University of Waterloo
Math 115 Midterm Examination

Spring 2015

Solutions

Exam details:

Course:	MATH 115 – Linear Algebra for Engineers
Instructor:	Laurent Poirrier
Date and time of exam:	June 18th, 2015 – 1:30 PM to 3:30 PM
Duration of exam:	120 minutes
Number of exam pages:	6 (includes cover page)
Exam type:	Closed book – no additional material allowed

Instructions:

Your answers must be stated and justified in a clear and logical form, and you must show all of your steps in order to receive full marks. You may use any result from class without proof, unless you are being asked to prove this result. Simplify your answers as much as possible.

1. [4 marks] Let $\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Compute $\operatorname{perp}_{\vec{w}} \vec{x}$.

   Solution:

   $$\operatorname{perp}_{\vec{w}} \vec{x} = \vec{x} - \operatorname{proj}_{\vec{w}} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{\vec{w} \cdot \vec{x}}{||\vec{w}||^2} \vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}{\left|\left| \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \right|\right|^2} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{2.3 + (-1).1 + 2.2}{2^2 + (-1)^2 + 2^2} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

   $$\operatorname{perp}_{\vec{w}} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

2. The following sets are *not* subspaces. For each set, find a counter-example that proves that it is not a subspace (use your counter-example to show that it does not satisfy the definition of a subspace).

   (a) [2 marks] $R = \left\{ x \in \mathbb{R}^2 \mid x_1 = 1 \right\}$.

   Solution:
   Take $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\vec{y} \in R$ because $y_1 = 1$. But let $t = 2$. Then, $\vec{z} = t\vec{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin R$ because $\vec{z}_1 \neq 1$.

(b) [2 marks] $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Solution:
Take $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The vector $\vec{y}$ is one of the three vectors in S. But let $t = 2$. $\vec{z} = t\vec{y} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \notin S$ because $\vec{z}$ is not one of the three vectors in S.

(c) [2 marks] $T = \left\{ x \in \mathbb{R}^2 \ \mid \ x_1 x_2 = 0 \right\}$.

Solution:
Let $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Both $\vec{y}, \vec{z} \in T$ because $y_1 y_2 = 0$ and $z_1 z_2 = 0$. But $\vec{w} = \vec{y} + \vec{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin T$ because $w_1 w_2 = 1.1 \neq 0$.

3. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$. Only one of the following statements is correct. For the one that is correct, just indicate that it is true (no justification necessary). For each of the other two, give a counter-example proving that it is false.

(a) [2 marks] If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, then $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$.

Solution:
False. Take any three vectors such that $\vec{v}_3$ is a scalar multiple of $\vec{v}_2$ but $\vec{v}_1$ is not. For example, take $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Then, by construction, the three vectors are linearly dependent. However, $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the whole $\mathbb{R}^2$ plane, while $\operatorname{span}\{\vec{v}_2, \vec{v}_3\}$ is only the line $\operatorname{span}\{\vec{v}_2\}$.

(b) [2 marks] If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, then $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \neq \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$.

Solution:
False. Take any three vectors that are scalar multiples of each other, for example $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. Then, they are clearly linearly dependent, by construction. However, $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_2, \vec{v}_3\}$, which in this case is even equal to $\operatorname{span}\{\vec{v}_1\} = \operatorname{span}\{\vec{v}_2\} = \operatorname{span}\{\vec{v}_3\}$.

(c) [2 marks] If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly dependent, then there exist $i, j \in \{1, 2, 3\}$ such that $\operatorname{span}\{\vec{v}_i, \vec{v}_j\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Solution:
True.

4. [2 marks] Write a transformation matrix $H \in \mathbb{R}^{2\times 2}$ such that $(H \cdot \vec{x})$ is $\vec{x}$ rotated (counter-clockwise) by an angle of $\frac{\pi}{6}$. Note: $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

Solution:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

5. (a) [1 mark] Write a matrix $A \in \mathbb{R}^{2\times 2}$ such that if $\vec{y} = A \cdot \vec{x}$, then $\vec{y}$ corresponds to scaling the components of $\vec{x}$ by factors a and b (reminder: it implies $y_1 = ax_1$ and $y_2 = bx_2$).

Solution:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

(b) [1 mark] Write a matrix $B \in \mathbb{R}^{2\times 2}$ such that if $\vec{z} = B \cdot \vec{y}$, then $\vec{z}$ corresponds to a shear of $\vec{y}$ with shear factor s (reminder: it implies $z_1 = y_1 + sy_2$ and $z_2 = y_2$).

Solution:

$$B = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

(c) [2 marks] Write a matrix $G \in \mathbb{R}^{2\times 2}$ such that if $\vec{z} = G\vec{x}$, then $\vec{z}$ corresponds to first scaling $\vec{x}$ by factors a and b, then shearing the result with a shear factor s. Note that this corresponds to applying the transformation represented by the matrix A, then the one represented by B. The variables s, a and b may appear in G.

