# The Strength of Multi-Row Relaxations

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August 2012

# Motivations

- Cuts viewed as facets of relaxations of the problem
- In particular, multi-row relaxations
- Focus on exact separation
- Evaluate any relaxation

# Plan

A. Separation over arbitrary mixed-integer sets

B. Application to two-row relaxations

# A. Separation over arbitrary mixed-integer sets

# Problem

Given

- a mixed-integer set $P \subseteq \mathbb{R}^n$,
- a point $x^* \in \mathbb{R}^n$,

find $(\alpha, \alpha_0) \in \mathbb{R}^{n+1}$ such that $\alpha^T x \geq \alpha_0$ is a valid inequality for $P$ that separates $x^*$,

or show that $x^* \in \text{conv}(P)$.

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or show that $x^* \in \text{conv}(P)$.

# General framework

Solve the optimization problem

$$
\begin{array}{lll}
\min & {x^*}^T \alpha & \\
\text{s.t.} & x^T \alpha \geq \alpha_0 & \text{for all } x \in P \\
 & \text{<norm.>} &
\end{array}
$$

Let $(\bar{\alpha}, \bar{\alpha}_0)$ be the optimal solution.

If ${x^*}^T \bar{\alpha} < \bar{\alpha}_0$, then $(\bar{\alpha}, \bar{\alpha}_0)$ separates $x^*$.

If ${x^*}^T \bar{\alpha} \geq \bar{\alpha}_0$, then $x^* \in \text{conv}(P)$.

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# Row generation

1. Consider the relaxation of the separation problem

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\end{array}
\qquad \text{(master)}
$$

Let $(\bar{\alpha}, \bar{\alpha}_0)$ be an optimal solution.

2. Now solve the MIP

$$
\begin{array}{ll}
\min & \bar{\alpha}^T x \\
\text{s.t.} & x \subseteq P
\end{array}
\qquad \text{(slave)}
$$

and let $x^p$ be a finite optimal solution.

$$\text{If } \bar{\alpha}^T x^p \geq \bar{\alpha}_0, \quad \text{then } (\bar{\alpha}, \bar{\alpha}_0) \text{ is valid for } P.$$

$$\text{If } \bar{\alpha}^T x^p < \bar{\alpha}_0, \quad \text{then } S := S \cup \{x^p\}.$$

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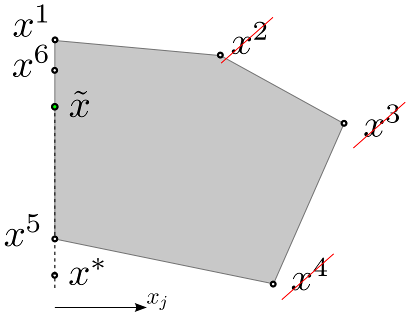
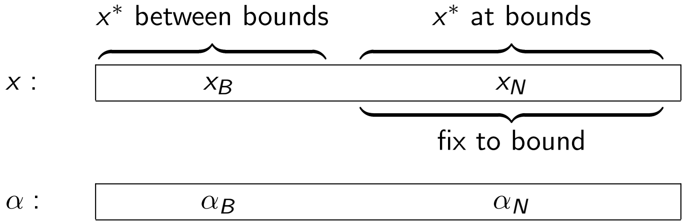
# Computational example

| | |
|---|---|
| Instance: | `bell3a` |
| Constraints: | 123 |
| Variables: | 133 (71 integer: 32 general, 39 binaries) |
| Models: | 82 five-row models read from an optimal tableau |
| Cuts: | 37 (17 tight at the end) |
| Gap closed: | 59.02% (from 39.03% by GMIs) |

