# A separation method for two-row cuts

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Joint work with Quentin Louveaux

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# Background

# The mixed integer linear problem

$$
(\text{MIP}) \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & A\,x \geq b \\ & x_j \in \mathbb{Z}, \text{for } j \in J \end{aligned}
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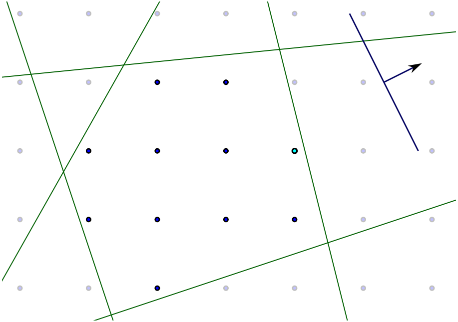

# The linear relaxation

$$
(\text{LP}) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & A x \geq b \\ & \cancel{x_j \in \mathbb{Z}, \text{for } j \in J} \end{array}
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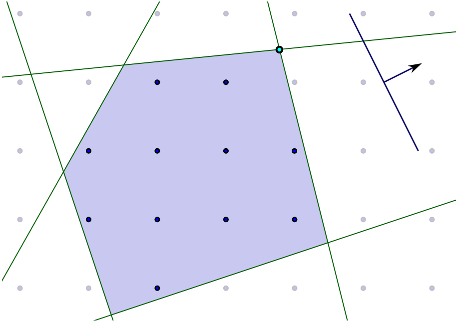
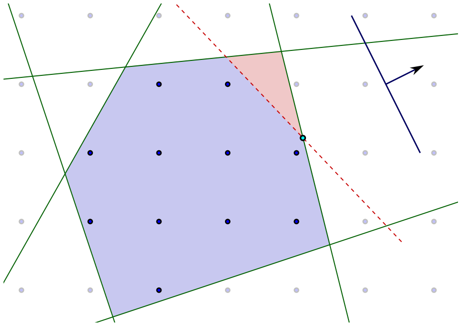


# Cuts

$$
\begin{array}{lll}
\text{(LP)} & \min & c^T x \\
 & \text{s.t.} & Ax \geq b \\
 & & \cancel{x_j \in \mathbb{Z}, \text{ for } j \in J} \\
\\
\text{(cut)} & & \alpha x \geq 1
\end{array}
$$



# Single-row cuts

From one (re)formulation of the problem

$$\text{(MIP)} \quad \begin{array}{ll} \min & \overline{c}^T x \\ \text{s.t.} & \overline{A} x \geq \overline{b} \\ & x_J \in \mathbb{Z} \end{array}$$

we extract **one** constraint $\overline{A}_i x \geq \overline{b}_i$.

- Knowing that $x_j \in \mathbb{Z}$, we construct a stronger inequality.
- In some cases, the cut can *separate* a given MIP-infeasible point $x^*$.

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# Two-row cuts

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$$(\text{MIP}) \quad \begin{aligned} \min \quad & \bar{c}^T x \\ \text{s.t.} \quad & \bar{A}x = \bar{b} \\ & x \geq 0 \\ & x_J \in \mathbb{Z} \end{aligned}$$

we extract **two** constraints

$$\begin{array}{rcl} x_1 \qquad + \sum_j \bar{a}_{1j} s_j & = & f_1 \\ + x_2 + \sum_j \bar{a}_{2j} s_j & = & f_2 \end{array}, \qquad \begin{array}{l} x_1, x_2 \in \mathbb{Z} \\ s_j \in \mathbb{R}_+ \end{array}$$

As a vector equation,

$$(P_I) \qquad x = f + \sum_j r^j s_j, \qquad \begin{array}{l} x \in \mathbb{Z}^2 \\ s \in \mathbb{R}^n_+ \end{array}$$

In case (MIP) describes the optimal simplex tableau, $(x^*_{LP}, s^*_{LP}) = (f, 0)$.

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# The two-row model

$$(\text{MIP}) \begin{cases} \overline{A}x = \overline{b} \\ x \geq 0 \\ x_J \in \mathbb{Z} \end{cases} \qquad (P_I) \begin{cases} x = f + \sum_j r^j s_j \\ x \in \mathbb{Z}^2 \\ s \in \mathbb{R}^n_+ \end{cases}$$

From (MIP) to $(P_I)$ we dropped:

- all equality constraints except two
- the integrality constraints on all variables except two
- the nonegativity constraints on these two variables

Therefore,

- $(P_I)$ is a relaxation of (MIP), i.e. (MIP) $\subseteq$ $(P_I)$.
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# Problem statement

# The two-row model

$$\begin{aligned} x &= f + \textstyle\sum_j r^j s_j \\ x &\in \mathbb{Z}^2 \\ s_j &\geq 0 \end{aligned}$$

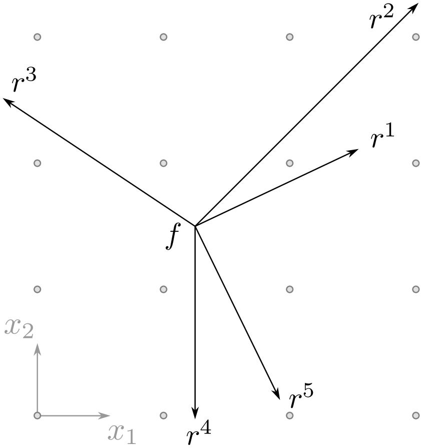
We want to separate

$$(x_1, x_2, s_1, \ldots) = (f_1, f_2, 0, \ldots).$$

Consider

$$\alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1,$$

with $v^i = f + \frac{1}{\alpha_i} r^i$, $\alpha_i \geq 0$.



