Multi-row approaches to cutting plane generation

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Tuesday, December 18th, 2012

Example: The university is hiring

	Junior	**Senior**
Teaching	40 hours	80 hours
Pay	\$ 31	\$ 45
Hire	at least one third	

Have as many taught hours as possible, with a budget of $ 239.

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\begin{aligned}
\max \quad & 40\,x_1 + 80\,x_2 & \\
\text{s.t.} \quad & 31\,x_1 + 45\,x_2 & \leq 239 \\
& x_1 \geq \tfrac{1}{2}\,x_2 & \\
& x_1 & \geq 0 \\
& x_2 & \geq 0 \\
& x_1 \;,\; x_2 & \in \mathbb{Z}
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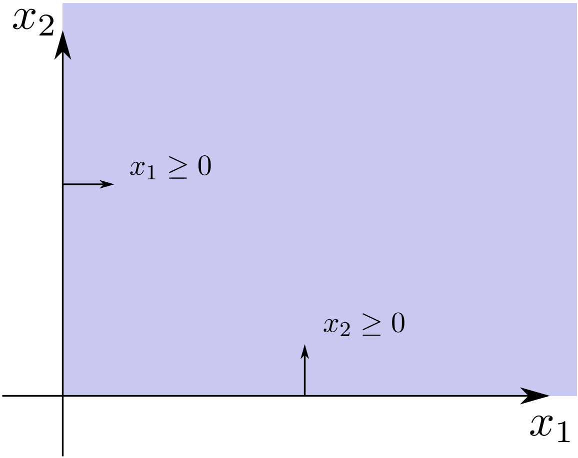

x_2
$x_1 \geq 0$
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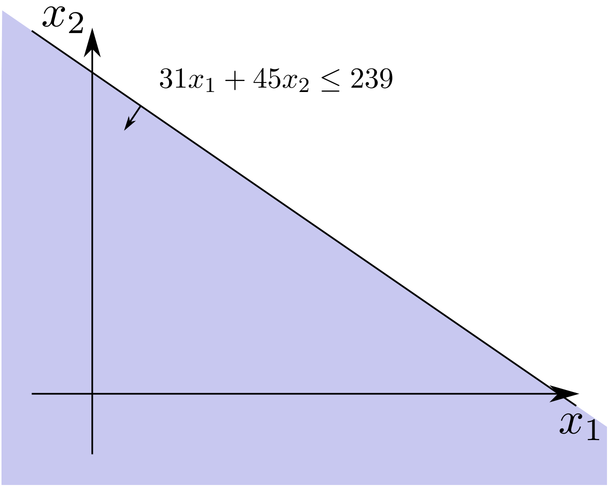
x_1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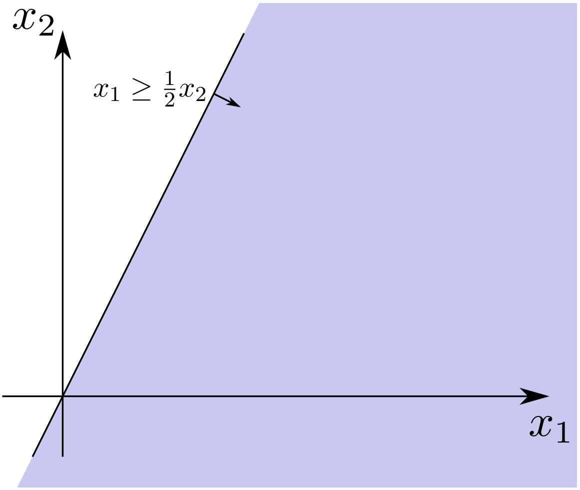
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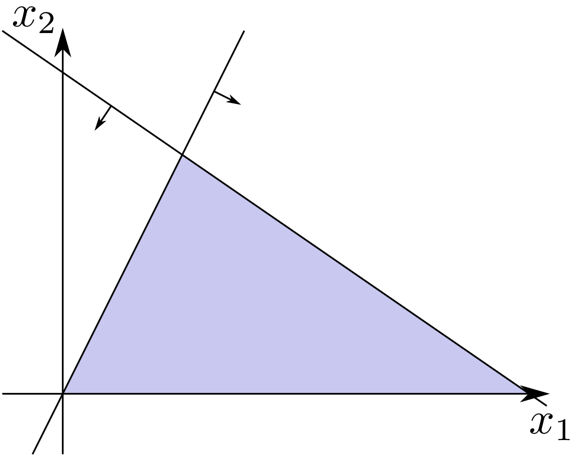
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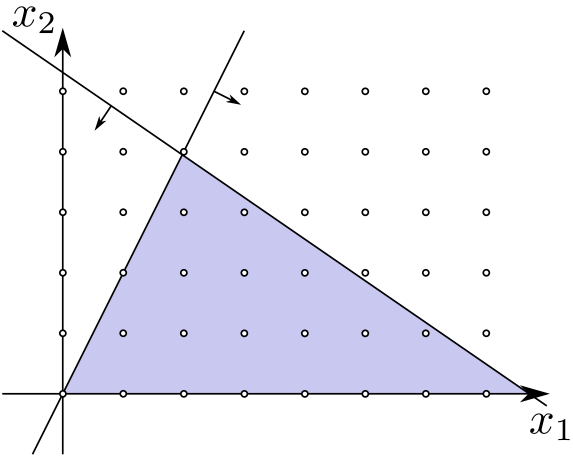
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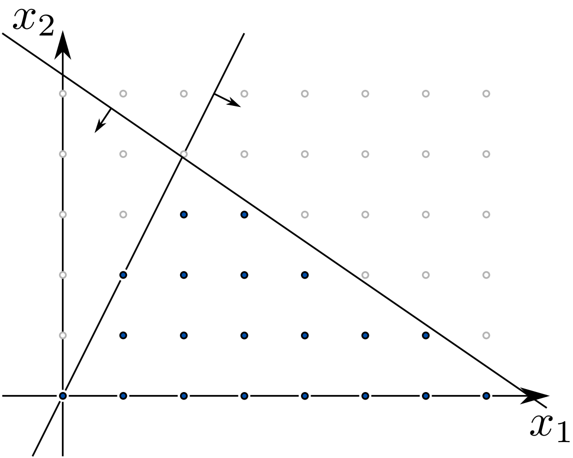

x_2
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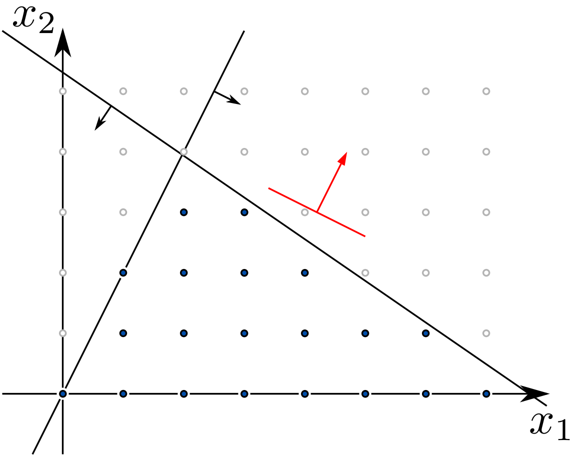
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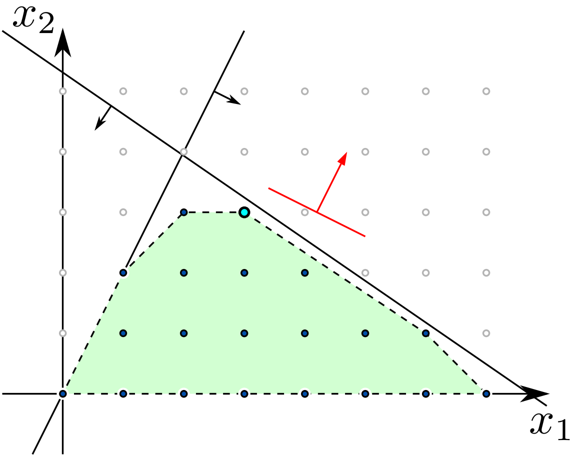

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Applications

- Scheduling (timetable building, machine tool switching, . . .)
- Bin-packing (chipset floor planning, . . .)
- Traveling Salesman Problem (ICs soldering and drilling)
- Discrete flow problems (power and energy distribution, . . .)
- Assignment
- Lot-sizing
- Transportation problems
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Most are NP-hard, and computationally difficult to solve.