| | |
|---|---|
| Time: | 1615.70s |
| Iterations: | 107647 |

# Two-phases: Phase one



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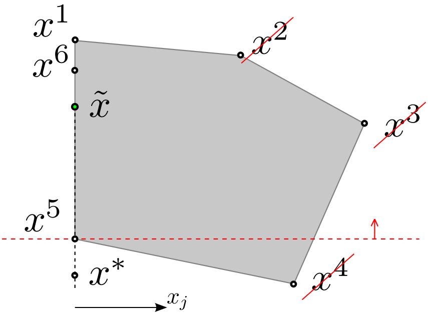
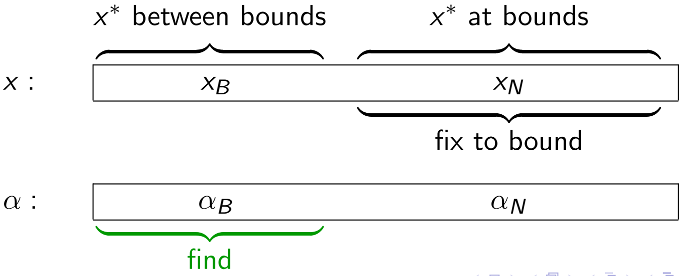
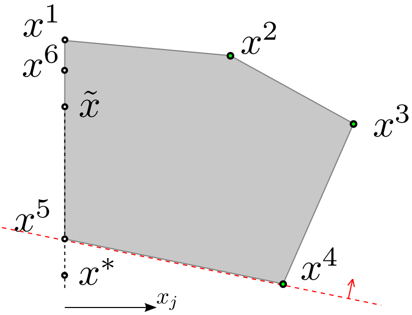
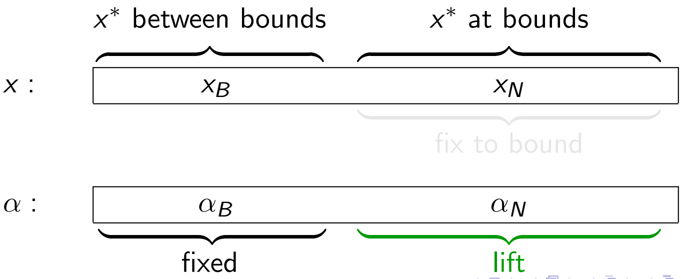


# Two-phases: Phase two



# Two-phases summary

- The feasible region of phase-1 slave is $P \cap \{x : x_N = x^*_N\}$
- "phase-1 separates" iff "phase-2 separates"
  $\longrightarrow$ whenever $x^* \in \text{conv}(P)$, phase-2 is avoided
- Optimal objective function values are the same
  $\longrightarrow$ phase-2 master objective function is 0

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# Computational example (2-phases)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

| | original | 2-phases |
|---|---|---|
| Time: | 1615.70s | 161.15s |
| Iterations: | 107647 | 23822 |

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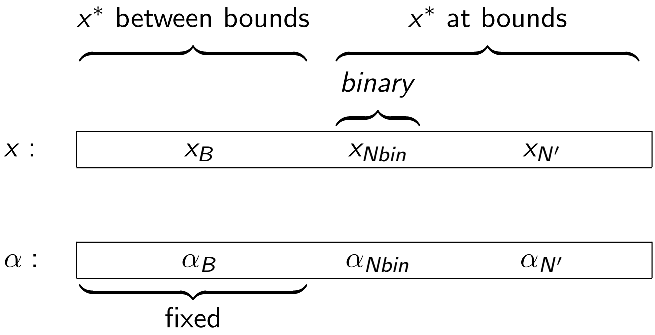

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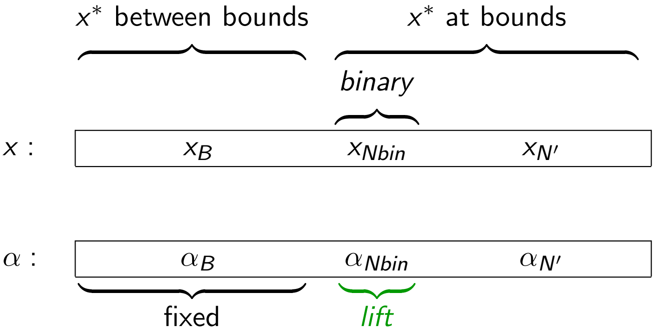

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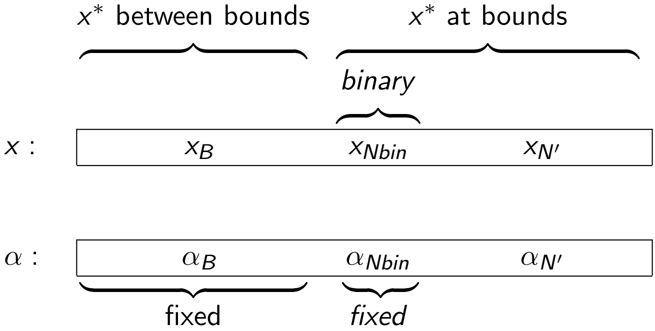