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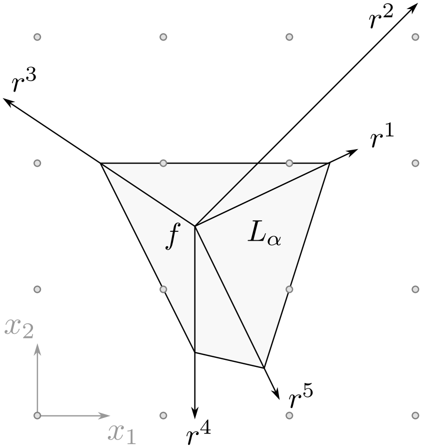

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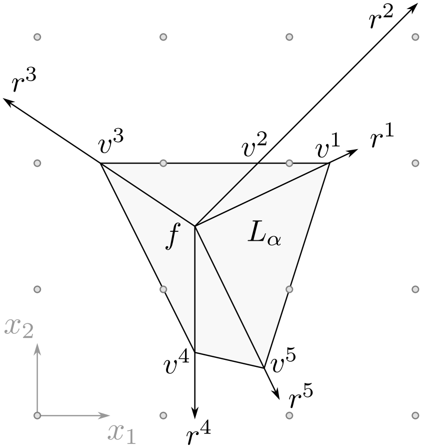

# The two-row separation problem

We have a model $P_I :=$

$$\left\{ (x,s) : \begin{array}{rcl} x & = & f + \sum_j r^j s_j \\ x & \in & \mathbb{Z}^2 \\ s_j & \geq & 0 \end{array} \right\}$$

and the general form of a cut

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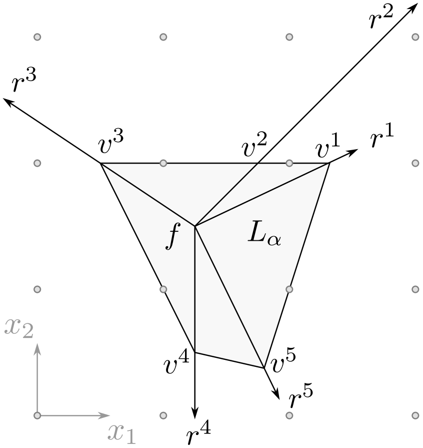
- How to compute a valid $\alpha$?
- How to choose among valid $\alpha$s?



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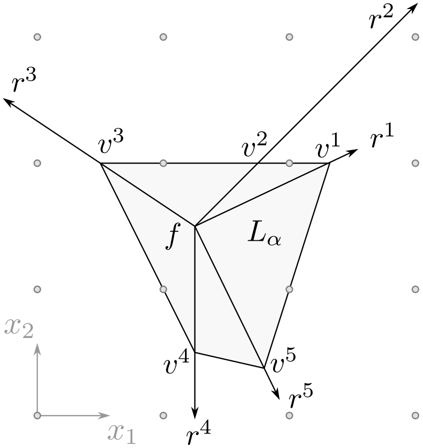

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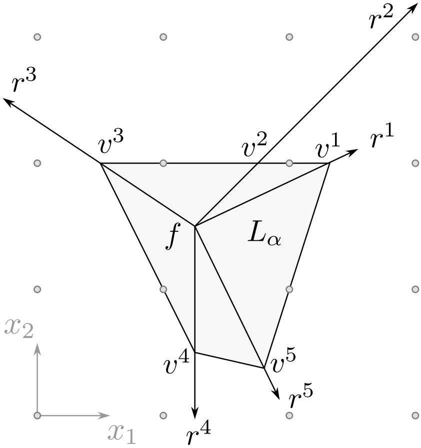

# Lattice-free sets – the LP intuition

Given $\overline{x} \in \mathbb{Z}^2$, we want, for all $s \in \mathbb{R}^n_+$ such that $\overline{x} = f + Rs$, that

$$\alpha_1 s_1 + \ldots + \alpha_n s_n \geq 1$$

i.e. we want

$$\begin{aligned} \min \quad & \alpha_1 s_1 + \ldots + \alpha_n s_n \quad \geq 1 \\ \text{s.t.} \quad & Rs = \overline{x} - f \\ & s \geq 0 \end{aligned}$$

therefore, for all $i, j, s_i^{\overline{x}}, s_j^{\overline{x}}$ such that $\overline{x} = f + s_i^{\overline{x}} r^i + s_j^{\overline{x}} r^j$, we must have

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# Lattice-free sets – the geometrical intuition

Given $\overline{x} \in \mathbb{Z}^2$,

for all $i, j$ : $\overline{x} \in f + \text{cone}(r^i, r^j)$,

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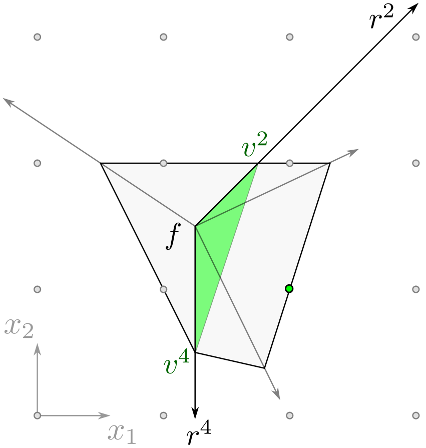

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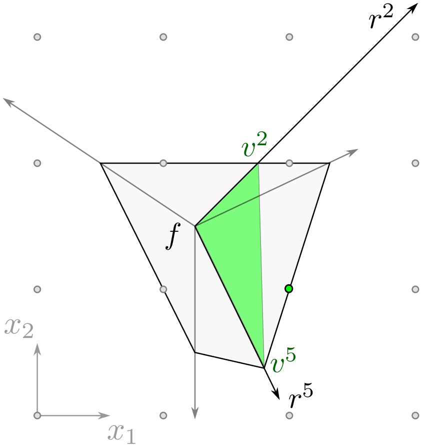

