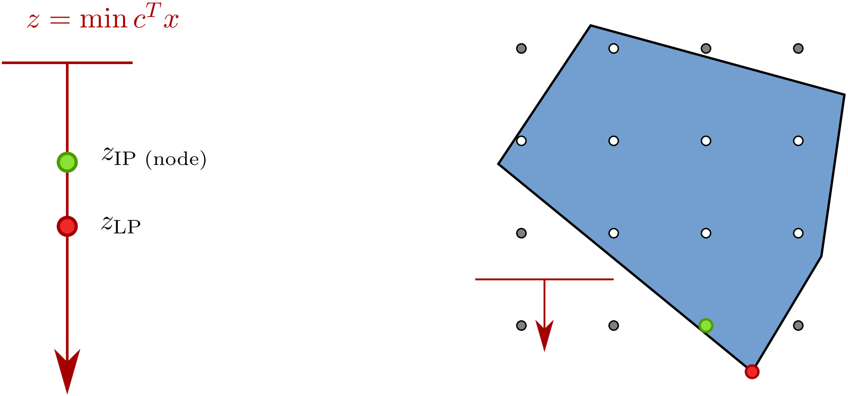
Numerically safe lower bounds for the Capacitated Vehicle Routing Problem

Laurent Poirrier*

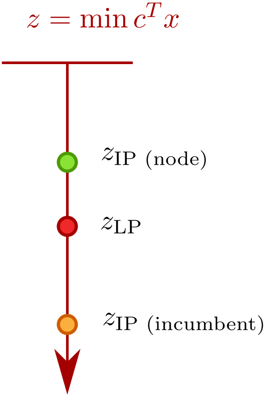
Joint work with Ricardo Fukasawa*

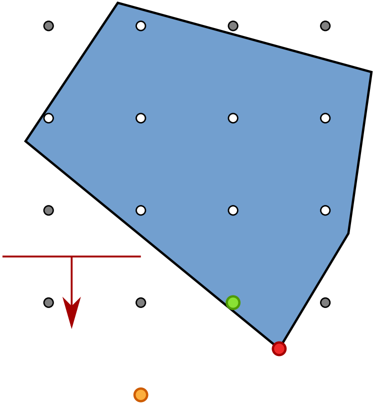
* Combinatorics & Optimization, UWaterloo