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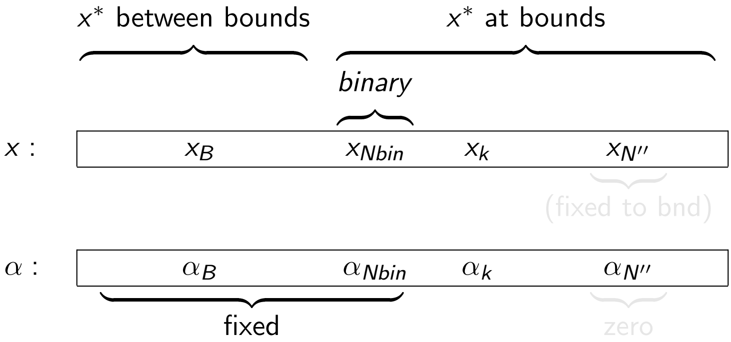

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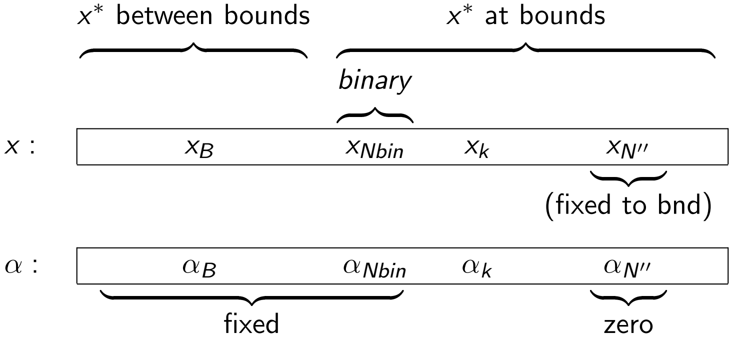

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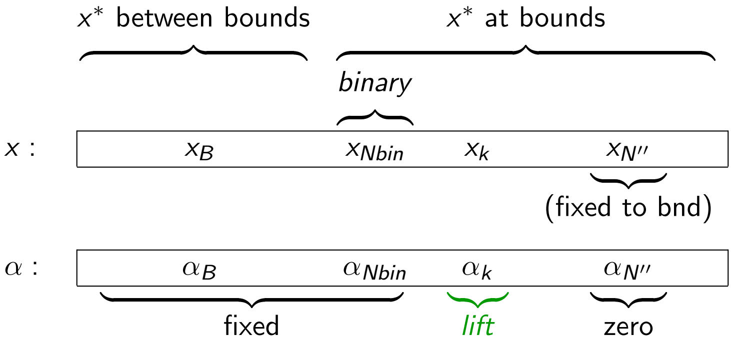

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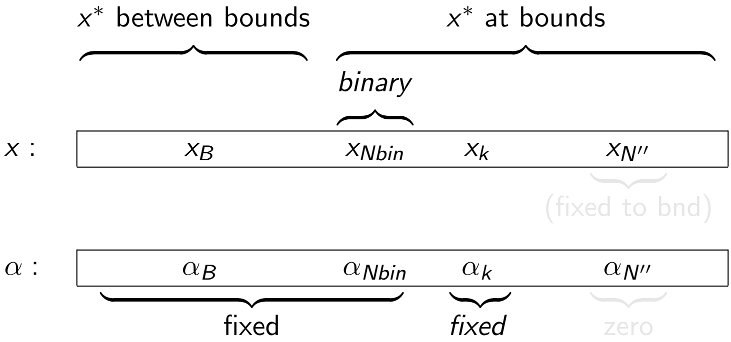

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# Solver tricks: callbacks

Solving slave MIPs

$$\begin{aligned} \min \quad & \bar{\alpha}^T x \\ \text{s.t.} \quad & x \subseteq P, \end{aligned}$$

- Feasible solution $\hat{x}$ with $\bar{\alpha}^T \hat{x} < \bar{\alpha}_0$

  $\rightarrow$ $\hat{x}$ can be added to $S$.