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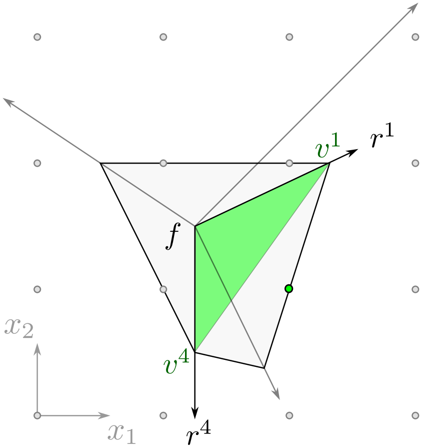

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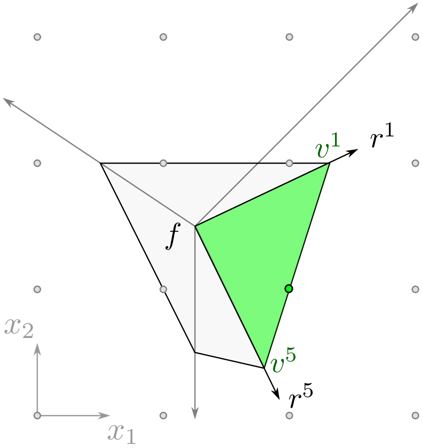

# Lattice-free sets – the intuition, for all $x$

For all $x \in \mathbb{Z}^2$,

for all $i, j$ : $x \in f + \text{cone}(r^i, r^j)$,

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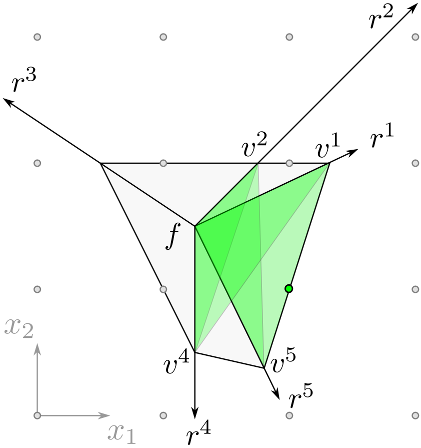

# Lattice-free sets – the intuition, for every cone

For all $i, j$,

for all $x \in \mathbb{Z}^2 \cap (f + \text{cone}(r^i, r^j))$,

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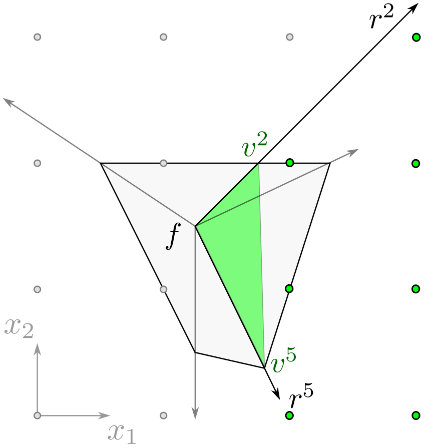


# Lattice-free sets – the set $\mathcal{X}_{ij}$

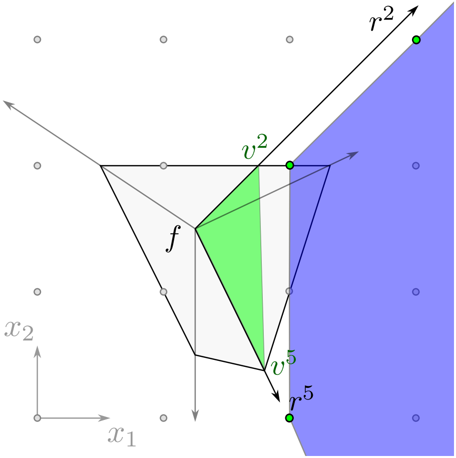
For all $i, j$,

for all $x \in \mathcal{X}_{ij}$,

$$s_i^x \alpha_i + s_j^x \alpha_j \geq 1,$$

with $s_i^x, s_j^x \,:\, x = f + s_i^x r^i + s_j^x r^j$.

- we can restrict $x \in \mathbb{Z}^2$ to $x \in \mathcal{X}_{ij}$ where $\mathcal{X}_{ij}$ is the set of the vertices of $\mathbb{Z}^2 \cap (f + \text{conv}(r^i, r^j))$.



# Polarity

Let $P \subseteq \mathbb{R}^N$ be a polyhedron and $Q \subseteq \mathbb{R}^N$ its polar.
There is a correspondance between

| | | |
|---|---|---|
| Extreme point $\overline{x} \in P$ | and | Facet of $Q$: $\overline{x}^T a \geq 1$ |
| Extreme ray $\overline{x} \in P$ | and | Facet of $Q$: $\overline{x}^T a \geq 0$ |
| Facet of $P$: $\overline{a}^T x \geq 1$ | and | Extreme point $\overline{a} \in Q$ |
| Facet of $P$: $\overline{a}^T x \geq 0$ | and | Extreme ray $\overline{a} \in Q$ |

# Polarity, applied

- We have a polyhedron $\mathrm{conv}(P_I) = \mathrm{conv}\left(\left\{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \mid \ x = f + \sum_j r^j s_j\right\}\right).$
- $\mathrm{conv}(P_I) \subseteq \mathbb{R}^{2+n}$ is of dimensionality $n$.
- We know the extreme points and rays of $\mathrm{conv}(P_I)$.
- We can build the polar $Q \subseteq \mathbb{R}^n$ of $\mathrm{conv}(P_I)$.
- We can optimize over $Q$ to find facets $\mathrm{conv}(P_I)$.