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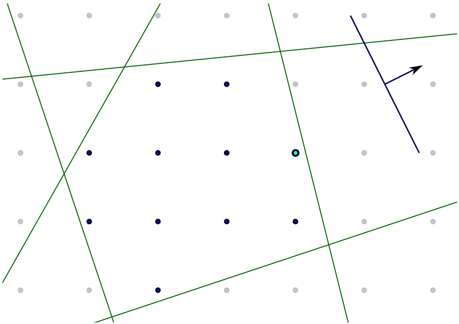
A Mixed Integer linear Programming problem

$$\text{(MIP)} \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & A\,x \geq b \\ & x_j \in \mathbb{Z}, \text{for } j \in J \end{array}$$



Solving MIPs: branch and bound

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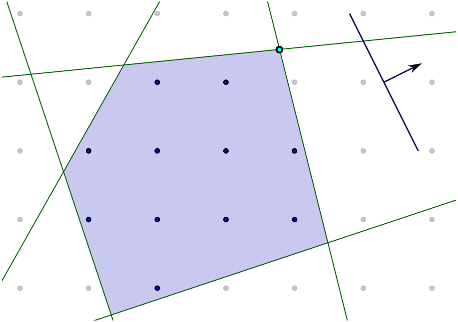

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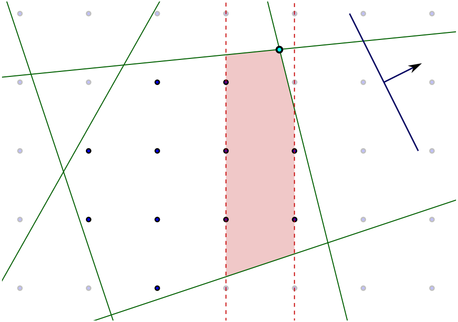
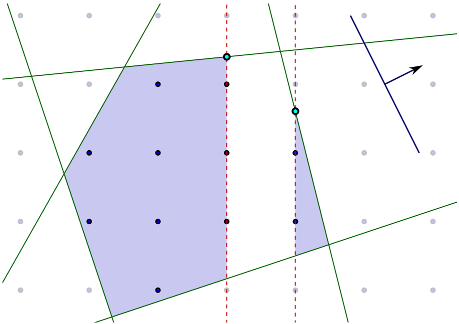
$x_i^* \notin \mathbb{Z}$



Solving MIPs: branch and bound

$$\text{(MIP1)}\quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & A\,x \geq b \\ & x_i \leq \lfloor x_i^* \rfloor \\ & x_j \in \mathbb{Z}, \text{for } j \in J \end{array}$$

$$\text{(MIP2)}\quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & A\,x \geq b \\ & x_i \geq \lceil x_i^* \rceil \\ & x_j \in \mathbb{Z}, \text{for } j \in J \end{array}$$



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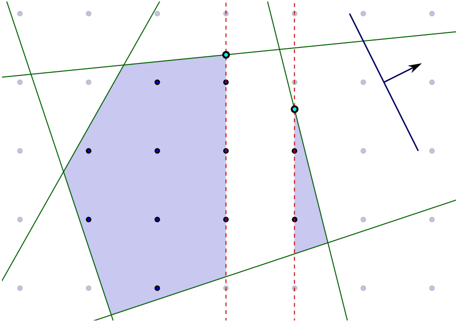

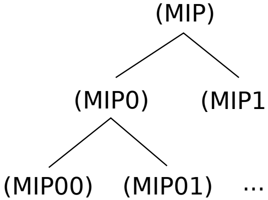

Cuts / Valid inequalities

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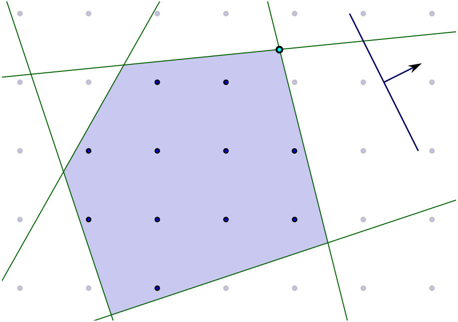

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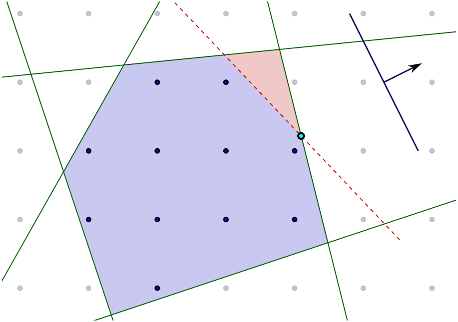

Why cut?

Most often,

	no cuts	↔	more cuts
computing cuts	0		more time
each b&b node	faster		slower
b&b nodes	more		less

In practice,

disabling cuts $\rightarrow 54\times$ slower

(geometric mean over 719 instances [Bixby, Rothberg, 2007]).

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Example of cut

Let $x \in \mathbb{Z}^3_+$,

$$3x_1 + 4x_2 - 5x_3 \leq 4.5$$

$$\Downarrow$$

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Chvatál-Gomory cut

What cuts?

Disabled cut	Performance degradation
Gomory mixed-integer	2.52 $\times$
Mixed-integer rounding	1.83 $\times$
Knapsack cover	1.40 $\times$
Flow cover	1.22 $\times$
Implied bound	1.19 $\times$
Flow path	1.04 $\times$
Clique	1.02 $\times$
GUB cover	1.02 $\times$

(geometric mean over 106 medium-sized instances [Bixby, Rothberg, 2007]).

A. Two-row cuts

A.1. Background

Single-row cuts

From one (re)formulation of the problem

$$\text{(MIP)} \quad \begin{aligned} \min \quad & \bar{c}^T x \\ \text{s.t.} \quad & \bar{A}\, x \geq \bar{b} \\ & x_J \in \mathbb{Z} \end{aligned}$$

we extract **one** constraint $\bar{A}_i\, x \geq \bar{b}_i$.

- Knowing that $x_j \in \mathbb{Z}$, we construct a stronger inequality.
- In some cases, the cut can *separate* a specific point x^*.

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Two-row cuts

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$$(\text{MIP}) \quad \begin{aligned} \min \quad & \bar{c}^T x \\ \text{s.t.} \quad & \bar{A}x = \bar{b} \\ & x \geq 0 \\ & x_J \in \mathbb{Z} \end{aligned}$$

we extract **two** constraints

$$\begin{array}{rcl} x_1 \qquad\; + \sum_j \bar{a}_{1j} s_j & = & f_1 \\ + x_2 + \sum_j \bar{a}_{2j} s_j & = & f_2 \end{array}, \qquad \begin{array}{l} x_1, x_2 \in \mathbb{Z} \\ s_j \in \mathbb{R}_+ \end{array}$$

As a vector equation,

$$(P_I) \qquad x = f + \sum_j r^j s_j, \qquad \begin{array}{l} x \in \mathbb{Z}^2 \\ s \in \mathbb{R}^n_+ \end{array}$$

In case (MIP) describes a simplex tableau, $(x^*_{LP}, s^*_{LP}) = (f, 0)$.

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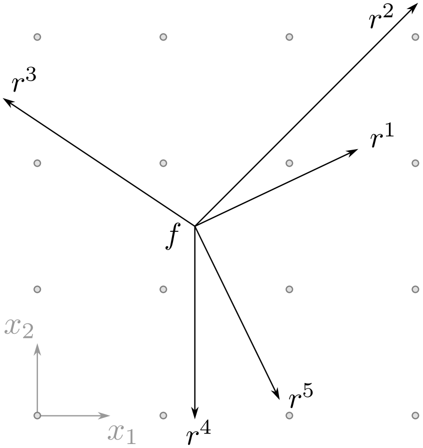
A.2. Problem statement

The two-row model

$$\begin{aligned} x &= f + \textstyle\sum_j r^j s_j \\ x &\in \mathbb{Z}^2 \\ s_j &\geq 0 \end{aligned}$$



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Example:

$$s = \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$x = f + \frac{1}{2}r^1 + \frac{1}{2}r^5 + \frac{1}{2}r^4 + \frac{1}{2}r^3$$

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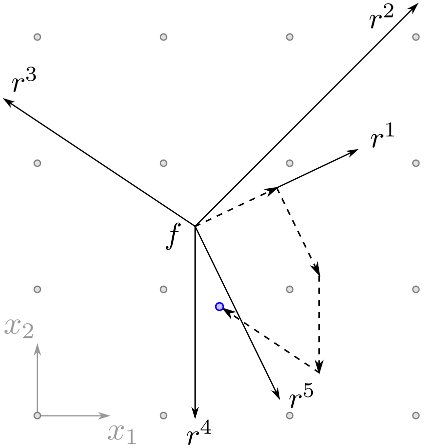

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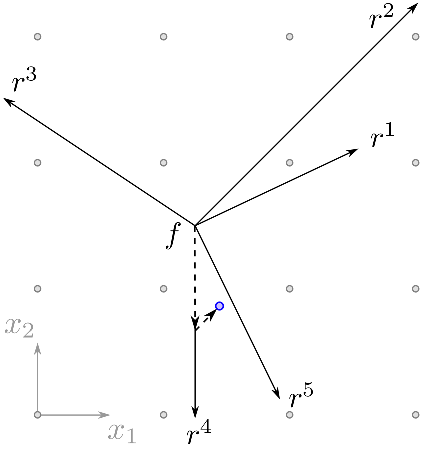

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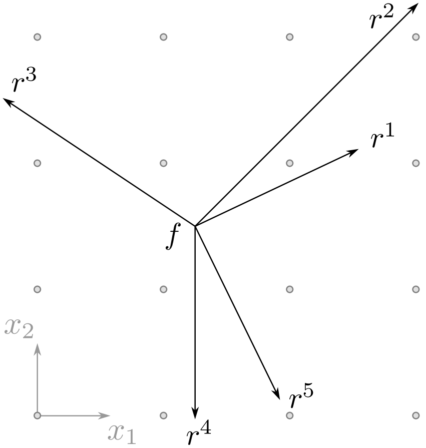
An inequality of the form

$$\alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1$$

with $\alpha_i \geq 0$, cuts off

interior(L_α)

in the x space
where $v^i = f + \frac{1}{\alpha_i} r^i$.