- Dual bound $\underline{z}$ reaches $\bar{\alpha}_0$,

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| | original | 2-phases | lifting | phase S | cb |
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| Time: | 1615.70s | 161.15s | 136.54s | 5.84s | 4.65s |
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# Computational example (summary)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

| | original | 2-phases | lifting | phase S | cb |
|---|---|---|---|---|---|
| | 347× | 35× | 29× | 1.26× | 1 |
| Time: | 1615.70s | 161.15s | 136.54s | 5.84s | 4.65s |
| Iterations: | 107647 | 23822 | 23231 | 2497 | 2497 |
| | 43× | 10× | 9× | 1 | 1 |

# B. Application to two-row relaxations

# Objectives

Mainly, evaluate and compare

- the intersection cut model
- a few strengthenings of it
- a full two-row model

# Two-row relaxation: which models?

- We are still far from a closure
- What reasonable set of two-models can we select?

  $\longrightarrow$ All models read from a simplex tableau

  $\longrightarrow$ $O(m^2)$ two-row models

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# “all” two-row models: separation loop

Let $x^* \leftarrow$ LP optimium
Read the two-row models from optimal tableau.
Read and add GMIs from that tableau.

**do** {
  Let $x^* \leftarrow$ new LP optimum.
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# “all” two-row models: results

Computations on the 62 MIPLIB 3.0 (preprocessed) instances for which

(a). the integrality gap is not zero, and

(b). the optimal MIP solution is known.

# “all” two-row models: results

We have a result for 55/62 instances (4 numerical, 3 memory).

| | cuts | gc% |
|---|---|---|
| GMI | 24.800 | 22.60% |
| All 2-row | 72.382 | 37.49% |

For 13 instances, the separation is exact.

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# Heuristic selection of two-row models

Issue:

- $O(m^2)$ is already a large number of models

Hypothesis:

- Not all models are necessary to achieve good separation

Rationale:

- MIPLIB models are mostly sparse
- Multi-cuts from rows with no common support are linear combinations of the corresponding one-row cuts

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With an arbitrary limit of $m$ two-row models,
we have a result for 58/62 instances (1 numerical, 3 memory).

On the 55 common results,

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For 25 instances, the separation is exact.

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# Two-row intersection cuts

$$\begin{array}{rrrrrrcr}
\overbrace{x_1 \quad\quad}^{\text{basic}} & \multicolumn{4}{c}{\overbrace{+2x_3 \quad -x_4 \quad +x_5 \quad +3x_6}^{\text{nonbasic}}} & & = & 2.4 \\
\quad\quad x_2 & -x_3 & & -x_5 & +2x_6 & & = & -0.2
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{Z}$$

$$0 \leq x_1 \leq 1$$

$$0 \leq x_2 \leq 1$$

$$0 \leq x_3 \leq 1$$

$$0 \leq x_4 \leq 1$$

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# Two-row intersection cuts

$$\overbrace{x_1 \quad\quad x_2}^{\text{basic}} \quad \overbrace{+2x_3 - x_4 + x_5 + 3x_6}^{\text{nonbasic}}$$

$$\begin{aligned}
x_1 \quad\quad & +2\,x_3 - x_4 + x_5 + 3\,x_6 = 3 \\
x_2 & - x_3 - x_5 + 2\,x_6 = -1 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{Z} \\
& 0 \le x_1 \le 1 \\
& 0 \le x_2 \le 1 \\
& 0 \le x_3 \le 1 \\
& 0 \le x_4 \le 1 \\
& 0 \le x_5 \le 1 \\
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$$
\begin{array}{cccccccc}
\overbrace{x_1 \quad\quad}^{\text{basic}} & \overbrace{+2\,x_3 & -\,x_4 & +\,x_5 & +3\,x_6}^{\text{nonbasic}} & = & 2.4 \\
\quad\quad {\color{red}x_2} & - \quad x_3 & & -\,x_5 & +2\,x_6 & = & -\,0.2
\end{array}
$$

$$x_1, {\color{red}-}, x_3, x_4, x_5, x_6 \in \mathbb{Z}$$

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$$x_1, x_2, \textcolor{blue}{-}, \textcolor{blue}{-}, \textcolor{blue}{-}, \textcolor{blue}{-} \in \mathbb{Z}$$