Branch and bound, pruning



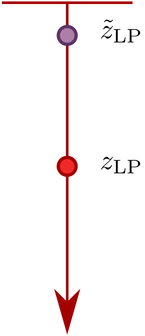
Branch and bound, pruning

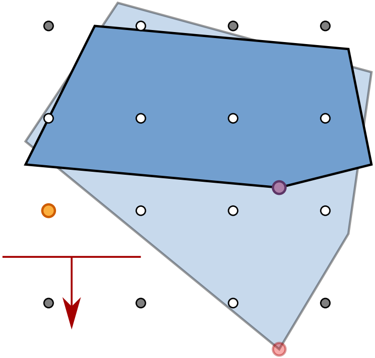




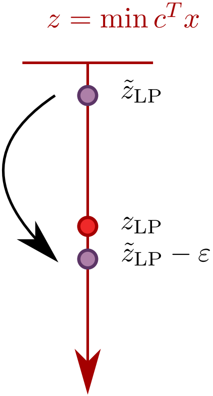
Round-off errors

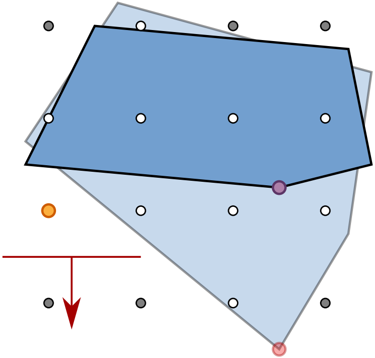
$$z = \min c^T x$$



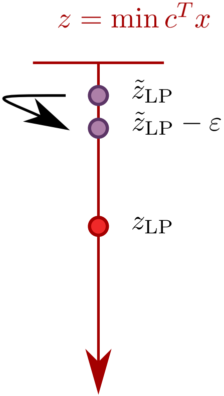


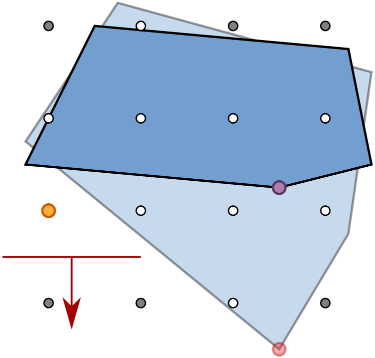
Round-off errors



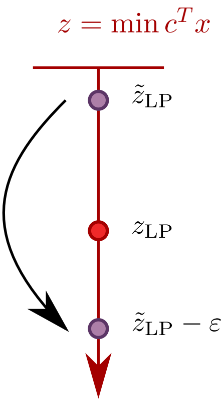


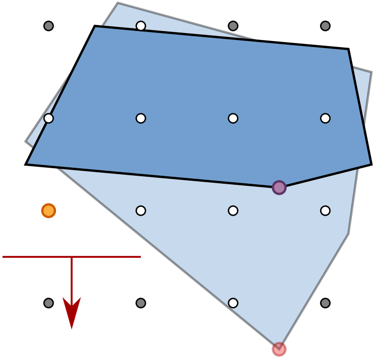
Round-off errors





Round-off errors





Fighting round-off errors

Exact LP solving:

- Applegate, Cook, Dash, Espinoza (2007): QSopt_ex
- Gleixner, Steffy, Wolter (2012): SoPlex

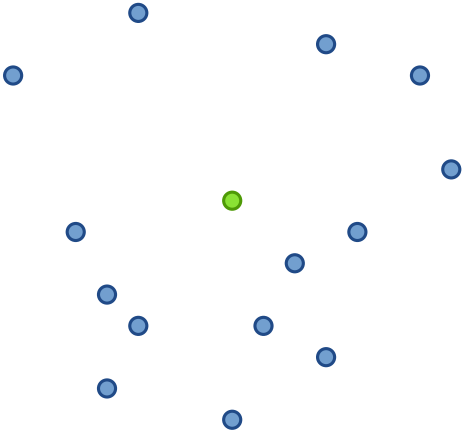
Capacitated vehicle routing problem (CVRP)

Trucks: K

Capacity: C

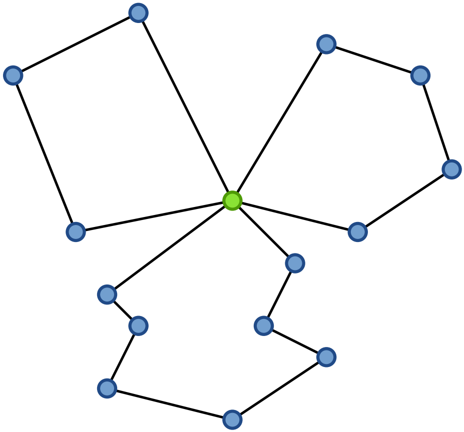
Client demand: d_i

Edge cost: l_e



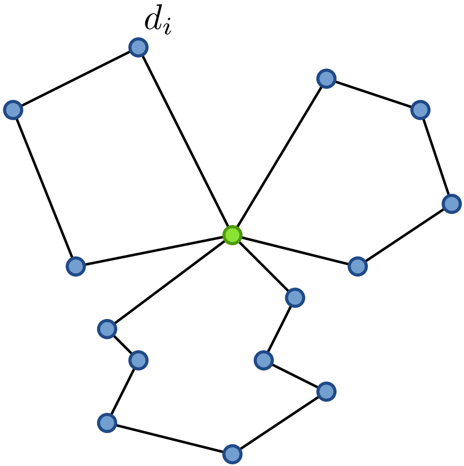
Capacitated vehicle routing problem (CVRP)