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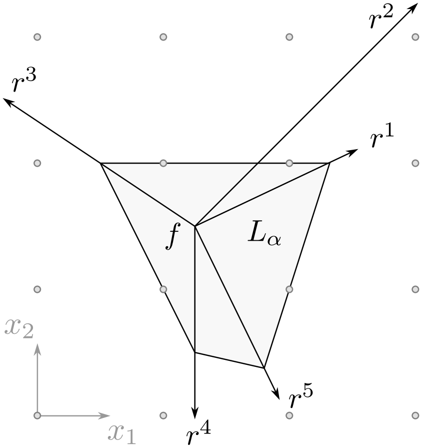

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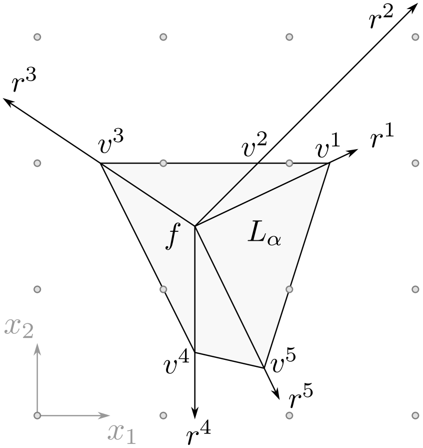

Validity: The linear programming intuition

Given $\overline{x} \in \mathbb{Z}^2$, we want that

$$\forall s \in \mathbb{R}^n_+ \;:\; \overline{x} = f + Rs, \qquad \alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1$$

i.e. we want

$$\begin{array}{rrcl} \min & \alpha_1 s_1 + \ldots + \alpha_n s_n & \geq & 1 \\ \text{s.t.} & Rs = \overline{x} - f & & \\ & s \geq 0 & & \end{array}$$

therefore we need

$$\forall i, j, s_i^{\overline{x}}, s_j^{\overline{x}} \;:\; \overline{x} = f + s_i^{\overline{x}} r^i + s_j^{\overline{x}} r^j, \qquad s_i^{\overline{x}} \alpha_i + s_j^{\overline{x}} \alpha_j \geq 1.$$

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Lattice-free sets – the geometrical intuition

Given $\overline{x} \in \mathbb{Z}^2$,

for all $i, j : \overline{x} \in f + \text{cone}(r^i, r^j)$,

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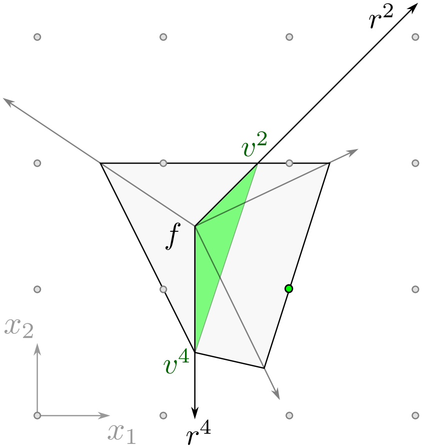

Lattice-free sets – the geometrical intuition

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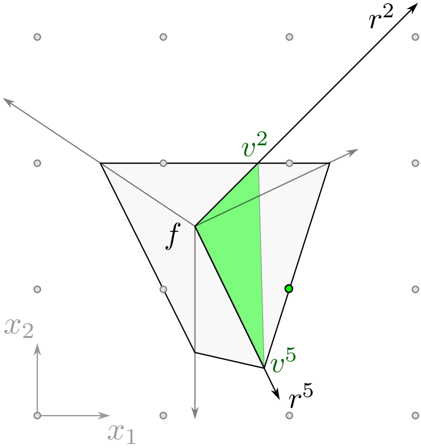

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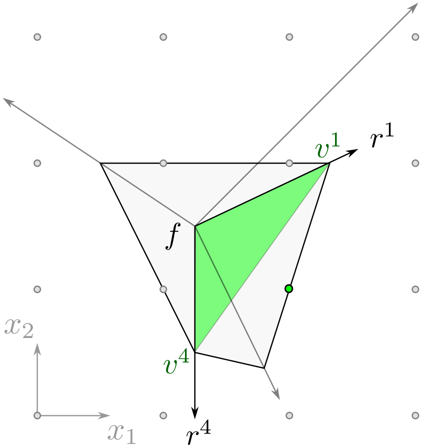

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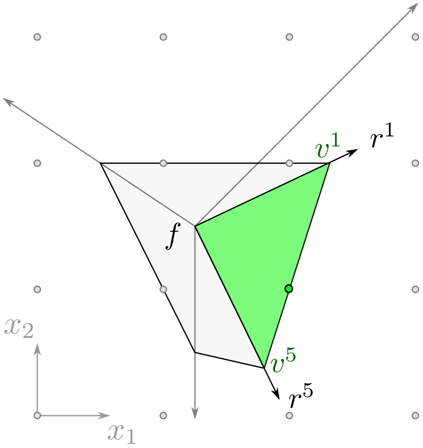

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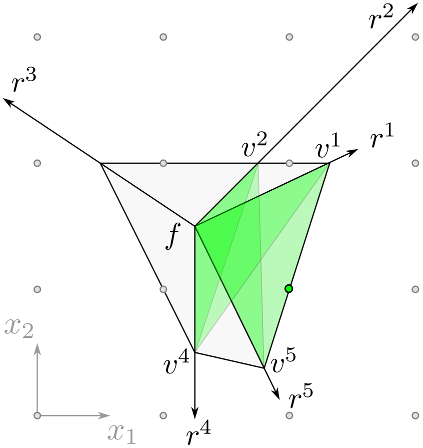

Lattice-free sets – the intuition, for all x

For all $x \in \mathbb{Z}^2$,

for all $i, j : x \in f + \text{cone}(r^i, r^j)$,

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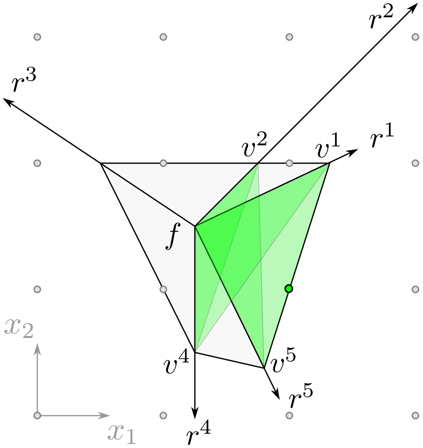


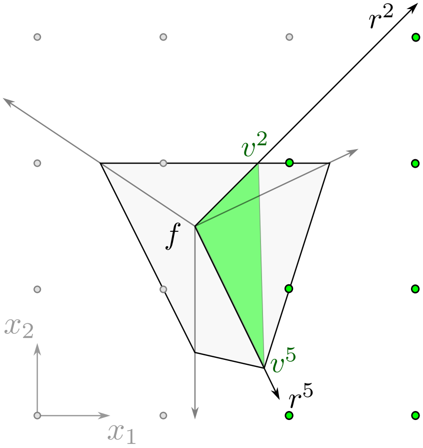
Lattice-free sets – the intuition, for every cone

For all i, j,

for all $x \in \mathbb{Z}^2 \cap (f + \mathrm{cone}(r^i, r^j))$,

$$s_i^x \alpha_i + s_j^x \alpha_j \geq 1,$$

with $s_i^x, s_j^x : x = f + s_i^x r^i + s_j^x r^j$.



Lattice-free sets – the set $\mathcal{X}_{ij}$

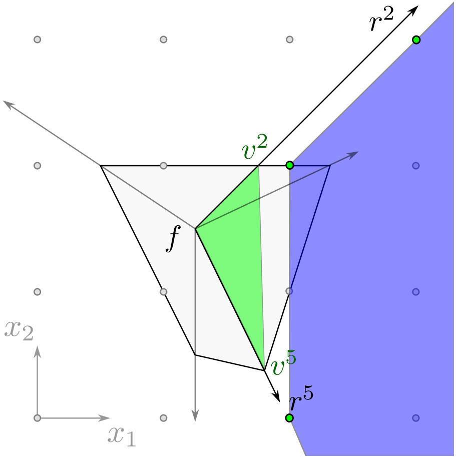
For all i, j,

for all $x \in \mathcal{X}_{ij}$,

$$s_i^x \alpha_i + s_j^x \alpha_j \geq 1,$$

with $s_i^x, s_j^x \,:\, x = f + s_i^x r^i + s_j^x r^j$.

- we can restrict $x \in \mathbb{Z}^2$ to $x \in \mathcal{X}_{ij}$ where $\mathcal{X}_{ij}$ is the set of the vertices of $\mathbb{Z}^2 \cap (f + \mathrm{conv}(r^i, r^j))$.



Polarity

Let $P \subseteq \mathbb{R}^N$ be a **radial** polyhedron and $Q \subseteq \mathbb{R}^N$ its polar. There is a correspondance between

Extreme point $\overline{x} \in P$	and	Facet of Q: $\overline{x}^T a \geq 1$
Extreme ray $\overline{x} \in P$	and	Facet of Q: $\overline{x}^T a \geq 0$
Facet of P: $\overline{a}^T x \geq 1$	and	Extreme point $\overline{a} \in Q$
Facet of P: $\overline{a}^T x \geq 0$	and	Extreme ray $\overline{a} \in Q$

Polarity, applied

- We have a polyhedron $\mathrm{conv}(P_I) = \mathrm{conv}\left(\left\{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \mid x = f + \sum_j r^j s_j\right\}\right)$.
- $\mathrm{conv}(P_I) \subseteq \mathbb{R}^{2+n}$ is of dimensionality n.
- We know the extreme points and rays of $\mathrm{conv}(P_I)$.
- We can build the polar $Q \subseteq \mathbb{R}^n$ of $\mathrm{conv}(P_I)$.
- We can optimize over Q to find facets $\mathrm{conv}(P_I)$.

Extreme point $\overline{x} \in \mathrm{conv}(P_I)$	$\longrightarrow$	Facet of Q: $\overline{x}^T \alpha \geq 1$
Extreme ray $\overline{x} \in \mathrm{conv}(P_I)$	$\longrightarrow$	Facet of Q: $\overline{x}^T \alpha \geq 0$
		$\downarrow\ \downarrow$
Facet of $\mathrm{conv}(P_I)$: $\overline{\alpha}^T x \geq 1$	$\longleftarrow$	Extreme point $\overline{\alpha} \in Q$
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Finding facets of $\operatorname{conv} P_I$

The polar of $\operatorname{conv}(P_I)$ is

$$Q = \{ \alpha \in \mathbb{R}^n_+ \mid \forall i, j,\ \forall x \in \mathcal{X}_{ij},\ \ s_i^x \alpha_i + s_j^x \alpha_j \geq 1 \}.$$

We find facets of $\operatorname{conv}(P_I)$ by choosing an objective function $c^T\alpha$ and optimizing over Q:

$$\begin{aligned} \min \quad & c^T\alpha \\ \text{s.t.} \quad & s_i^x \alpha_i + s_j^x \alpha_j \geq 1, \quad \forall i, j,\ \forall x \in \mathcal{X}_{ij} \\ & \alpha \geq 0 \end{aligned}$$

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A.3. New developments

Complexity of writing the polar (1)

- For each cone, compute integer hull.
- For each vertex, write one constraint.