| | | |
|---|---|---|
| Extreme point $\overline{x} \in \mathrm{conv}(P_I)$ | $\longrightarrow$ | Facet of $Q$: $\overline{x}^T\alpha \geq 1$ |
| Extreme ray $\overline{x} \in \mathrm{conv}(P_I)$ | $\longrightarrow$ | Facet of $Q$: $\overline{x}^T\alpha \geq 0$ |
| | | $\downarrow \ \downarrow$ |
| Facet of $\mathrm{conv}(P_I)$: $\overline{\alpha}^T x \geq 1$ | $\longleftarrow$ | Extreme point $\overline{\alpha} \in Q$ |
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# Polarity, applied

- We have a polyhedron $\text{conv}(P_I) = \text{conv}\left(\left\{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \mid x = f + \sum_j r^j s_j\right\}\right)$.
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| | | |
|---|---|---|
| Extreme point $\bar{x} \in \text{conv}(P_I)$ | $\longrightarrow$ | Facet of $Q$: $\bar{x}^T\alpha \geq 1$ |
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$$
\begin{array}{lcl}
\hline
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# Finding facets of conv $P_I$

The polar of conv$(P_I)$ is

$$Q = \{ \, \alpha \in \mathbb{R}^n_+ \mid \forall i, j, \; \forall x \in \mathcal{X}_{ij}, \;\; s^x_i \alpha_i + s^x_j \alpha_j \geq 1 \, \}.$$

We find facets of conv$(P_I)$ by choosing an objective function $c^T\alpha$ and optimizing over $Q$:

$$\begin{array}{rl} \min & c^T\alpha \\ \text{s.t.} & s^x_i\alpha_i + s^x_j\alpha_j \geq 1, \quad \forall i, j, \; \forall x \in \mathcal{X}_{ij} \\ & \alpha \geq 0 \end{array}$$

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# New developments

# Complexity of writing the polar (1)

- For each cone, compute the integer hull.
- For each vertex of each integer hull, compute its representation in the corresponding cone and write one inequality of the polar.

1. The complexity is quadratic in the number of rays.
2. We have a polynomial (but possibly large) number of integer vertices in each cone.

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# The complexity of the polar – the intuition

$$P_I = \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ :$$
$$x = f + \textstyle\sum_{j=1}^n r^j s_j\}$$

$$Q = \{\ \alpha \in \mathbb{R}^n_+ \mid \forall i, j,\ \forall x \in \mathcal{X}_{ij},$$
$$s_i^x \alpha_i + s_j^x \alpha_j \geq 1\ \}$$

$$L_\alpha = \operatorname{conv}\left(\left\{f, v^1, \ldots, v^n\right\}\right)$$
$$\text{with } v^i = f + \frac{1}{\alpha_i} r^i$$

$$Q' = \{\ \alpha \in \mathbb{R}^n_+ \mid \forall i,\ \forall x \in \mathcal{X}_{i,i+1},$$
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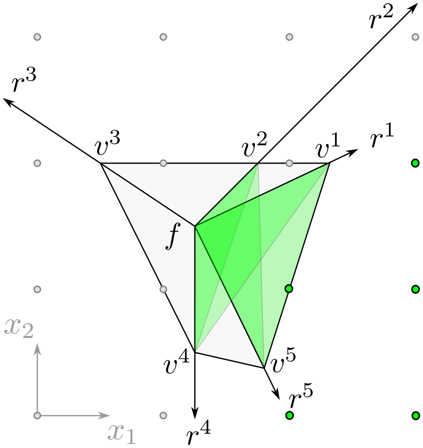

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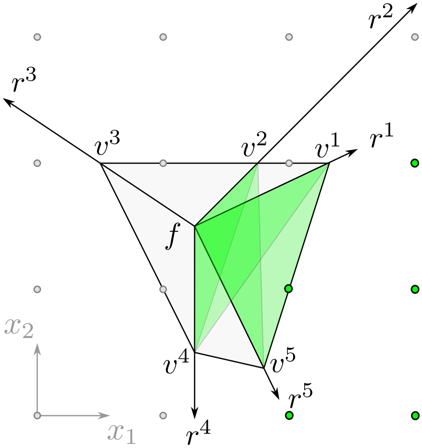

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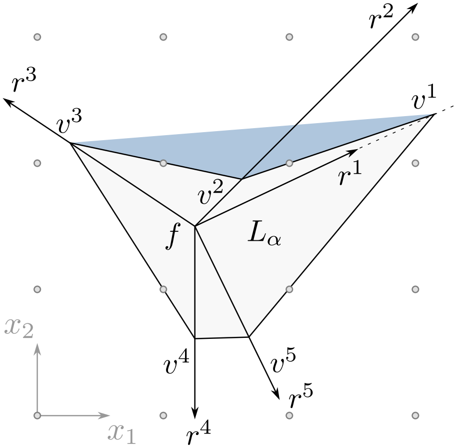

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$\overline{Q} = \{\ \alpha \in \mathbb{R}^n_+ \mid$

$\quad \forall i,\ \forall x \in \mathcal{X}_{i,i+1},$

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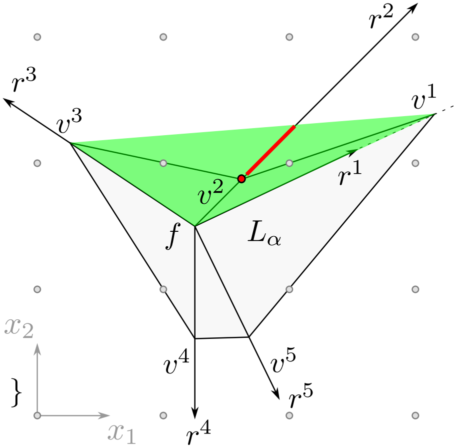


Note: $r^j = \lambda^j_i r^i + \lambda^j_k r^k$

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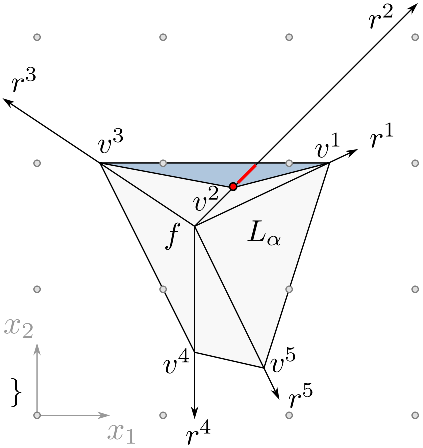


- What is $Q \setminus \overline{Q}$?