Trucks:	K
Capacity:	C
Client demand:	d_i
Edge cost:	l_e



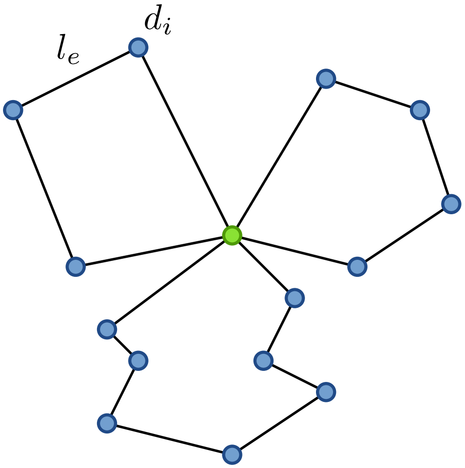
Capacitated vehicle routing problem (CVRP)

Trucks:	K
Capacity:	C
Client demand:	d_i
Edge cost:	l_e



Capacitated vehicle routing problem (CVRP)

Trucks:	K
Capacity:	C
Client demand:	d_i
Edge cost:	l_e



Capacitated vehicle routing problem (CVRP)

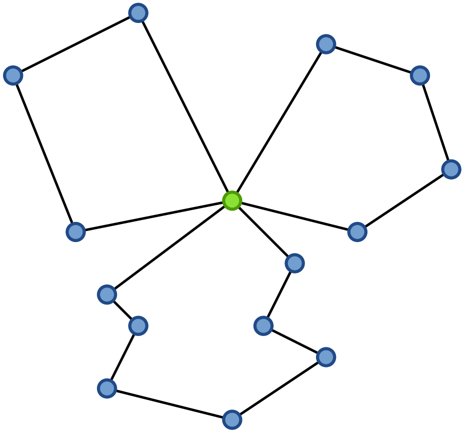
$$
\begin{aligned}
\min \quad & \sum_{e} l_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = b_i, && \forall i \\
& \sum_{e} \alpha_e^k x_e \geq \alpha_0^k, && \forall k \\
& x_e \leq u_e, && \forall e \\
& x_e \geq 0, && \forall e
\end{aligned}
$$

Dantzig-Wolfe Formulation

Express the solution as a linear combination of all possible q-routes.

$q_j^e = 1$ if route j uses edge e

$$x_e = \sum_j q_j^e \lambda_j$$

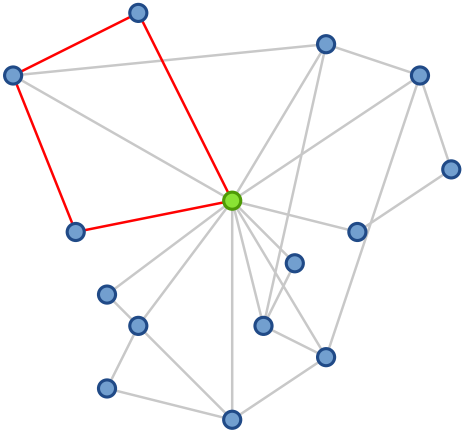


Dantzig-Wolfe Formulation

Express the solution as a linear combination of all possible q-routes.

$q_j^e = 1$ if route j uses edge e

$$x_e = \sum_j q_j^e \lambda_j$$



Dantzig-Wolfe Formulation

$$
\begin{aligned}
\min \quad & \sum_{e} l_e \, x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = b_i, && \forall i \\
& \sum_{e} \alpha_e^k \, x_e \geq \alpha_0^k, && \forall k \\
& x_e \leq u_e, && \forall e \\
& x_e \geq 0, && \forall e
\end{aligned}
$$

Dantzig-Wolfe Formulation

$$
\begin{array}{llllllll}
\min & \sum_{e} & l_e & \sum_{j} & q_j^e & \lambda_j & & & \\
\text{s.t.} & \sum_{e \in \delta(i)} & & \sum_{j} & q_j^e & \lambda_j & = & b_i, & \forall i \\
& \sum_{e} & \alpha_e^k & \sum_{j} & q_j^e & \lambda_j & \geq & \alpha_0^k, & \forall k \\
& & & \sum_{j} & q_j^e & \lambda_j & \leq & u_e, & \forall e \\
& & & & & \lambda_j & \geq & 0, & \forall j
\end{array}
$$

Pricing

$$
\begin{aligned}
\bar{c}_j &= \sum_e \left(l_e - \sum_{i:e\in\delta(i)} \omega_i - \sum_k \alpha_e^k \pi_k - \rho_e \right) q_j^e \\
&= \sum_e \bar{c}_e q_j^e \\
\bar{c}_{\min} &= \min_j \left\{ \sum_e \bar{c}_e q_j^e \right\}
\end{aligned}
$$