1. Cones: quadratic in the number of rays.
2. Vertices: polynomial (but possibly large) number in each cone.

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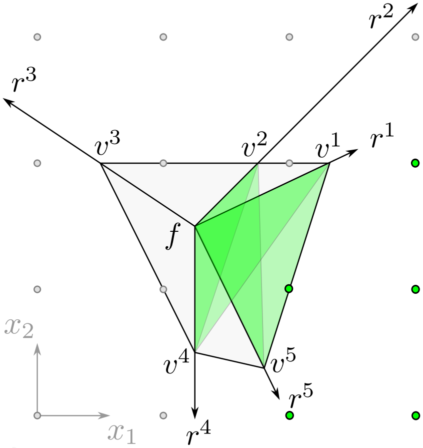

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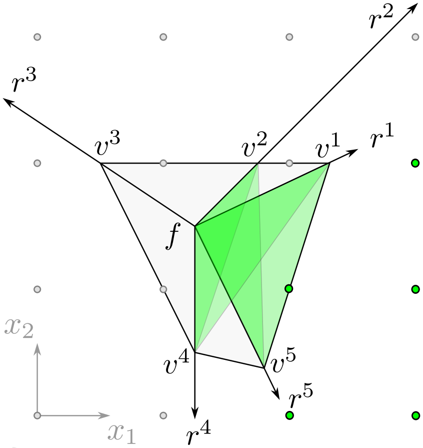

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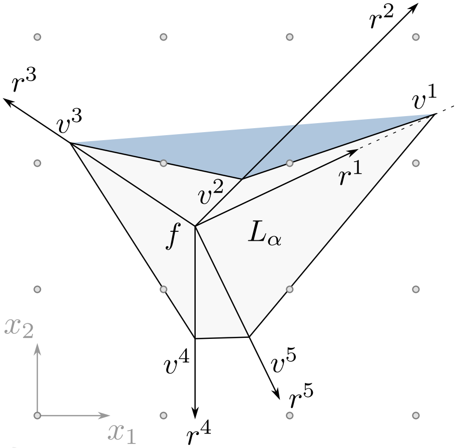

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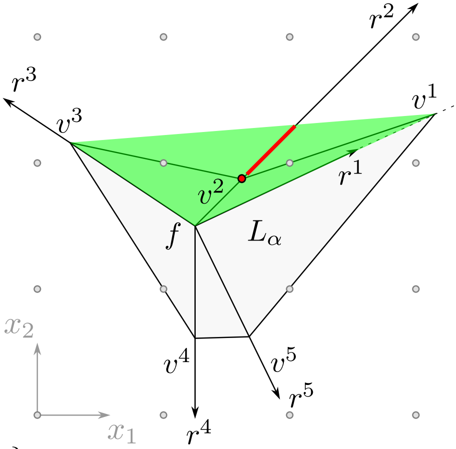

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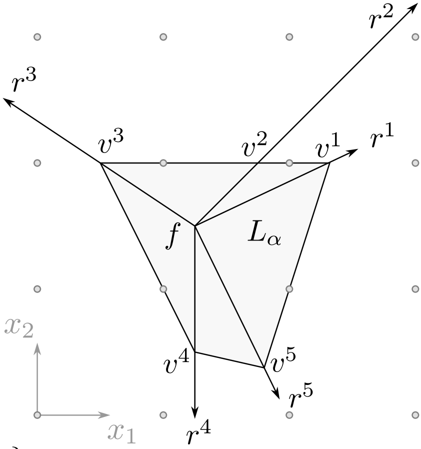

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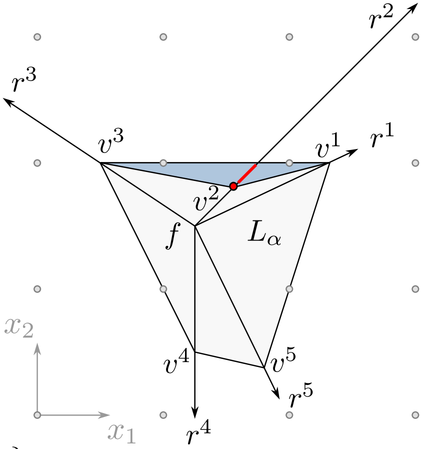

- What is $Q \setminus \overline{Q}$?

The complexity of the polar – the theory

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Theorem

$\overline{Q} \subseteq Q$, and all vertices of Q are in $\overline{Q}$.

Corollary

If $c > 0$, $\begin{array}{ll} \min & c^T \alpha \\ \text{s.t.} & \alpha \in Q \end{array}$ and $\begin{array}{ll} \min & c^T \alpha \\ \text{s.t.} & \alpha \in \overline{Q} \end{array}$ share the same set of optimal solutions.

If $c_i < 0$, then $\begin{array}{ll} \min & c^T \alpha \\ \text{s.t.} & \alpha \in Q \end{array}$ is unbounded.

The complexity of the polar – the theory

$$Q = \{\ \alpha \in \mathbb{R}^n_+ \ \mid \forall i, j,\ \forall x \in \mathcal{X}_{ij}, \qquad s_i^x \alpha_i + s_j^x \alpha_j \geq 1\ \}$$

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If $c > 0$, $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in Q \end{array}$ *and* $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in \overline{Q} \end{array}$ *share the same set of optimal solutions.*

If $c_i < 0$, then $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in Q \end{array}$ *is unbounded.*

The complexity of the polar – the theory

$$Q = \{ \alpha \in \mathbb{R}^n_+ \mid \forall i,j,\ \forall x \in \mathcal{X}_{ij}, \quad s^x_i \alpha_i + s^x_j \alpha_j \geq 1 \}$$

$$\begin{aligned} \overline{Q} = \{ \alpha \in \mathbb{R}^n_+ \mid & \\ & \forall i,\ \forall x \in \mathcal{X}_{i,i+1}, \quad s^x_i \alpha_i + s^x_{i+1} \alpha_{i+1} \geq 1 \\ & \forall i : r^i \in \mathrm{cone}(r^{i-1}, r^{i+1}), \quad \alpha_i \leq \lambda^i_{i-1} \alpha_{i-1} + \lambda^i_{i+1} \alpha_{i+1} \}. \end{aligned}$$

Theorem

$\overline{Q} \subseteq Q$, and all vertices of Q are in $\overline{Q}$.

Corollary

If $c > 0$, $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in Q \end{array}$ and $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in \overline{Q} \end{array}$ share the same set of optimal solutions.

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Complexity of writing the polar (2)

- For each cone, compute integer hull.
- For each vertex, write one constraint.

1. Cones: ~~quadratic~~ linear in the number of rays.
2. Vertices: polynomial (but possibly large) number in each cone.
3. In practice, generate the constraints of $\overline{Q}$ by row generation.

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A.4. Results

Computational results

	Average iter. per cut	Average time (ms) per cut
MIPLIB 3	3.1	1.8 ms
MIPLIB 2003	15.6	24.3 ms

	one-row		two-row (split sets)		two-row	
	Average sep. cuts	Average %gc	Average sep. cuts	Average %gc	Average sep. cuts	Average %gc
MIPLIB 3	695.0	29.4 %	39.7	34.8 %	232.7	36.2 %
MIPLIB 2003	4465.3	31.3 %	465.5	33.0 %	600.7	34.5 %

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Conclusions

- We have a fast separation for two-row cuts.
- These cuts are the strongest for the two-row model.
- They close more gap than one-row (intersection) cuts.

But

- they do not close much more gap than two-row intersection cuts from split sets.

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B. Separation over arbitrary mixed-integer sets

Motivations

- We want to test stronger relaxations
- We still want exact separation

B.1. Separation method

Problem

Given

- a relaxation P of mixed-integer set in $\mathbb{R}^n$,
- a point $x^* \in \mathbb{R}^n$,

find $(\alpha, \alpha_0) \in \mathbb{R}^{n+1}$ such that

$$\alpha^T x \geq \alpha_0$$

is a valid inequality for P that separates x^*,

or show that $x^* \in \text{conv}(P)$.

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General framework

Solve the optimization problem

$$\begin{array}{ll} \min & {x^*}^T \alpha \\ \text{s.t.} & x^T \alpha \geq \alpha_0 \quad \text{for all } x \in P \end{array} \qquad \text{(Sep. LP)}$$

<norm.>

Let $(\bar{\alpha}, \bar{\alpha}_0)$ be an optimal solution.

If ${x^*}^T \bar{\alpha} < \bar{\alpha}_0$, then $(\bar{\alpha}, \bar{\alpha}_0)$ separates x^*.

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Row generation

1. Consider the relaxation of the separation problem

$$\begin{aligned} \min \quad & {x^*}^T \alpha \\ \text{s.t.} \quad & x^T \alpha \geq \alpha_0 \quad \text{for all } x \in S \subseteq P \\ & \text{<norm.>} \end{aligned} \qquad \text{(master)}$$

Let $(\bar{\alpha}, \bar{\alpha}_0)$ be an optimal solution.

2. Now solve the MIP

$$\begin{aligned} \min \quad & \bar{\alpha}^T x \\ \text{s.t.} \quad & x \subseteq P \end{aligned} \qquad \text{(slave)}$$

and let x^p be a finite optimal solution.

If $\bar{\alpha}^T x^p \geq \bar{\alpha}_0$, then $(\bar{\alpha}, \bar{\alpha}_0)$ is valid for P.