# The complexity of the polar – the theory

$$Q = \{\, \alpha \in \mathbb{R}^n_+ \;\mid \forall i, j,\ \forall x \in \mathcal{X}_{ij}, \qquad s^x_i \alpha_i + s^x_j \alpha_j \geq 1 \,\}$$

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**Theorem**

*$\overline{Q} \subseteq Q$, and all vertices of $Q$ are in $\overline{Q}$.*

**Corollary**

*If $c > 0$,* $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in Q \end{array}$ *and* $\begin{array}{ll} \min & c^T\alpha \\ \text{s.t.} & \alpha \in \overline{Q} \end{array}$ *share the same set of optimal solutions.*

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# Complexity of writing the polar (2)

- For each cone, compute the integer hull.
- For each integer point in each integer hull, compute its representation in the corresponding cone and write one inequality of of the polar.

1. The complexity is ~~quadratic~~ linear in the number of rays.
2. We have a polynomial (but possibly large) number of integer vertices in each cone.

# Can we avoid the integer hulls $\mathcal{X}_{ij}$?

$$\overline{Q} = \{ \alpha \in \mathbb{R}^n_+ \mid$$
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$$\overline{Q}(S) = \{ \alpha \in \mathbb{R}^n_+ \mid$$
$$\forall i,\ \forall x \in S \cap (f + \mathrm{cone}(r^i, r^{i+1})), \quad s_i^x \alpha_i + s_{i+1}^x \alpha_{i+1} \geq 1$$
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with $S \subset \mathbb{Z}^2$.

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with $S \subset \mathbb{Z}^2$.

# Separation algorithm

$S := S_0$

do {

$\alpha \;:=\; \text{argmin} \quad c^T\alpha$

$\qquad\qquad \text{s.t.} \quad \alpha \in \overline{Q}(S)$

if $\alpha \in \overline{Q}$

OK, valid cut, exit.

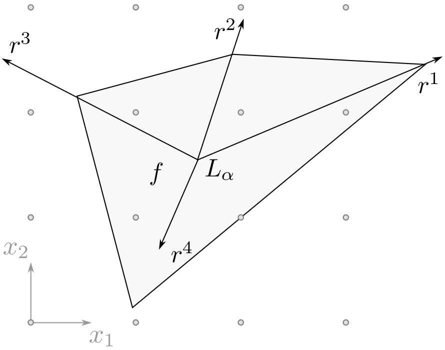
else

Find a constraint of $\overline{Q}$

violated by $\alpha$.

Add constraints to $S$.

}

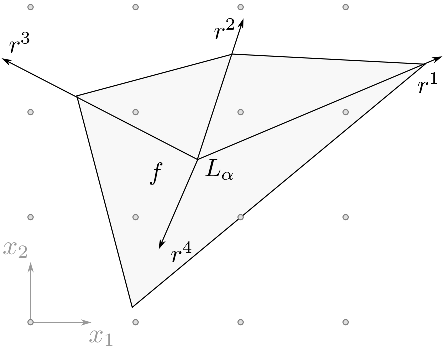


# Separation algorithm

$S := S_0$
do {
  $\alpha$ := argmin $c^T\alpha$
    s.t. $\alpha \in \overline{Q}(S)$

  if $L_\alpha$ is lattice-free
    OK, valid cut, exit.
  else
    Find $x \in \mathbb{Z}^2 \cap \text{interior}(L_\alpha)$.

    Add $x$ to $S$.
}



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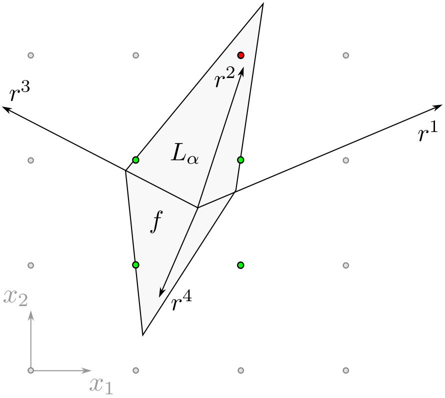

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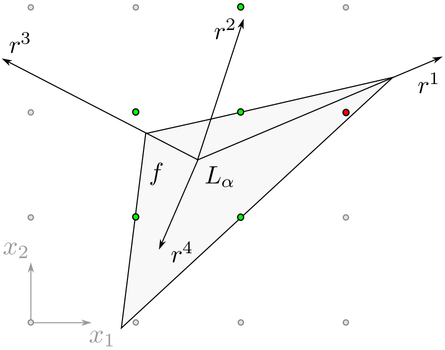

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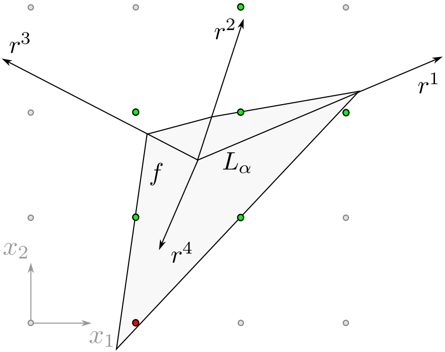