Pricing

$$\bar{c}_j = \sum_e \left(l_e - \sum_{i:e\in\delta(i)} \omega_i - \sum_k \alpha_e^k \pi_k - \rho_e \right) q_j^e$$

$$= \sum_e \bar{c}_e q_j^e$$

$$\bar{c}_{\min} = \min_j \left\{ \sum_e \bar{c}_e q_j^e \right\}$$

Pricing

$$\begin{aligned}\bar{c}_j &= \sum_e \left(l_e - \sum_{i:e\in\delta(i)} \omega_i - \sum_k \alpha_e^k \pi_k - \rho_e \right) q_j^e \\ &= \sum_e \bar{c}_e q_j^e \\ \bar{c}_{\min} &= \min_j \left\{ \sum_e \bar{c}_e q_j^e \right\}\end{aligned}$$

The problem

Dantzig-Wolfe formulation:

$$\begin{aligned}
\min \quad & \sum_j \left(\sum_e l_e q_j^e \right) \lambda_j \\
\text{s.t.} \quad & \sum_j \left(\sum_{e \in \delta(i)} q_j^e \right) \lambda_j = b_i && \forall i \\
& \sum_j \left(\sum_e \alpha_e^k q_j^e \right) \lambda_j \geq \alpha_0^k && \forall k \\
& \sum_j q_j^e \lambda_j \leq u_e && \forall e \\
& \lambda_j \geq 0 && \forall j
\end{aligned}$$

Column generation:

$$\bar{c}_{\min} = \min_j \left\{ \sum_e \bar{c}_e q_j^e \right\}$$

Solution 1. Scaling approach

- Held, Cook, Sewell (2012): graph coloring

$$\begin{array}{llcll} \min & c^T x & \quad & \max & b^T y \\ \text{s.t.} & Ax \geq b & & \text{s.t.} & A^T y \leq c \\ & x \geq 0 & & & y \geq 0 \end{array}$$

Given $\bar{y} \geq 0$ and largest dual violation Δ,

$$y' = \frac{c_{min}}{c_{min} + \Delta} \bar{y}$$

is dual feasible.

Proof:

$$A_j^T y' = \frac{c_{min}}{c_{min} + \Delta} A_j^T \bar{y} \leq \frac{c_{min}}{c_{min} + \Delta}(c_j + \Delta) \leq \frac{c_j}{c_j + \Delta}(c_j + \Delta) = c_j$$

Solution 1. Scaling approach

- Held, Cook, Sewell (2012): graph coloring

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Given $\bar{y} \geq 0$ and largest dual violation Δ,

$$y' = \frac{c_{min}}{c_{min} + \Delta} \bar{y}$$

is dual feasible.

Proof:

$$A_j^T y' = \frac{c_{min}}{c_{min} + \Delta} A_j^T \bar{y} \leq \frac{c_{min}}{c_{min} + \Delta} (c_j + \Delta) \leq \frac{c_j}{c_j + \Delta} (c_j + \Delta) = c_j$$

Solution 1. Scaling approach

- Held, Cook, Sewell (2012): graph coloring

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Given $\bar{y} \geq 0$ and largest dual violation Δ,

$$y' = \frac{c_{min}}{c_{min} + \Delta} \bar{y}$$

is dual feasible.

Proof:

$$A_j^T y' = \frac{c_{min}}{c_{min} + \Delta} A_j^T \bar{y} \leq \frac{c_{min}}{c_{min} + \Delta} (c_j + \Delta) \leq \frac{c_j}{c_j + \Delta} (c_j + \Delta) = c_j$$

Solution 1. Scaling approach

- Held, Cook, Sewell (2012): graph coloring

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \qquad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Given $\bar{y} \geq 0$ and largest dual violation Δ,

$$y' = \frac{c_{min}}{c_{min} + \Delta} \bar{y}$$

is dual feasible.