If $\bar{\alpha}^T x^p < \bar{\alpha}_0$, then $S := S \cup \{x^p\}$.

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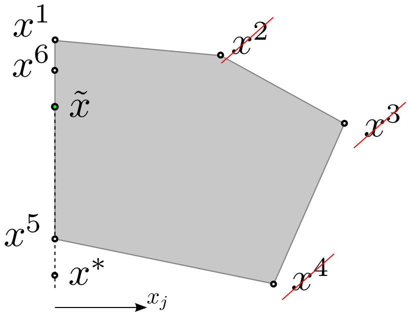
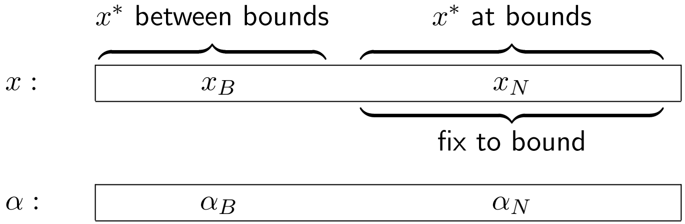
Computational example

Instance:	`bell3a`
Constraints:	123
Variables:	133 (71 integer: 32 general, 39 binaries)
Models:	82 five-row models read from an optimal tableau
Cuts:	37 (17 tight at the end)
Gap closed:	59.02% (from 39.03% by GMIs)

Time:	1615.70s
Iterations:	107647

Two-phases: Phase one





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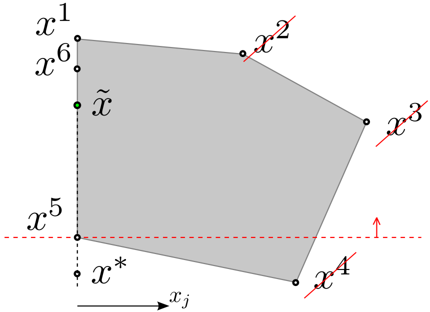
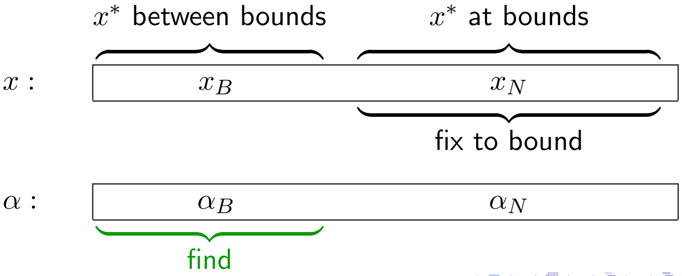
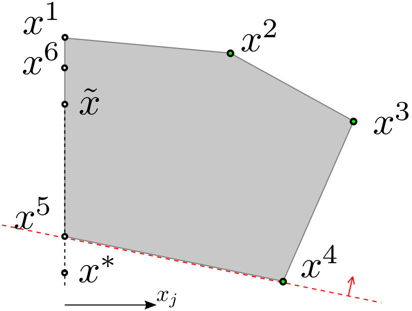
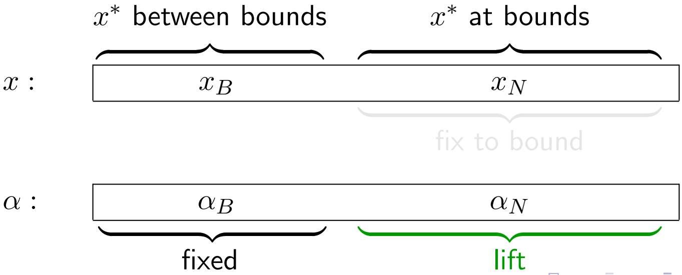




Two-phases: Phase two





Computational example (2-phases)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases
Time:	1615.70s	161.15s
Iterations:	107647	23822

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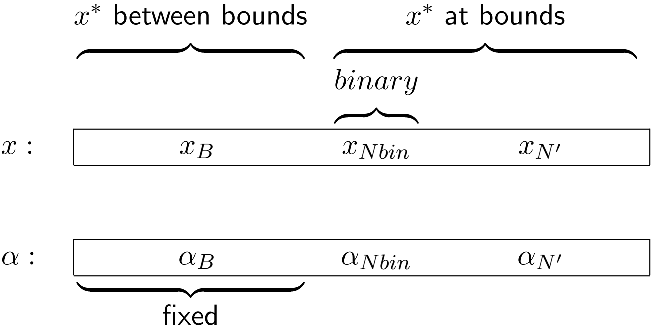
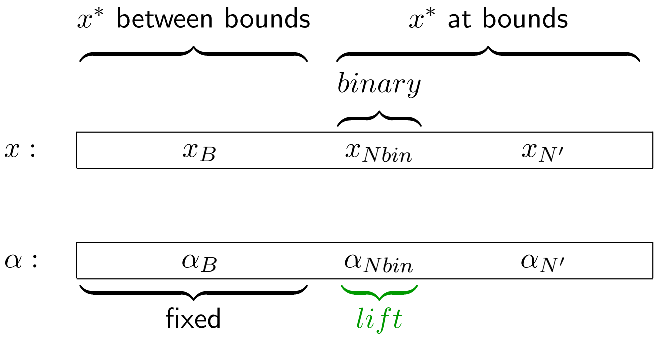
Lifting binary variables



Lifting binary variables



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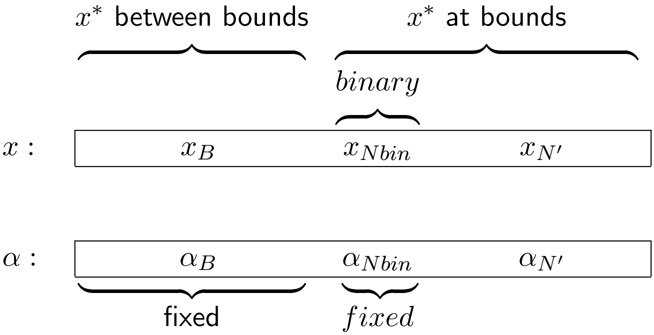

Computational example (lifting binaries)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases	lifting
Time:	1615.70s	161.15s	136.54s
Iterations:	107647	23822	23231

Computational example (lifting binaries)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases	lifting
Time:	1615.70s	161.15s	136.54s
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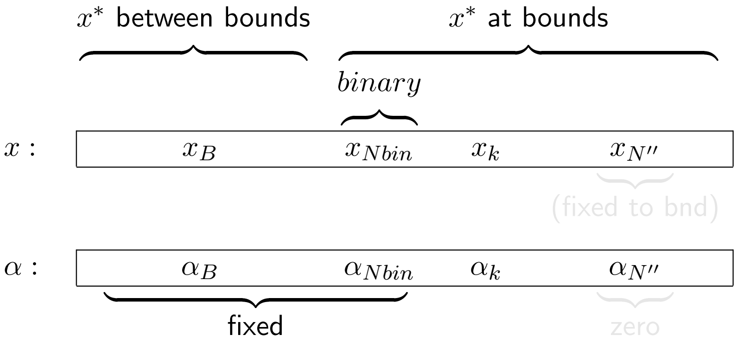
Sequential phase-2 ("phase-S")



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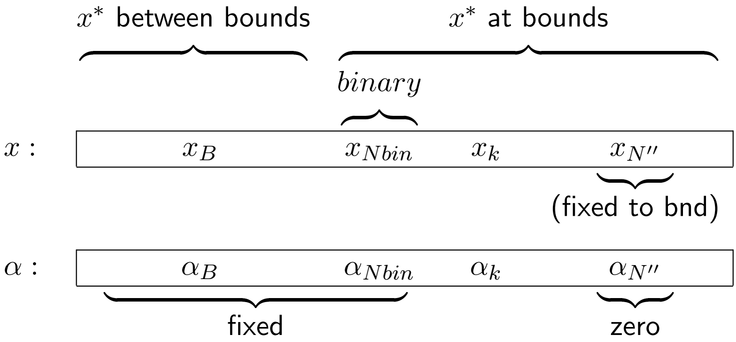

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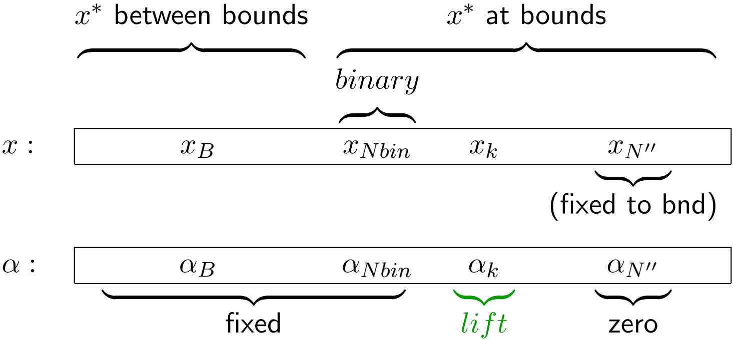

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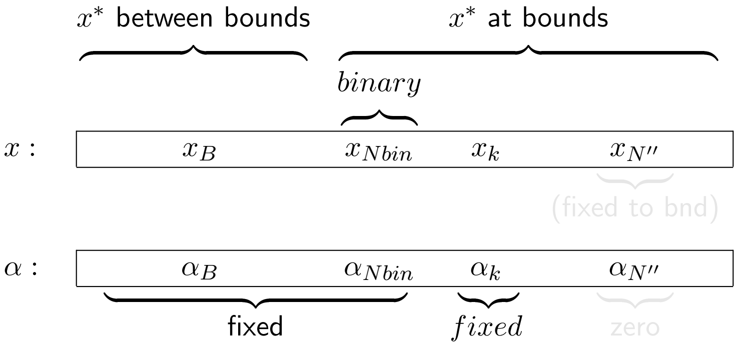