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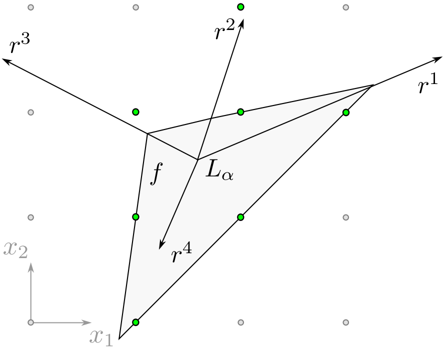
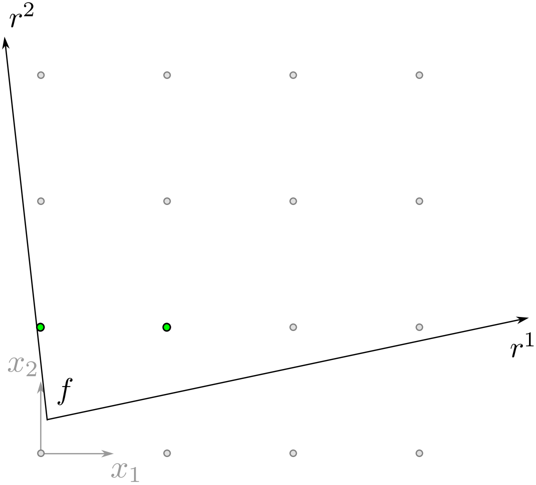


# Integer pair extension



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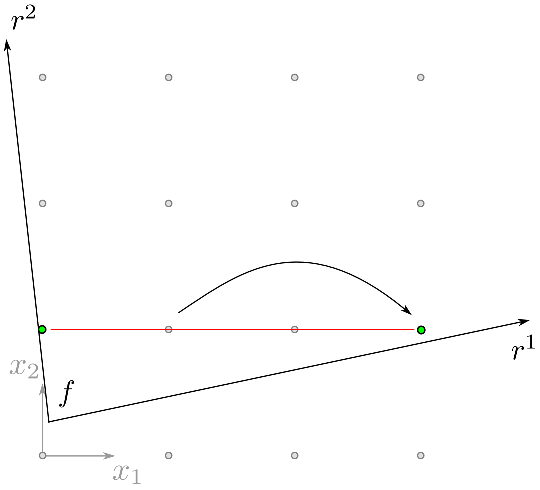

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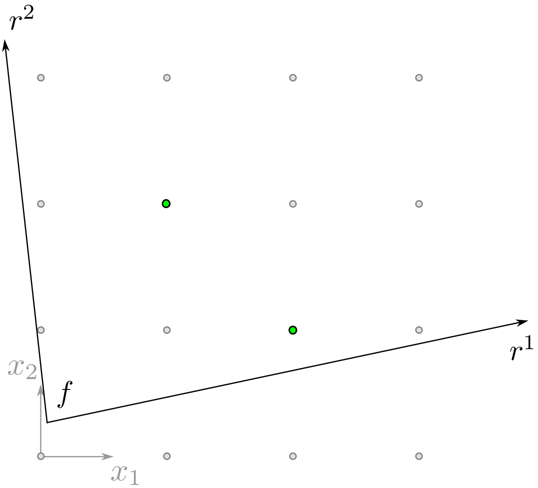

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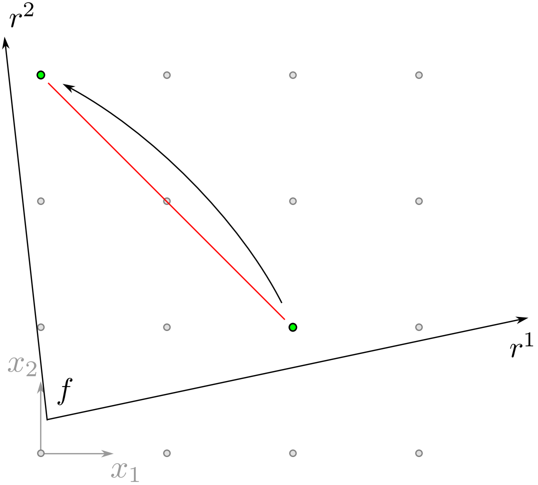

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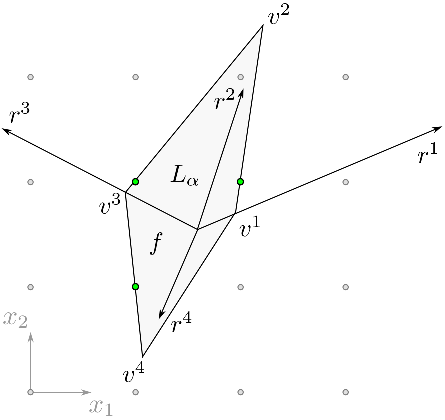

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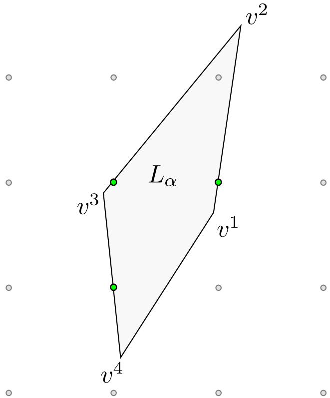

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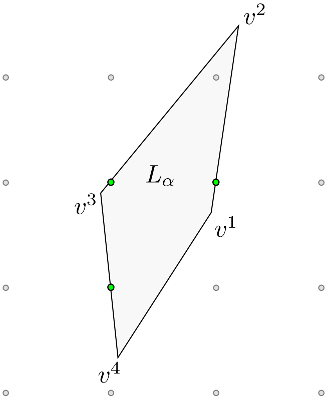

