr>
<td>Biochemical oscillations</td>
<td>1 min to 20 min</td>
</tr>
<tr>
<td>Mitotic cycle</td>
<td>10 min to 24 h</td>
</tr>
<tr>
<td>Hormonal rhythms</td>
<td>10 min to several hours</td>
</tr>
<tr>
<td>Circadian rhythm</td>
<td>24 h</td>
</tr>
<tr>
<td>Ovarian cycle</td>
<td>28 days</td>
</tr>
</tbody>
</table>

Interconnected oscillators

- Compelling physical and natural phenomena.
- Organized collective phenomena emerge from synchronization, phase locking, resonance, excitability
- Engineering applications: laser technology, superconductivity, NMR, MEMS, ...
Contents

- Two motivating examples
  - Rhythmic control
  - Group path planning
- The oscillator and the integrator
- The action potential: a paradigm for biological oscillators
- The dissipative oscillator
- Conclusions
The wiper: a rhythmic control testbed

(with M. Gerard, R. Ronsse, Ph. Lefevre)
Simplest cartoon for the wiper model

wiper = 2 orthogonal bouncing balls (i.e. impact oscillators)

Design principle: stable motions result from the right interconnections between oscillators
Why study the wiper?

Rhythmic control is fundamental to biolocomotion: walking, swimming, hopping, flying

How much and what type of feedback is needed?

Experimentation with human subjects reveals connections between dynamical properties and compliance to the juggling task.

More: Dynamics and control of bounce juggling (MTNS talk)
Collective motion and synchronization

(with D. Paley and N. Leonard, Princeton)

Applications: optimal sensor coverage, data collection in the ocean, coordinated control . . .
Key property for group path planning

$N$ particles in the plane moving at unit speed:

\[ \dot{r}_k = e^{i\theta_k} \]
\[ \dot{\theta}_k = u_k, \quad 1 \leq k \leq N. \]

\[ \dot{\theta}_k = +\frac{1}{N} \sum_{i=1}^{N} \sin(\theta_i - \theta_k) + \epsilon \omega^\text{par}_k \]
parallel ('sync')

\[ \dot{\theta}_k = -\frac{1}{N} \sum_{i=1}^{N} \sin(\theta_i - \theta_k) + \epsilon \omega^\text{rot}_k \]
circular ('antisync')

Stable group motions have a lot in common with phase-locked oscillators . . .
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System theory

Complex systems = interconnections of simpler systems

Stability, robustness, performance dictated by external properties of elements + interconnection properties

The fundamental building block:
storage element = energy accumulator

LTI theory: storage=integrator
dissipativity theory: generalized storage concept
Dissipativity theory

Lyapunov-like characterization:

\[ \dot{S} \leq w(u, y) \]

Storage elements: inductor, capacitor, heat-tank, spring, inertia wheel, stable mechanical system, . . .

Complex storage elements are built from simpler storage elements by interconnections

system analysis matches physical modelling; increasingly used in applications.
What if ‘building block’ = an oscillator?

- Neuron
- quantum bit
- two-state memory
- resonant circuit
- biological clocks
- moving particle

≈ integrators? Storage elements? Interconnections?
The neuron: a paradigm for oscillators

- External characterization: system view is common
  - step response
  - building block of networks
- Internal characterization: neurophysics provides accurate dynamical models
- Quest for system analysis: function is central to neuroscience and beyond

Yet ... the puzzle largely persists ...
Action potential: external description

[Neuroscience. M. Bear et al.]

A two-port circuit
Classified according to step response
Action potential: modelling

[Hodgkin-Huxley, 1952]
Cell membrane acts as capacitor

\[ C \frac{dV}{dt} = -I_{Na} - I_{K} - I_{leak} + I_{input} \]

Each input current obeys Ohm law:

\[ I_{k} = g_{k}(R_{k}) \left( V - E_{k} \right) \]

Conductance \( g_{k} \) adapts to voltage:

\[ \tau \dot{R}_{k} = -R_{k} + F_{k}(V) \]
Action potential: oscillation mechanism

Oscillation ≈ periodic switch between (quasi) equilibrium potentials:

\[ E_{Na} \approx 55\, mV \quad ; \quad E_K \approx -92\, mV \quad ; \quad V \approx -60\, mV \quad \text{(in average)} \]

Two currents for basic oscillator; additional currents responsible for different step responses;
Interconnections: the integrator model

[Hopfield, ’84]

\[ \tau y_k = -y_k + \text{sat}(u_k) \]