Computational example (phase S)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases	lifting	phase S
Time:	1615.70s	161.15s	136.54s	5.84s
Iterations:	107647	23822	23231	2497

Computational example (phase S)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

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Time:	1615.70s	161.15s	136.54s	5.84s
Iterations:	107647	23822	23231	2497

Computational example (solver tricks)

(bell3a, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases	lifting	phase S	cb
Time:	1615.70s	161.15s	136.54s	5.84s	4.65s
Iterations:	107647	23822	23231	2497	2497

Computational example (solver tricks)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases	lifting	phase S	cb
Time:	1615.70s	161.15s	136.54s	5.84s	4.65s
Iterations:	107647	23822	23231	2497	2497

Computational example (summary)

(`bell3a`, 82 five-row models, 37 cuts, 59.02%gc)

	original	2-phases	lifting	phase S	cb
	$347\times$	$35\times$	$29\times$	$1.26\times$	1
Time:	1615.70s	161.15s	136.54s	5.84s	4.65s
Iterations:	107647	23822	23231	2497	2497
	$43\times$	$10\times$	$9\times$	1	1

B.2. Application to two-row relaxations

Two-row intersection cuts + strengthening

	basic		nonbasic	
	$\in \mathbb{Z}$	bnd.	$\in \mathbb{Z}$	bnd.
P_I	$\surd$	$\times$	$\times$	B
S-free	$\surd$	$\surd$	$\times$	B
lifting	$\surd$	$\times$	$\surd$	B
P_{IU}	$\surd$	$\times$	$\times$	$\surd$
full 2-row	$\surd$	$\surd$	$\surd$	$\surd$

$\surd$:	keep
B:	keep binding
$\times$:	drop

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full 2-row	$\surd$	$\surd$	$\surd$	$\surd$

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P_{IU}	$\surd$	$\times$	$\times$	$\surd$
full 2-row	$\surd$	$\surd$	$\surd$	$\surd$

$\surd$:	keep
B:	keep binding
$\times$:	drop

Two-row intersection cuts and strengthenings

51 common instances:

	cuts	gc%	exact
GMI	28.240	22.46%	all
P_I	29.420	27.65%	42
S-free	38.380	30.22%	29
lifting	22.700	27.35%	10
P_{IU}	42.640	28.56%	25
full 2-row	55.500	35.66%	22

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Two-row intersection cuts and strengthenings

51 common instances:

	cuts	gc%	exact
GMI	[illegible]8.240	22[illegible]%	all
P_I	29[illegible]420	[illegible].65%	42
S-free	38.3[illegible]	30.22%	29
lifting	22[illegible]0	[illegible]7.35%	10
P_{IU}	[illegible].640	2[illegible]56%	25
full 2-row	55.500	35.6[illegible]%	22

Two-row intersection cuts and strengthenings

15 common instances:

	cuts	gc%	exact
GMI	20.667	26.541	all
P_I	20.933	33.535	all
S-free	25.400	35.229	all
P_{IU}	36.600	36.257	all
full 2-row	57.267	43.956	all

Two-row intersection cuts and strengthenings

7 common instances:
[bell5, blend2, egout, khb05250, misc03, misc07, set1ch]

	cuts	gc%	exact
GMI	25.571	24.744	all
P_I	25.143	33.641	all
S-free	28.714	33.836	all
lifting	25.571	33.716	all
P_{IU}	47.857	37.531	all
full 2-row	48.000	37.583	all

Bases

- We depend on a specific optimal basis
- Will the gap closed by two-row cuts survive more GMIs?

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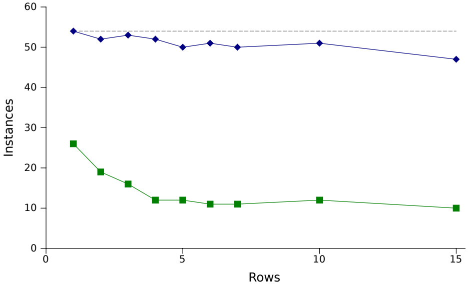
Relax and cut: results

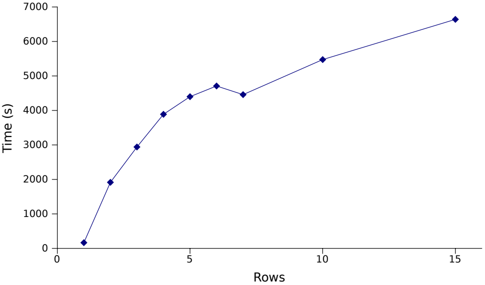
43 common instances:

		cuts	gc%	exact
	GMI	24.814	23.282	all
	2-row i.c.	31.884	28.838	42
	full 2-row	62.140	36.080	22
relax&cut	GMI	60.372	34.970	all
relax&cut	2-row i.c.	63.163	41.951	37
relax&cut	full 2-row	76.767	47.277	12

More rows: Computing time

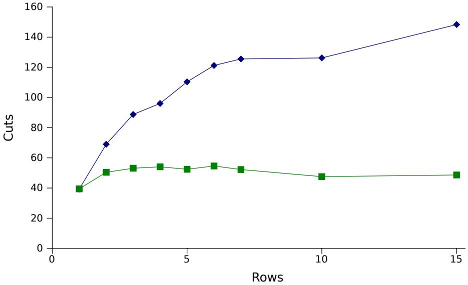


instances with result, and
instances with exact separation

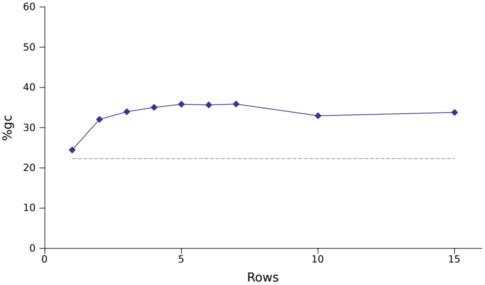


geometric mean of time
(on 42 common instances)

More rows: Gap closed



number of cuts generated
(on 42 common instances)



average %gc
(on 42 common instances)

Overall summary

- a (quick) two-row intersection cut separator
- assessment: strength of the two-row model
- a (slow) generic arbitrary-MIP cut separator
- assessment: strength of multi-row model and variants

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///////

|||||||

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Adding all valid inequalities for (MIP), we obtain:

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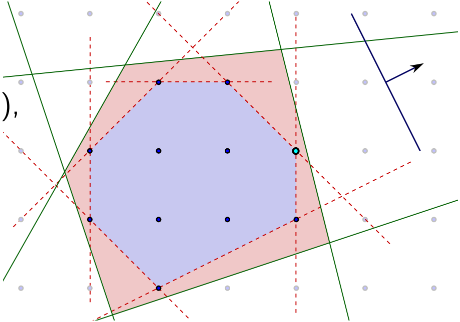
In theory: as hard as solving (MIP)

In practice: much harder



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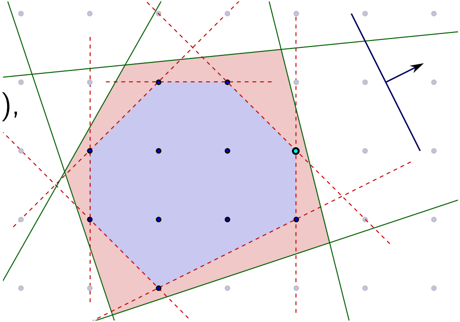

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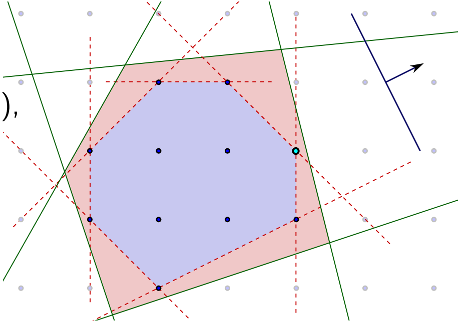

Can we avoid the integer hulls $\mathcal{X}_{ij}$?

$$\overline{Q} = \{ \, \alpha \in \mathbb{R}^n_+ \; | \;\; \forall i, \; \forall x \in \mathcal{X}_{i,i+1}, \quad s^x_i \alpha_i + s^x_{i+1}\alpha_{i+1} \geq 1 \;\; \forall i : \; r^i \in \mathrm{cone}(r^{i-1}, r^{i+1}), \quad \alpha_i \leq \lambda^i_{i-1}\alpha_{i-1} + \lambda^i_{i+1}\alpha_{i+1} \, \}$$

$$\overline{Q}(S) = \{ \, \alpha \in \mathbb{R}^n_+ \; | \;\; \forall i, \; \forall x \in S \cap (f + \mathrm{cone}(r^i, r^{i+1})), \quad s^x_i \alpha_i + s^x_{i+1}\alpha_{i+1} \geq 1 \;\; \forall i : \; r^i \in \mathrm{cone}(r^{i-1}, r^{i+1}), \quad \alpha_i \leq \lambda^i_{i-1}\alpha_{i-1} + \lambda^i_{i+1}\alpha_{i+1} \, \},$$

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Separation algorithm

$S := S_0$

do {

$$\alpha := \operatorname{argmin} \; c^T\alpha \quad \text{s.t.} \quad \alpha \in \overline{Q}(S)$$

if $\alpha \in \overline{Q}$

OK, valid cut, exit.

else

Find a constraint of $\overline{Q}$ violated by α.

Add constraints to S.