Proof:

$$A_j^T y' = \frac{c_{min}}{c_{min} + \Delta} A_j^T \bar{y} \leq \frac{c_{min}}{c_{min} + \Delta} (c_j + \Delta) \leq \frac{c_j}{c_j + \Delta} (c_j + \Delta) = c_j$$

Solution 1. Scaling approach

In general,

$$z \geq \frac{c_{min}}{c_{min} + \Delta} b^T \bar{y}$$

For CVRP,

$$z \geq \frac{2l_f}{2l_f + \Delta} \bar{z}$$

where $l_f := \min\{l_e : e \in \delta(0)\}$ and

$$\Delta = |\bar{c}_{\min}|$$

Solution 1. Scaling approach

In general,

$$z \geq \frac{c_{min}}{c_{min} + \Delta} b^T \bar{y}$$

For CVRP,

$$z \geq \frac{2l_f}{2l_f + \Delta} \bar{z}$$

where $l_f := \min\{l_e : e \in \delta(0)\}$ and

$$\Delta = |\bar{c}_{\min}|$$

Solution 1. Scaling approach

In general,

$$z \geq \frac{c_{min}}{c_{min} + \Delta} b^T \bar{y}$$

For CVRP,

$$z \geq \frac{2l_f}{2l_f + \Delta} \bar{z}$$

where $l_f := \min\{l_e : e \in \delta(0)\}$ and

$$\Delta = |\bar{c}_{\min}|$$

Solution 2. Primal bound shift

- Neumaier, Shcherbina (2004), Steffy, Wolter (2014): MIP
- Applegate, Bixby, Chvátal, Cook (2006): TSP

Solution 2. Primal bound shift

$$
\begin{array}{lllllllll}
\min & \sum_j & \sum_e & l_e & q_j^e & \lambda_j & & & \\
\text{s.t.} & \sum_j & \sum_{e \in \delta(i)} & & q_j^e & \lambda_j & = & b_i, & \forall i \\
& \sum_j & \sum_e & \alpha_e^k & q_j^e & \lambda_j & \geq & \alpha_0^k, & \forall k \\
& \sum_j & & & q_j^e & \lambda_j & \leq & u_e, & \forall e \\
& & & & & \lambda_j & \geq & 0, & \forall j
\end{array}
$$

Solution 2. Primal bound shift

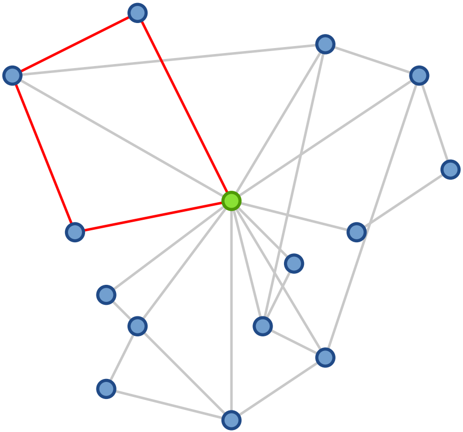
$$
\begin{aligned}
\max \quad & \sum_i b_i \omega_i + \sum_k \alpha_0^k \pi_k + \sum_e u_e \rho_e \\
\text{s.t.} \quad & \sum_i \left(\sum_e q_j^e \right) \omega_i + \sum_k \left(\sum_e \alpha_e^k q_j^e \right) \pi_k + \sum_e q_j^e \rho_e \leq \left(\sum_e l_e q_j^e \right) \quad \forall j \\
& \pi \geq 0 \qquad \rho \leq 0
\end{aligned}
$$

Solution 2. Primal bound shift

$$
\begin{aligned}
\max \quad & \cdots \quad + \sum_e u_e \rho_e \\
\text{s.t.} \quad & \cdots \quad + \sum_e q_j^e \rho_e \quad \le c_j \qquad \forall j \\
& \cdots \qquad \rho \le 0
\end{aligned}
$$

Solution 2. Primal bound shift

$$
\begin{array}{llll}
\max & \cdots & + \sum_e u_e \rho_e & \\
\text{s.t.} & \cdots & + \sum_e q_j^e \rho_e \le c_j & \forall j \\
& \cdots & \rho \le 0 &
\end{array}
$$



Dual feasible:

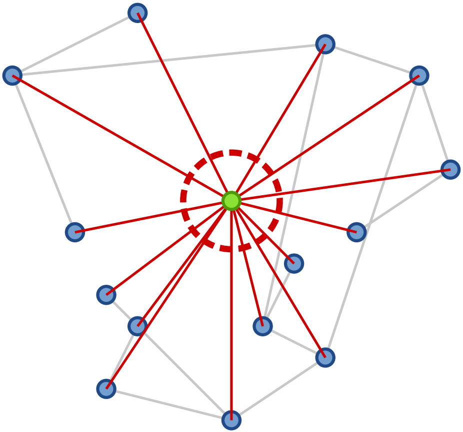
$$\rho'_e = \bar{\rho}_e - \frac{1}{2}\Delta \text{ for } e \in \delta(0)$$

Solution 2. Primal bound shift

$$
\begin{aligned}
\max \quad & \cdots \quad + \sum_e u_e \rho_e \\
\text{s.t.} \quad & \cdots \quad + \sum_e q_j^e \rho_e \quad \le c_j \qquad \forall j \\
& \cdots \qquad \rho \le 0
\end{aligned}
$$

Dual feasible:

$\rho'_e = \bar{\rho}_e - \frac{1}{2}\Delta$ for $e \in \delta(0)$

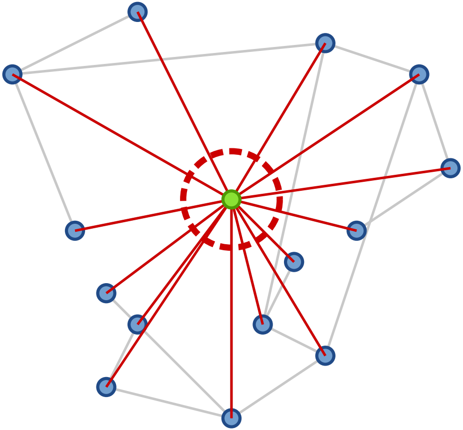


Solution 2. Primal bound shift

$$\begin{array}{llll} \max & \cdots & +\sum_e u_e \rho_e & \\ \text{s.t.} & \cdots & +\sum_e q_j^e \rho_e \leq c_j & \forall j \\ & \cdots & \rho \leq 0 & \end{array}$$

Dual feasible:

$$\rho'_e = \bar{\rho}_e - \frac{1}{2}\Delta \text{ for } e \in \delta(0)$$



Solution 2. Primal bound shift

Direct bound:

$$z \geq \bar{z} - n\Delta$$

Tighter bound:

$$z \geq \bar{z} - \sum_i \Delta^i$$

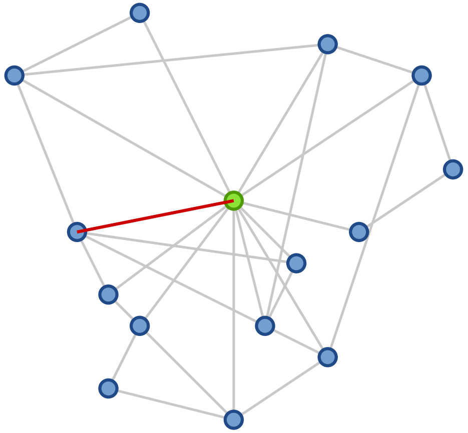
Solution 2. Primal bound shift

Direct bound:

$$z \geq \bar{z} - n\Delta$$

Tighter bound:

$$z \geq \bar{z} - \sum_i \Delta^i$$



Solution 3. Primal bound shift

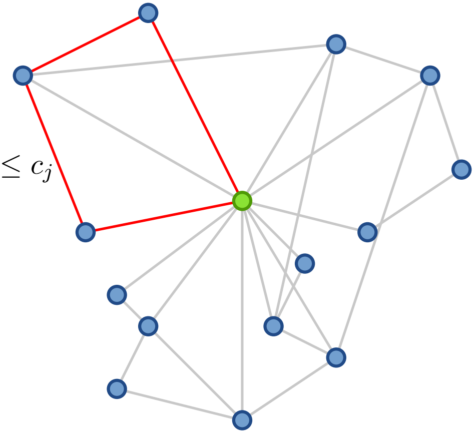
$$
\begin{aligned}
\max \quad & \sum_i b_i \omega_i + \sum_k \alpha_0^k \pi_k + \sum_e u_e \rho_e \\
\text{s.t.} \quad & \sum_i \left(\sum_e q_j^e \right) \omega_i + \sum_k \left(\sum_e \alpha_e^k q_j^e \right) \pi_k + \sum_e q_j^e \rho_e \leq \left(\sum_e l_e q_j^e \right) \quad \forall j \\
& \omega \text{ free} \qquad \pi \geq 0 \qquad \rho \leq 0
\end{aligned}
$$