\( y_k \): average activity (firing rate?)
\( u_k \): average stimulus

Symmetric networks = passive interconnections of passive elements
Simplification is adequate if neuron = storage element
Interconnections: the phasor model

[Kuramoto, ’84]

\[ \dot{\theta}_k = \omega_k + u_k, \quad \theta_k \in S^1 \]

- State \(\approx\) oscillator phase
- all-to-all sinusoidal phase coupling:
  \[ u_k = \frac{K}{N} \sum_{i=1}^{N} \sin(\theta_i - \theta_k) \]
  central to mathematical results on synchronization
- Adequate simplification in the limit of weak coupling
An incomplete picture

In a number of situations, integrator and phasor models are too simplified. The network dynamical properties depend on the individual spike timing. Poorly understood at present time.

Phase?
average firing rate?
The neuron: a paradigm for oscillators

- External characterization: system view is common
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- Internal characterization: neurophysics provides accurate dynamical models

- Quest for system analysis: function is central to neuroscience and beyond

  Yet ... the puzzle largely persists ...
“A “neurodynamics" including action potentials displays much richer collective phenomena than do models that represent action potentials only by their statistical average effect. Previously, this richness has prevented a thorough understanding of how to compute in a useful fashion with a feedback network of such (model) neurons. On the other hand, the notion of attractors and a Lyapunov function has provided powerful methods to analyze neural networks based on a firing-rate description. The convergence proof (…) and the existence of a Lyapunov function (…) for special cases leads us to hope that it will be possible to achieve a quantitative understanding of the relationship between connections, initial conditions, and computation in these systems as well." [Hopfield and Herz, PNAS 92, 1995].
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The dissipative oscillator

(with Guy-Bart Stan)

\[ u \rightarrow O \rightarrow y \]

- Internal description: ‘global’ limit cycle, not restricted to low-dimensional models
- External description: a dissipation inequality, suitable for analysis of interconnections
- Includes prototype oscillators as simplest examples.
A dissipation inequality:

\[ \dot{S} \leq a(y) - d(y) + w(u, y) \]

storage variation  local activation  global dissipation  external supply

oscillation results from competition between local activation and global dissipation
a parameter ($k$) controls the "local activation" (linear positive feedback)

**Absolute stability for** $k = 0$; equilibrium loses stability at $k = k^* \geq 0$.

**Bifurcation codimension (one or two) dictates one of two scenarios for global oscillations near** $k^*$
Codimension two scenario (Hopf)

Absolute stability for $k \leq k^*$ and codimension two bifurcation implies a global oscillation for $k \gtrsim k^*$.

Van der Pol: Energy flows between two storage elements; static NL element regulates the sign of dissipation. A ‘basic’ mechanism in electro-mechanical systems.
Codimension one scenario (pitchfork)

Absolute stability for $k \leq k^*$ and codimension one bifurcation implies global bistability for $k \gtrsim k^*$.

Center manifold dynamics: $\dot{y} = (k - k^*)y - y^3 + 0(y^4)$
(Slow) adaptation converts the bistable system into a relaxation oscillation: a ‘basic’ oscillation mechanism in biology.

\[
\dot{y} = ky - y^3 + R \\
\tau \dot{R} = -R + y
\]

Fitzugh-Nagumo model: mimics the periodic switch in Hodgkin-Huxley model.
A dissipation inequality:

\[
\dot{S} \leq (k - k_{passive}) y^2 - y \phi(y) + uy
\]

with \( \phi(\cdot) \) a stiffening nonlinearity (e.g. cubic) and \( k_{passive} \) the parameter value at which \( \frac{\Sigma}{1 - k \Sigma} \) loses passivity.
If $\Gamma$ is passive, then the result for one oscillator holds for the network: codimension of bifurcation and absolute stability at the bifurcation determine two global oscillation mechanisms.

If $\Gamma = \Gamma^T \geq 0$ and corank $\Gamma = q \geq 1$ then the bifurcation value is unchanged and $q$-equivariant
Sync and incremental dissipativity

The dissipation inequality holds *incrementally* if $\Sigma$ is linear and $\phi$ is monotone.

Incremental dissipativity implies global synchronization for a broad class of interconnections.
Conclusions

- Oscillators are systems but not classical storage elements.
- Interconnected oscillators exhibit specific and important dynamical properties: sync, phase-locking, resonance, ... 
- The dissipative oscillator is richer than phase models and amenable to system analysis.
- Dissipativity theory extends beyond stability analysis of equilibria.

Oscillators will be important in tomorrow's dynamics and control applications. Beware!