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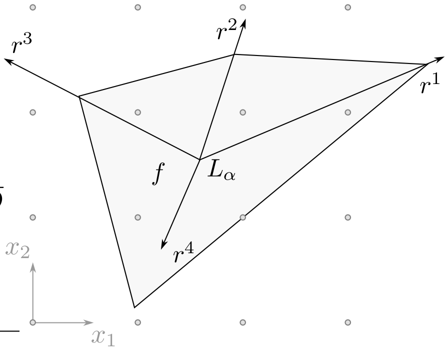

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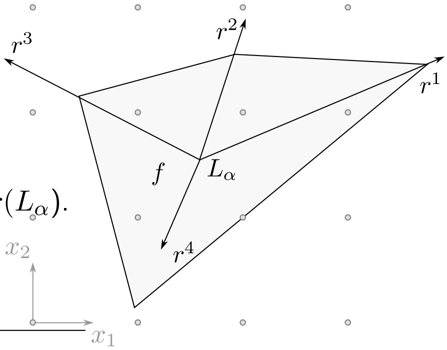

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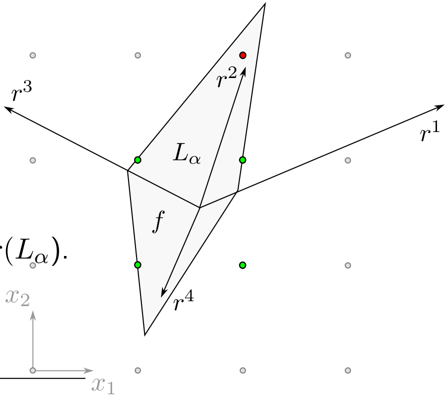

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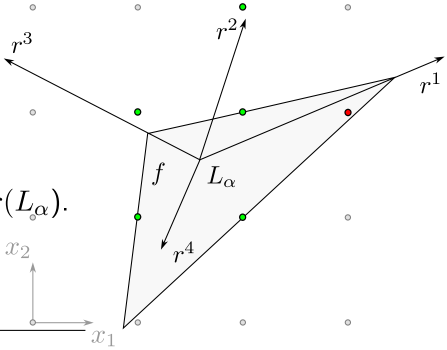

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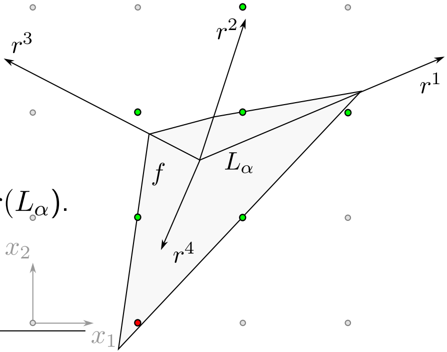

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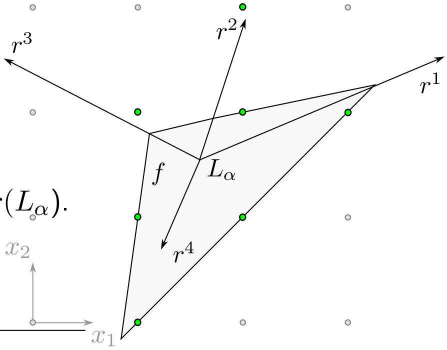
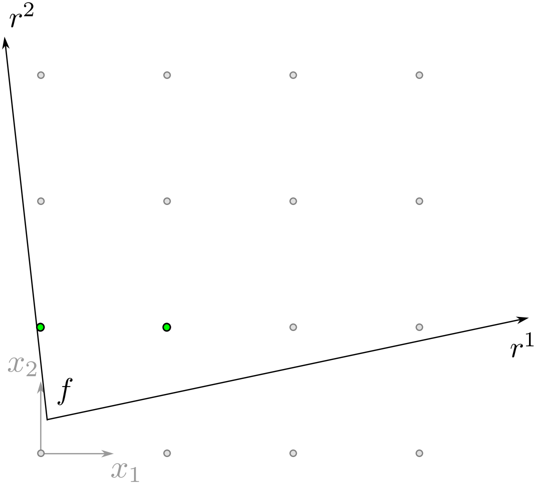


Integer pair extension



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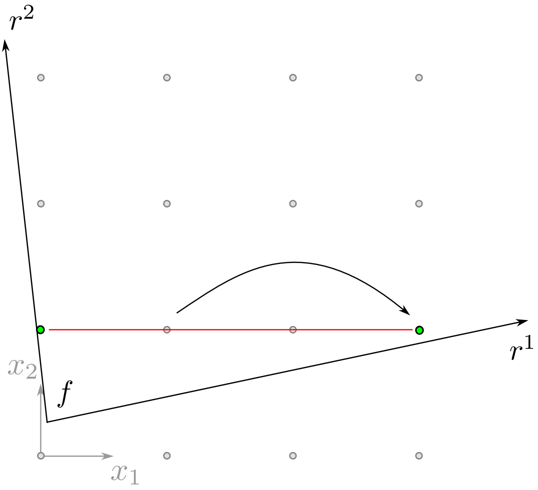

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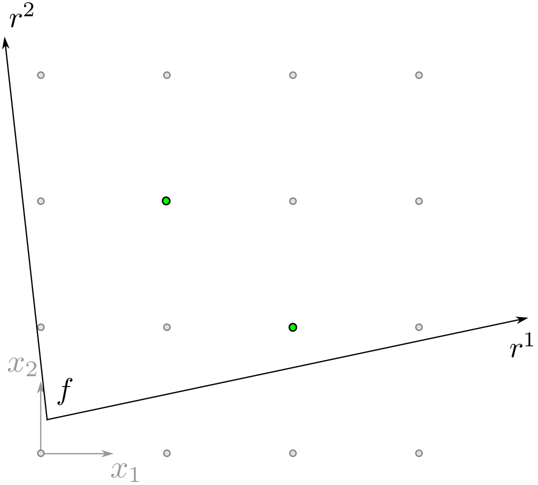

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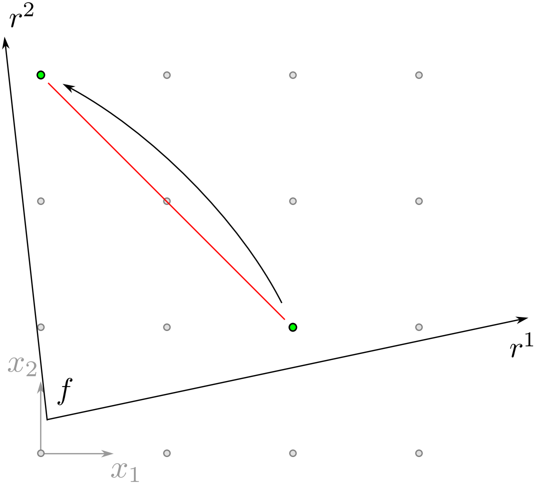

The oracle

Find an integer point in $\text{interior}(L_\alpha)$ or prove that L_α is lattice-free.

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- but $d = 2$
- we know $S \cap L_\alpha$
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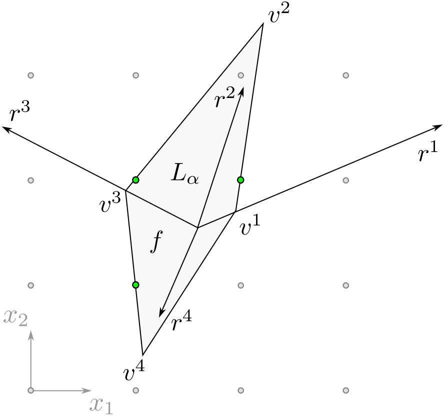

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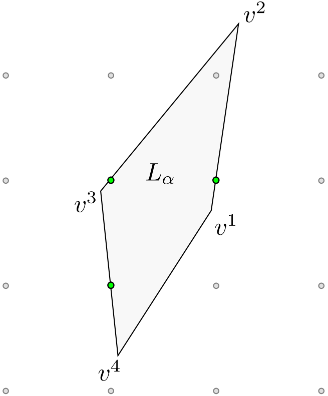

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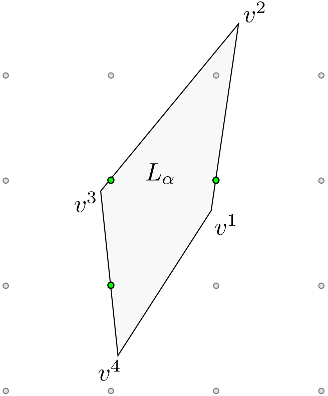

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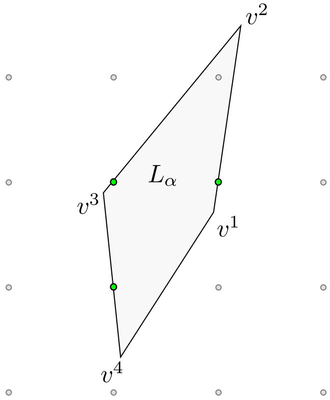

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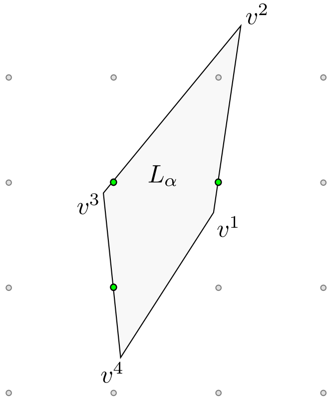

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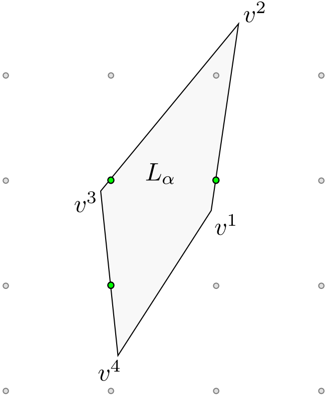

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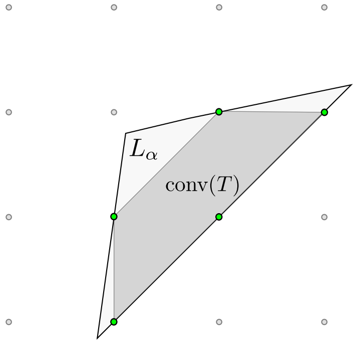