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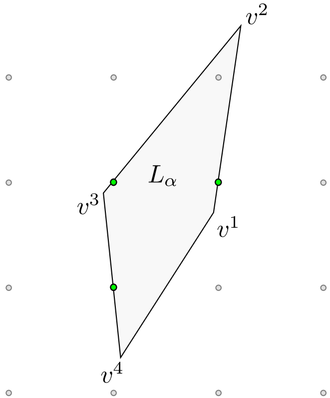

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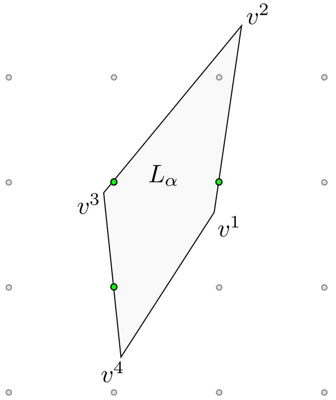

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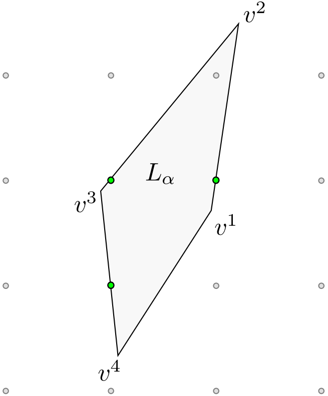

# The oracle: conv($T$)

Find an integer point in interior($L_\alpha$) or prove that $L_\alpha$ is lattice-free.

1. Consider the convex hull conv($T$) of the known tight integer points: $T := S \cap \text{boundary}(L_\alpha)$.
   - we triangularize conv($T$)
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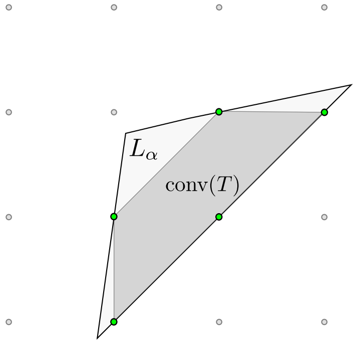
   - possible with modulo arithmetic



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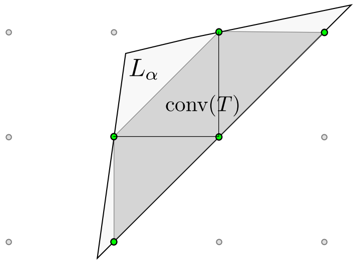

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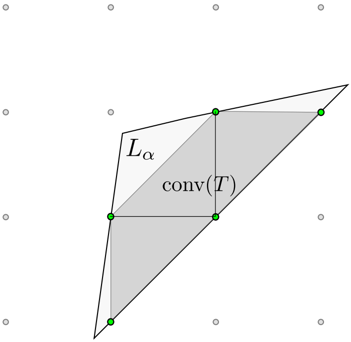

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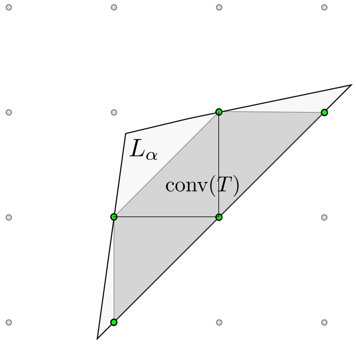
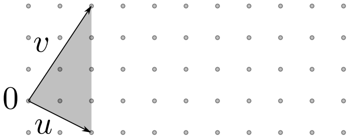


# The oracle: conv($T$), continued

Find an integer point in conv($T$) or prove that conv($T$) is lattice-free.

### Theorem

*Let $T$ be a triangle with vertices $(0, u, v)$ that has interior lattice points and such that $\gcd(u_1, u_2) = \gcd(v_1, v_2) = 1$. $T$ has an interior lattice point $w$ such that $w = \frac{\lambda}{\det([u|v])} u + \frac{1}{\det([u|v])} v$ with $\lambda \in \mathbb{Z}_+$.*




We therefore build $w$ by solving the diophantine system

$$\begin{cases} \lambda u_1 + v_1 = k_1 \det([u|v]) \\ \lambda u_2 + v_2 = k_2 \det([u|v]) \end{cases}, \quad \lambda, k_1, k_2 \in \mathbb{Z}$$

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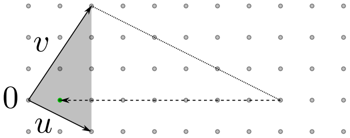


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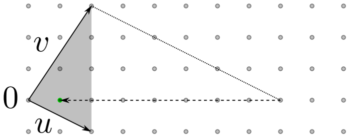


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# The oracle: Main Theorem

Find an integer point in interior($L_\alpha$) or prove that $L_\alpha$ is lattice-free.

2. Assume conv($T$) lattice-free.

Let us define:

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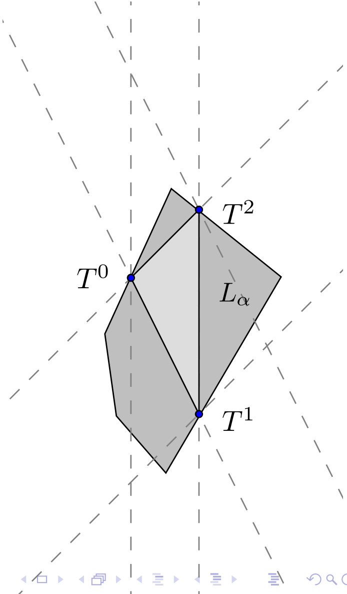
*If* interior($L_\alpha$) *contains an integer point, then at least one of these* $U^{ijk}$ *points must also belong to* interior($L_\alpha$).