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$$0 \le x_2 \le 1$$
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$$x_1, x_2, \_, \_, \_, \_ \in \mathbb{Z}$$

$$\_$$

$$\_$$

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\end{array}
$$

$$
\begin{aligned}
x_1, x_2, -, -, -, - &\in \mathbb{Z} \\
& - \\
& - \\
0 \leq x_3 &\leq \infty \\
0 \leq x_4 &\leq \infty \\
-\infty \leq x_5 &\leq 1 \\
-\infty \leq x_6 &\leq 1
\end{aligned}
$$

# Two-row intersection cuts + strengthening

| | basic | | nonbasic | |
|---|---|---|---|---|
| | $\in \mathbb{Z}$ | bnd. | $\in \mathbb{Z}$ | bnd. |
| $P_I$ | $\surd$ | $\times$ | $\times$ | B |
| $S$-free | $\surd$ | $\surd$ | $\times$ | B |
| lifting | $\surd$ | $\times$ | $\surd$ | B |
| $P_{IU}$ | $\surd$ | $\times$ | $\times$ | $\surd$ |
| full 2-row | $\surd$ | $\surd$ | $\surd$ | $\surd$ |

| | |
|---|---|
| $\surd$: | keep |
| B: | keep binding |
| $\times$: | drop |

# Two-row intersection cuts + strengthening

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| $S$-free | $\surd$ | $\surd$ | $\times$ | B |
| lifting | $\surd$ | $\times$ | $\surd$ | B |
| $P_{IU}$ | $\surd$ | $\times$ | $\times$ | $\surd$ |
| full 2-row | $\surd$ | $\surd$ | $\surd$ | $\surd$ |

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# Two-row intersection cuts and strengthenings

51 common instances:

| | cuts | gc% | exact |
|---|---|---|---|
| GMI | 28.240 | 22.46% | all |
| $P_I$ | 29.420 | 27.65% | 42 |
| $S$-free | 38.380 | 30.22% | 29 |
| lifting | 22.700 | 27.35% | 10 |
| $P_{IU}$ | 42.640 | 28.56% | 25 |
| full 2-row | 55.500 | 35.66% | 22 |

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| | cuts | gc% | exact |
|---|---|---|---|
| GMI | 28.240 | [illegible] | all |
| $P_I$ | 29.420 | [illegible] | 42 |
| $S$-free | [illegible] | 30.22% | 29 |
| lifting | [illegible] | 27.35% | 10 |
| $P_{IU}$ | [illegible] | [illegible] | 25 |
| full 2-row | 55.500 | [illegible] | 22 |

# Two-row intersection cuts and strengthenings

15 common instances:

| | cuts | gc% | exact |
|---|---|---|---|
| GMI | 20.667 | 26.541 | all |
| $P_I$ | 20.933 | 33.535 | all |
| $S$-free | 25.400 | 35.229 | all |
| $P_{IU}$ | 36.600 | 36.257 | all |
| full 2-row | 57.267 | 43.956 | all |

# Two-row intersection cuts and strengthenings

7 common instances:
[bell5, blend2, egout, khb05250, misc03, misc07, set1ch]

| | cuts | gc% | exact |
|---|---|---|---|
| GMI | 25.571 | 24.744 | all |
| $P_I$ | 25.143 | 33.641 | all |
| $S$-free | 28.714 | 33.836 | all |
| lifting | 25.571 | 33.716 | all |
| $P_{IU}$ | 47.857 | 37.531 | all |
| full 2-row | 48.000 | 37.583 | all |

# Bases

- We depend on the optimal basis
- Will the gap closed by two-row cuts survive more GMIs?

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# Relax and cut

- Convenient way to explore different (feasible) bases.
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# Relax and cut: results

43 common instances:

| | | cuts | gc% | exact |
|---|---|---|---|---|
| | GMI | 24.814 | 23.282 | all |
| | 2-row i.c. | 31.884 | 28.838 | 42 |
| | full 2-row | 62.140 | 36.080 | 22 |
| relax&cut | GMI | 60.372 | 34.970 | all |
| relax&cut | 2-row i.c. | 63.163 | 41.951 | 37 |
| relax&cut | full 2-row | 76.767 | 47.277 | 12 |

# Future

- More rows
- More tricks
- More tests

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