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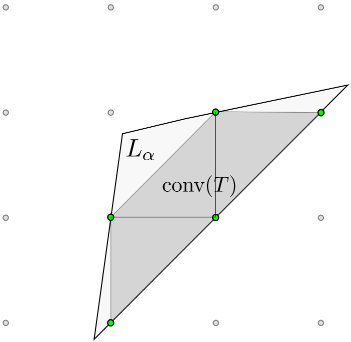

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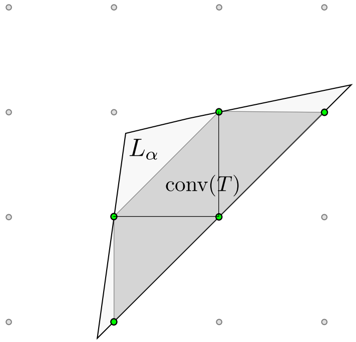
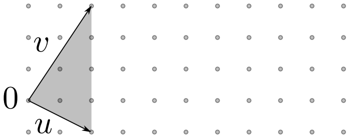


The oracle: $\mathrm{conv}(T)$, continued

Let Δ be $\mathrm{conv}(0, u, v)$ with $u, v \in \mathbb{Z}$ and $\gcd(u_1, u_2) = \gcd(v_1, v_2) = 1$.

$$\left\{ \frac{\lambda}{\det([u|v])} u + \frac{\mu}{\det([u|v])} v \;:\; \lambda, \mu \in \mathbb{Z}_+,\ 0 < \lambda + \mu < \det([u|v]) \right\}$$

Prop.: Δ has an interior lattice point with $\mu = 1$, or is lattice-free.



It is enough to solve the diophantine system

$$\begin{cases} \lambda u_1 + v_1 = k_1 \det([u|v]) \\ \lambda u_2 + v_2 = k_2 \det([u|v]) \end{cases}, \quad \lambda, k_1, k_2 \in \mathbb{Z}$$

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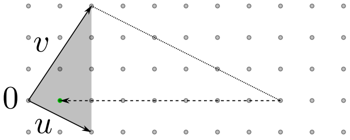

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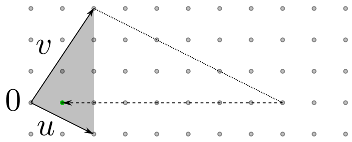

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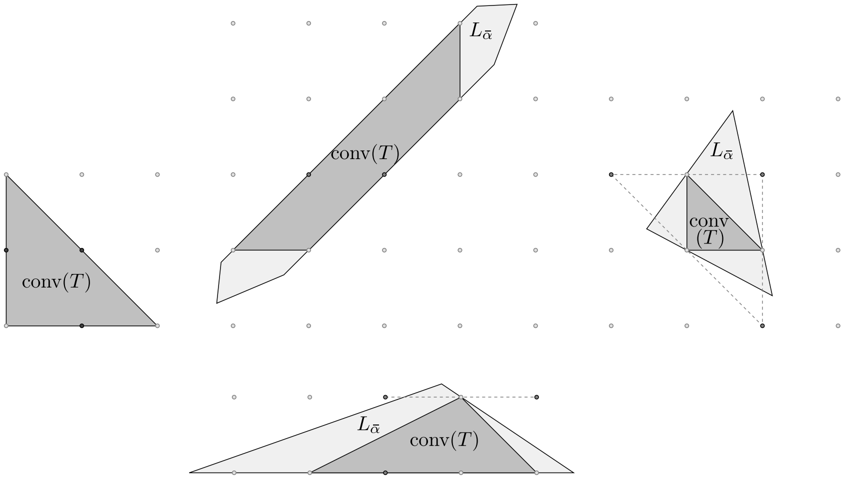
The oracle: $\text{interior}(L_\alpha)$

2. Assuming $\text{conv}(T)$ lattice-free,

Prop.: It is enough to check 2 or 3 specific integer points:



Solver tricks: callbacks

Solving slave MIPs

$$\begin{aligned} \min \quad & \bar{\alpha}^T x \\ \text{s.t.} \quad & x \subseteq P, \end{aligned}$$

- Feasible solution $\hat{x}$ with $\bar{\alpha}^T \hat{x} < \bar{\alpha}_0$
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- We are still far from a closure
- What reasonable set of two-models can we select?

  → All models read from a simplex tableau

  → $O(m^2)$ two-row models

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“all” two-row models: separation loop

Let $x^* \leftarrow$ LP optimium
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Read and add GMIs from that tableau.

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“all” two-row models: results

Computations on the 62 MIPLIB 3.0 (preprocessed) instances for which

(a). the integrality gap is not zero, and

(b). an optimal MIP solution is known.

"all" two-row models: results

We have a result for 55/62 instances (4 numerical, 3 memory).

	cuts	gc%
GMI	24.800	22.60%
All 2-row	72.382	37.49%

For 13 instances, the separation is exact.

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Rationale:

- MIPLIB models are mostly sparse
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For 25 instances, the separation is exact.

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Polyhedron P		Polyhedral cone P^+
vertex v	$\rightarrow$	extreme ray $(v, -1)$
extreme ray r	$\rightarrow$	extreme ray $(r, 0)$
l in the lineality space	$\rightarrow$	$(l, 0)$ in the lineality space
facet-defining $\alpha^T x \geq \alpha_0$	$\Leftrightarrow$	facet-defining $\alpha^T x + \alpha_0 x_0 \geq 0$
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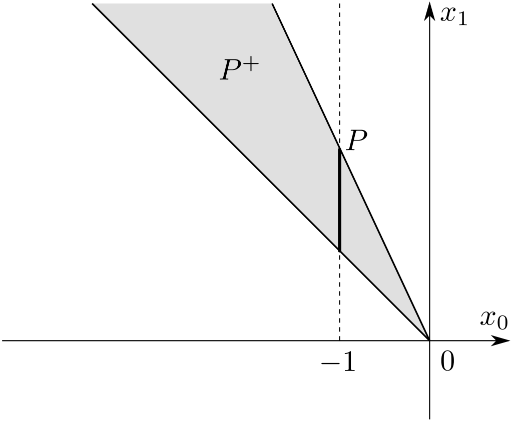
valid $\alpha^T x = \alpha_0$	$\Leftrightarrow$	valid $\alpha^T x + \alpha_0 x_0 = 0$

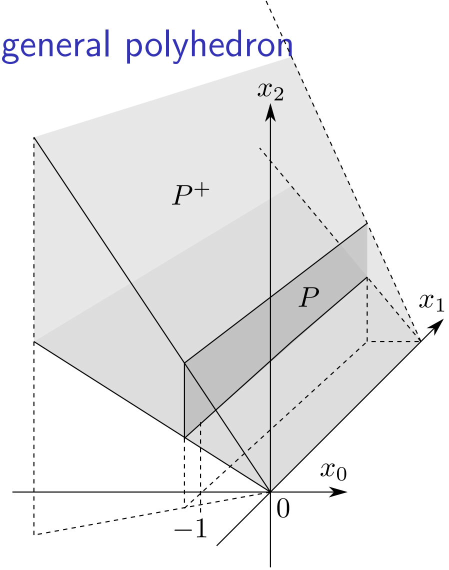
Conify: P is a polytope



Note: $P = \mathrm{proj}_x(P^+ \cap \{x_0 = -1\})$.

$P^+ = \{(x, x_0) \in R^{n+1} : x_0 \leq 0,\ x \in -x_0 P\}$

Conify: P is a general polyhedron



Note: $P = \text{proj}_x(P^+ \cap \{x_0 = -1\})$.

$P^+ = \{(x, x_0) \in R^{n+1} : x_0 \leq 0,\ \text{“}x \in -x_0 P + \text{recc}(P)''\}$

Polarity for full-dimensional polyhedral cones

P^+		Q
extreme ray r	$\Leftrightarrow$	facet-defining $r^T\alpha \geq 0$
l in the lineality space	$\Leftrightarrow$	valid $l^T\alpha = 0$

Q is the polar of P^+	$\Leftrightarrow$	P^+ is the polar of Q
facet-defining $\beta^T x \geq 0$	$\Leftrightarrow$	extreme ray β
valid $\gamma^T x = 0$	$\Leftrightarrow$	γ in the lineality space

Polarity for full-dimensional polyhedral cones

P^+		Q
extreme ray r	$\Leftrightarrow$	facet-defining $r^T\alpha \geq 0$
l in the lineality space	$\Leftrightarrow$	valid $l^T\alpha = 0$
Q is the polar of P^+	$\Leftrightarrow$	P^+ is the polar of Q
facet-defining $\beta^T x \geq 0$	$\Leftrightarrow$	extreme ray β
valid $\gamma^T x = 0$	$\Leftrightarrow$	γ in the lineality space

Polarity for full-dimensional polyhedral cones

P^+		Q
extreme ray r	$\Leftrightarrow$	facet-defining $r^T\alpha \geq 0$
l in the lineality space	$\Leftrightarrow$	valid $l^T\alpha = 0$
Q is the polar of P^+	$\Leftrightarrow$	P^+ is the polar of Q
facet-defining $\beta^T x \geq 0$	$\Leftrightarrow$	extreme ray β
valid $\gamma^T x = 0$	$\Leftrightarrow$	γ in the lineality space

Going back to general (full-dimensional) polyhedra

Polyhedron P	Polyhedral cone P^+	Polar of P^+ Q
vert. v	ray $(v, -1)$	$v^T\alpha - \alpha_0 \geq 0$
ray r	ray $(v, 0)$	$r^T\alpha \geq 0$
l in lin.sp.	$(l, 0)$ in lin.sp.	$l^T\alpha = 0$
$\alpha^T x \geq \alpha_0$	$\alpha^T x + \alpha_0 x_0 \geq 0$	ray (α, α_0)
$\alpha^T x = \alpha_0$	$\alpha^T x + \alpha_0 x_0 = 0$	(α, α_0) in lin.sp.