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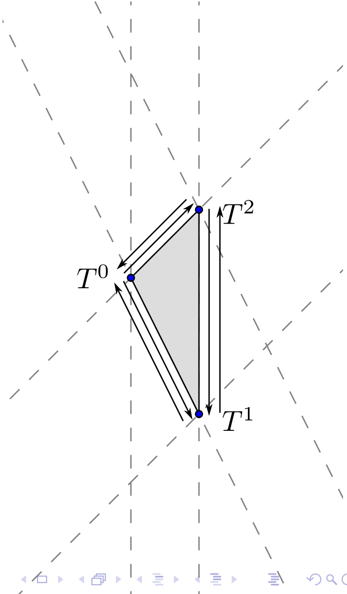


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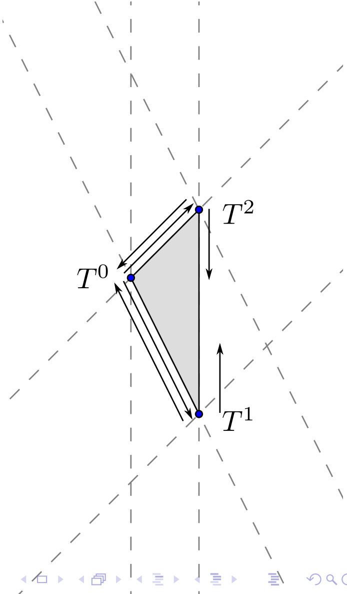

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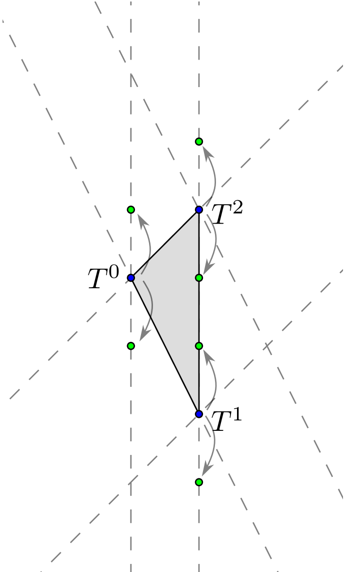

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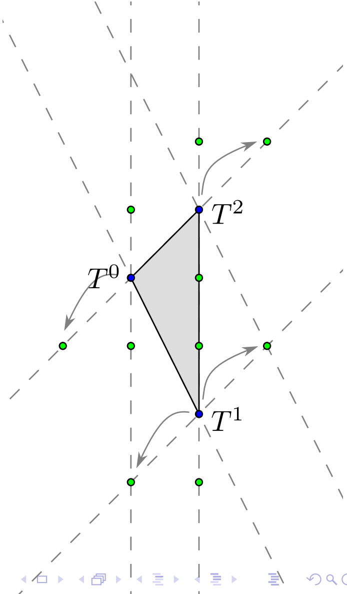


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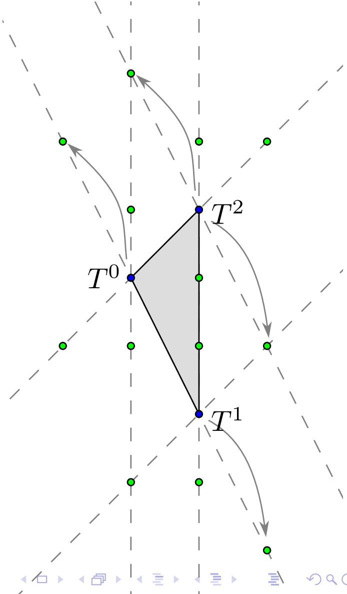


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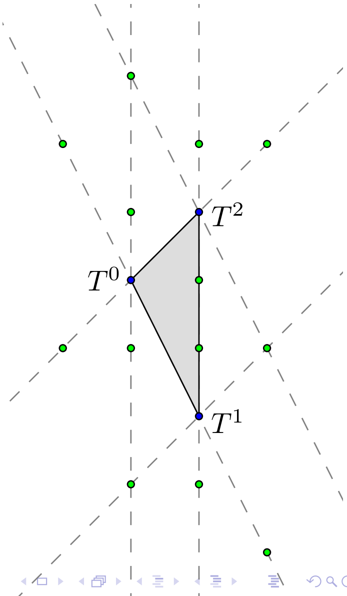


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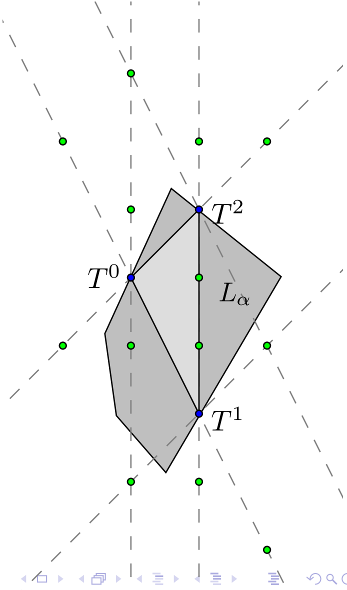


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# Results

# Preliminary Computational results

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| | two-row | | | CPLEX | |
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# Conclusions

We have

- a fast separation for two-row cuts
- a closed-form procedure for guaranteeing validity
- means for driving towards deep cuts (with $c^T\alpha$)

We need

- a better definition of a "deep" cut
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# Future directions

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- objective function
- basis choice: what is a good basis?
- basis choice: impact of a pivot on $Q$
- choice of the rows
- separation of $x \neq x^*_{LP}$
- cut handling
- (algorithm complexity)
- strengthening (lifting)
- more than two rows