Solution 3. Primal bound shift

$$\begin{aligned}
\max \quad & \sum_i b_i \omega_i && + \cdots \\
\text{s.t.} \quad & \sum_i \left(\sum_{e \in \delta(i)} q_j^e \right) \omega_i && + \cdots && \leq c_j && \forall j \\
& \omega \text{ free} && \cdots
\end{aligned}$$

Solution 3. Primal bound shift

$$
\begin{array}{lll}
\max & \sum_i b_i \omega_i & + \cdots \\
\text{s.t.} & \sum_i \left(\sum_{e \in \delta(i)} q_j^e \right) \omega_i & + \cdots \quad \leq c_j \\
 & \omega \text{ free} & \cdots
\end{array}
$$



Dual feasible:

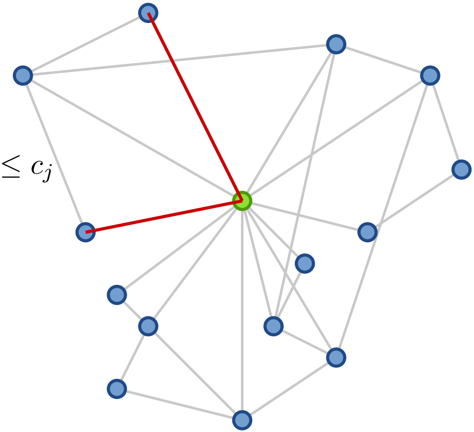
$$\omega_0' = \bar{\omega}_0 - \frac{1}{2}\Delta$$

Bound:

$$z \geq \bar{z} - K\Delta$$

Solution 3. Primal bound shift

$$
\begin{array}{llll}
\max & \sum_i b_i \omega_i & + \cdots & \\
\text{s.t.} & \sum_i \left(\sum_{e \in \delta(i)} q_j^e \right) \omega_i & + \cdots & \leq c_j \\
 & \omega \text{ free} & \cdots &
\end{array}
$$



Dual feasible:

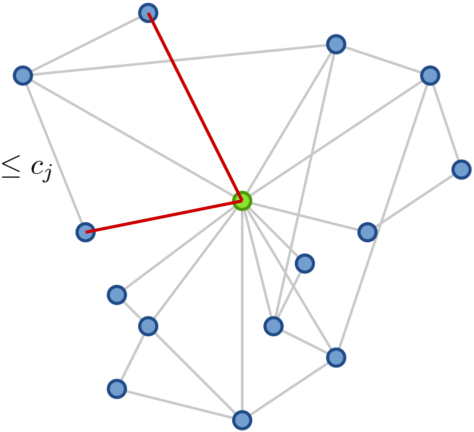
$$\omega_0' = \bar{\omega}_0 - \frac{1}{2}\Delta$$

Bound:

$$z \geq \bar{z} - K\Delta$$

Solution 3. Primal bound shift

$$\begin{array}{llll} \max & \sum_i b_i \omega_i & + \cdots & \\ \text{s.t.} & \sum_i \left(\sum_{e \in \delta(i)} q_j^e \right) \omega_i & + \cdots & \leq c_j \\ & \omega \text{ free} & \cdots & \end{array}$$



