One limitation of classical correlations

What if we randomly permute the fMRI volumes?
Dynamical properties of fMRI connectivity in neuronal networks mediating consciousness

R. Liégeois(1), M. Bahri(2), S. Laureys(2), M. Zorzi(3) and R. Sepulchre(1,3)

(1) Department of Electrical Engineering and Computer Science, University of Liège, Belgium
(2) Coma Science Group, Cyclotron Research Centre, University of Liège, Belgium
(3) Department of Engineering, Trumpington Street, University of Cambridge, United Kingdom

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Outline

1. Theoretical framework
   - Static and Dynamical models
   - Our approach

2. Application
   - A static approach
   - Contribution of a dynamic approach
Starting from \( m \) observations \( \{x_k\}_{k=1}^m, x_k \in \mathbb{R}^n \) of an unknown phenomenon \( F \), let’s compare the two approaches:

**Static models**
- \( F \) is modeled as a centered Gaussian random vector \( X \)
- \( X \) takes values in \( \mathbb{R}^n \)
- \( X \) is completely described by \( \Sigma = \mathbb{E}[XX^T] \)

**Dynamic models**
- \( F \) is modeled as a centered Gaussian, stationary, stochastic process \( \{X_t\}_{t \in \mathbb{Z}} \)
- \( X_i \) takes values in \( \mathbb{R}^n \) \( \forall i \in \mathbb{Z} \)
- \( \{X_t\} \) is completely described by the covariance lags sequence \( R_k = \mathbb{E}[X_{t+k}X_t^T] \forall k \in \mathbb{Z} \)
- Through the power spectral density \( \Phi(e^{i\theta}) = \sum_{k=-\infty}^{+\infty} R_k e^{-jk\theta} \)
Starting from $m$ observations $\{x_k\}_{k=1}^m$, $x_k \in \mathbb{R}^n$ of an unknown phenomenon $F$, let’s compare the two approaches:

**Static models**

- $\mathbb{F}$ is modeled as a centered Gaussian, stationary, stochastic process $\{X_t\}_{t \in \mathbb{Z}}$.
- $X_i$ takes values in $\mathbb{R}^n$ $\forall i \in \mathbb{Z}$.
- $\{X_t\}$ is completely described by the covariance lags sequence $R_k = \mathbb{E}[X_{t+k}X_t^T] \forall k \in \mathbb{Z}$.

**Dynamic models**

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- Through the power spectral density $\Phi(e^{j\theta}) = \sum_{k=-\infty}^{+\infty} R_k e^{-jk\theta}$.
Starting from $m$ observations $\{x_k\}_{k=1}^m, x_k \in \mathbb{R}^n$ of an unknown phenomenon $F$, let’s compare the two approaches:

### Static models

### Dynamic models
Our approach:

Important features:

1. The dynamical/ordering information is taken into account using an *autoregressive* (AR) model (e.g. Avventi et al. 2013).

2. The sparsity of direct effective connectivity is required through the sparsity of $\Phi(e^{i\theta})^{-1}$.
Our approach: an AR model encoding sparsity constraints

The AR model of order M

\[ x(t) = \sum_{j=1}^{M} A_j x(t - j) + e(t) \]

where:

- \( x(t) \) is the state vector of size \((n \times 1)\), at time \( t \)
- \( e(t) \sim N(0, \Sigma) \) is white noise
- \( A_j \) are \((n \times n)\) matrices that describe the model
- This model results in an estimated power spectral density
  \[ \tilde{\Phi}(e^{j\theta}) = \sum_{k=-M}^{M} \tilde{R}_k e^{-jk\theta} \]
  which has a sparse inverse

\[ \begin{align*}
\tilde{R}_0 & \leftrightarrow \Sigma & \leftrightarrow \text{static connectivity} \\
\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_M & \leftrightarrow \text{dynamic connectivity}
\end{align*} \]
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\end{cases}$$
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Connectivity in the Default Mode Network was computed in 18 controls undergoing four different states of consciousness: wakefulness (W), mild sedation (MS), deep sedation (U) and subsequent recovery of consciousness (R).

**Figure:** Boveroux et al., Anesthesiology 2010
The static connectivity encoded in $R_0$ is coherent with Boveroux et al.

The trend is almost opposite for the dynamical connectivity extracted from $R_1$
Theoretical framework

Application

Conclusions

Take home message

- Dynamical approaches allow more comprehensive analyses of connectivity.
- Several tools allow to take into account dynamics (sliding windows, wavelets, autoregressive models)

Perspectives

- A power spectral density $\Phi(e^{i\theta})$ is more difficult to interpret than a connectivity matrix...
- Computational challenges have to be addressed in order to largely apply those tools to neuronal networks (e.g. convex formulation).
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From this equation, we can derive the link between $\Phi(z)$ and $A(z)$:

$$A(z) X(z) = E(z) \iff X(z) E(z)^{-1} = A(z)^{-1}.$$ 

Hence, we have that $$\Phi(z) = W(z) W(z^{-1})^T = A(z)^{-1} A(z^{-1})^{-T}$$ where $W$ is the shaping filter.
In both the static and the dynamic cases the model is requested to be as simple as possible. This results for the static case (resp. dynamic) in a \emph{sparse conditional independency} pattern, encoded in the \textit{precision matrix} $\Sigma^{-1}$ (resp. \textit{inverse power spectral density} $\Phi^{-1}(e^{j\theta})$) :

$$[\Sigma^{-1}]_{kl} (\text{resp.} [\Phi^{-1}(e^{j\theta})]_{kl}) = 0 \iff X_k \perp X_l \mid X_{V \setminus \{k,l\}}$$

$$\equiv X_k \text{ and } X_l \text{ are conditionaly independent}$$

where $V$ is the set of all variables and $(k,l) \notin E$ where $E$ is the sparsity pattern of the model.

\[ \Sigma \text{ is full} \]
\[ \Sigma^{-1} \text{ is sparse} \]
App. C : An example of power spectral density

Evaluating $\Phi(z)$