The conic duality theorem

Consider the primal-dual pair

$$
\begin{aligned}
c^* &= \min\ c^T x & \qquad b^* &= \max\ p^T b \\
&\text{s.t. } (Ax - b) \in K & &\text{s.t. } p^T A = c^T \\
& & & p \in K_*
\end{aligned}
$$

PRIMAL DUAL

Theorem

(i) The value $p^T b$ for a feasible solution for the dual is less or equal to the value $c^T x$ for a feasible solution of the primal.
The duality gap $c^T x - p^T b \geq 0$ for every primal-dual pair (x, p).

(ii) If the primal is strictly feasible $(Ax - b \succ_K 0)$ and bounded below then the dual is feasible and $b^* = c^*$.

(iii) If the dual is strictly feasible and bounded above then the primal is feasible and $b^* = c^*$.

(iv) Assume one of the problems is bounded and strictly feasible then (x^*, p^*) is an optimal primal-dual pair

- if and only if $c^T x^* = (p^*)^T b$
- if and only if $p^T (Ax - b) = 0$ (complementarity slackness)

KKT for convex problems

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{0}. \end{aligned}$$

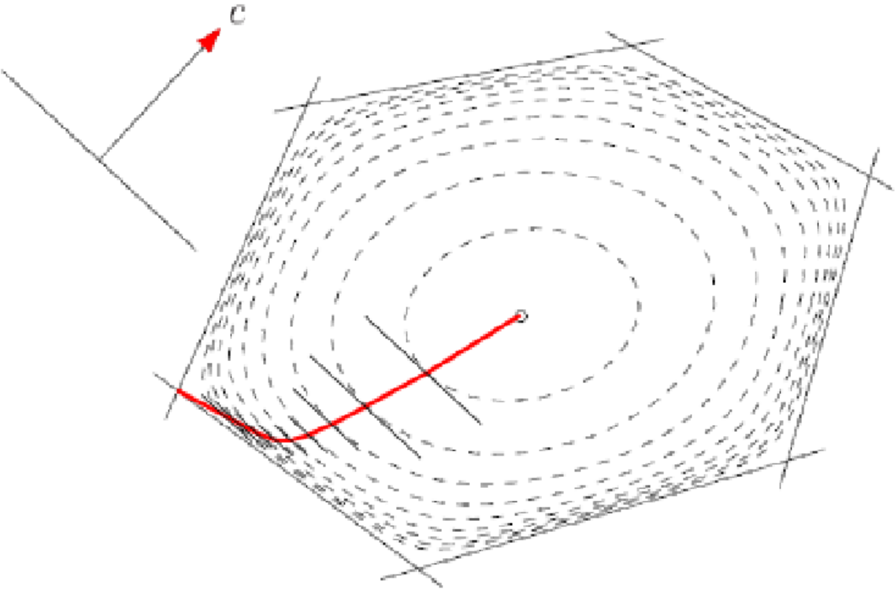
$\hat{\mathbf{x}}$ is a (global) optimal solution if, and only if, there exists a vector $\hat{\mathbf{y}} \in \mathbf{R}^m$ such that

(1) $\nabla f(\hat{\mathbf{x}}) + \sum_{i=1}^{m} \hat{y}_i \nabla g_i(\hat{\mathbf{x}}) = \mathbf{0}^\mathsf{T}$

(2) $\mathbf{g}(\hat{\mathbf{x}}) \leq \mathbf{0}$,

(3) $\hat{\mathbf{y}} \geq \mathbf{0}$,

(4) $\hat{\mathbf{y}}^\mathsf{T} \mathbf{g}(\hat{\mathbf{x}}) = 0$.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