Dual feasible:

$$\omega_0' = \bar{\omega}_0 - \frac{1}{2}\Delta$$

Bound:

$$z \geq \bar{z} - K\Delta$$

Solution 4. Lagrangian relaxation

$$
\begin{array}{llllll}
\min & \sum_{e} & l_e & x_e & & \\
\text{s.t.} & \sum_{e \in \delta(i)} & & x_e & = & b_i, & \forall i \\
 & \sum_{e} & \alpha_e^k & x_e & \geq & \alpha_0^k, & \forall k \\
 & & & x_e & \leq & u_e, & \forall e \\
 & & & x_e & \geq & 0, & \forall e
\end{array}
$$

$$
\begin{array}{ll}
\min & \sum_{e \in E} l_e x_e + \sum_{i \in V} \left(b_i - \sum_{e \in \delta(i)} x_e \right) \omega_i + \sum_{k} \left(\alpha_0^k - \sum_{e \in E} \alpha_e^k x_e \right) \pi_k \\
 & + \sum_{e \in E} (u_e - x_e) \rho_e \\
\text{s.t.} & x \in \operatorname{conv}(\{K q_j : \text{ for all } j\}).
\end{array}
$$

Solution 4. Lagrangian relaxation

$$
\begin{aligned}
\min \quad & \sum_{e} l_e \, x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(i)} x_e = b_i, && \forall i \\
& \sum_{e} \alpha_e^k \, x_e \geq \alpha_0^k, && \forall k \\
& x_e \leq u_e, && \forall e \\
& x_e \geq 0, && \forall e
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \sum_{e \in E} l_e x_e + \sum_{i \in V} \left(b_i - \sum_{e \in \delta(i)} x_e \right) \omega_i + \sum_{k} \left(\alpha_0^k - \sum_{e \in E} \alpha_e^k x_e \right) \pi_k \\
& + \sum_{e \in E} (u_e - x_e) \, \rho_e \\
\text{s.t.} \quad & x \in \operatorname{conv}(\{K q_j : \text{ for all } j\}).
\end{aligned}
$$

Solution 4. Lagrangian relaxation

$$z \geq \sum_{i \in V} b_i \omega_i + \sum_{k} \alpha_0^k \pi_k + \sum_{e \in E} u_e \rho_e + K \min_j \left\{ \sum_{e \in E} \bar{c}_e q_j^e \right\} = \bar{z} - K\Delta$$

Solution 4. Lagrangian relaxation

$$z \geq \sum_{i\in V} b_i\omega_i + \sum_k \alpha_0^k \pi_k + \sum_{e\in E} u_e\rho_e + K \min_j \left\{ \sum_{e\in E} \bar{c}_e q_j^e \right\} = \bar{z} - K\Delta$$

Summary of safe lower bounds

$$z \quad \geq \quad \max\left\{z_{\mathsf{scale}}, \quad z_\rho, \quad z_\omega = z_{\mathsf{Lagrange}}\right\}$$

$$z \quad \geq \quad \max\left\{\frac{2l_f}{2l_f+\Delta}\bar{z}, \quad \bar{z} - \sum_i \Delta^i, \quad \bar{z} - K\Delta\right\}$$

Computational comparison

98 CVRP instances, total 68939 branch and bound nodes.

Strictly strongest bound:

z_{scale}	z_ρ	z_ω
0	57779	10623

Comparison with unsafe bounds

98 CVRP instances, total 68939 branch and bound nodes.

Strictly strongest bound:

unsafe ε	$z_{\text{safe}} > z_{\text{unsafe}}$	$z_{\text{safe}} \leq z_{\text{unsafe}}$
10^{-6}	68901 (0)	38 (0)
10^{-9}	59188 (0)	9751 (8)

Comparison with unsafe bounds

98 CVRP instances, total 68939 branch and bound nodes.

Strictly strongest bound:

unsafe ε	$z_{\text{safe}} > z_{\text{unsafe}}$	$z_{\text{safe}} \leq z_{\text{unsafe}}$
10^{-6}	68901 (0)	38 (0)
10^{-9}	59188 (0)	9751 (8)

Conclusion

- Safe lower bounds for CVRP
- Computational cost almost negligible
- Safe bounds are stronger than unsafe bounds

Thanks!

Conclusion

- Safe lower bounds for CVRP
- Computational cost almost negligible
- Safe bounds are stronger than unsafe bounds

Thanks!

Integer conversion

$$M \leq (T - C - 1) / \left(C \max_{e \in E} \left\{ \frac{|\bar{c}_e|}{d_{\mathsf{min}}(e)} \right\} + \max_{e \in \delta^-(0)} |\bar{c}_e| \right)$$