Solution:

$$G = BA = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & bs \\ 0 & b \end{bmatrix}$$

(d) [4 marks] Find values of s, a and b such that the following conditions both hold:
$\begin{bmatrix} 13 \\ 2 \end{bmatrix} = G \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \end{bmatrix} = G \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:
We have

$$\begin{bmatrix} a & bs \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & bs \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

i.e.

$$\begin{cases} 1.a + 1.bs = 13 \\ 1.0 + 1.b = 2 \\ 1.a + 0.bs = 3 \\ 1.0 + 0.b = 0 \end{cases} \text{therefore} \begin{cases} a = 3 \\ b = 2 \\ s = 5 \end{cases}.$$

6. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, and $v_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

(a) [2 marks] Write a matrix whose columnspace is span$\{v_1, v_2, v_3, v_4\}$.

Solution:

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

(b) [2 marks] Write a matrix whose rowspace is span$\{v_1, v_2, v_3, v_4\}$.

Solution:

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

7. The matrix $A \in \mathbb{R}^{3\times 6}$ and its reduced row echelon form B are given as follows:

$$A = \begin{bmatrix} 1 & 2 & 1 & -2 & -1 & 1 \\ 2 & 4 & 1 & -3 & -3 & -2 \\ -1 & -2 & 0 & 1 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) [2 marks] Determine the rank of A.

Solution:
There are 3 leading ones in the RREF of A so rank$(A) = 3$.

(b) [2 marks] Determine a basis for the columnspace of A.

Solution:
We take the columns of A that correspond to columns its RREF with leading ones:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \right\}.$$

(c) [2 marks] Determine a basis for the rowspace of A.

Solution:
We take the rows of the RREF that are not all zeros (in this case, all of them):

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

(d) [5 marks] The set $S = \{x \in \mathbb{R}^6 | A\vec{x} = \vec{0}\}$ is the set of all solutions to the system $A\vec{x} = \vec{0}$. This set S is a subspace. Determine a basis for S.

Solution:
The augmented matrix of the system $A\vec{x} = \vec{0}$ is

$$\left[\begin{array}{cccccc|c} 1 & 2 & 1 & -2 & -1 & 1 & 0 \\ 2 & 4 & 1 & -3 & -3 & -2 & 0 \\ -1 & -2 & 0 & 1 & 3 & 5 & 0 \end{array}\right] \text{ and its RREF is } \left[\begin{array}{cccccc|c} 1 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right].$$

Therefore, the general solution is

$$\left\{\begin{array}{rcr} x_1 & = & -2x_2 + 1x_4 - 1x_5 \\ x_3 & = & 1x_4 - 2x_5 \\ x_5 & = & -2x_5 \end{array}\right. .$$

We have 3 parameters x_2, x_4, x_6 (i.e. free variables), which we rename $r = x_2$, $s = x_4$ and $t = x_6$. We rewrite the general solution

$$\left\{\begin{array}{rcl} x_1 & = & -2r + s - t \\ x_2 & = & r \\ x_3 & = & s - 2t \\ x_4 & = & s \\ x_5 & = & -2t \\ x_6 & = & t \end{array}\right. ,$$

which yields the vector equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

A basis of S is given by the three linearly independent vectors

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

8. [5 marks] A matrix G is called skew-symmetric if $G = -G^T$. Prove that for any square matrix $A \in \mathbb{R}^{n\times n}$, the matrix $(A - A^T)$ is skew-symmetric.

Solution:
We need to show that $(A - A^T) = -(A - A^T)^T$. We start from the right-hand side:

$$-(A - A^T)^T = -(A^T - {A^T}^T) = -(A^T - A) = -A^T + A = A - A^T$$

and obtain the left-hand side $A - A^T$.

9. [8 marks] Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be such that $||\vec{u}|| = ||\vec{v}||$ and $\vec{u} \cdot \vec{v} = 0$.
Prove that if $\vec{x} \cdot \vec{w} \neq 0$ for all $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$, then $\vec{x} \notin \text{span}\{\vec{u}, \vec{v}\}$.

Solution:
We prove that if $\vec{x} \in \text{span}\{\vec{u}, \vec{v}\}$, then $\vec{x} \cdot \vec{w}$ is not different from 0 for all $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$.

If $\vec{x} \in \text{span}\{\vec{u}, \vec{v}\}$, then there exist $s, t \in \mathbb{R}$ such that $\vec{x} = s\vec{u} + t\vec{v}$. We will now construct $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$ such that $\vec{x} \cdot \vec{w} = 0$. Let $\vec{w} = -t\vec{u} + s\vec{v}$. Clearly, $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$. Then,

$$\begin{aligned}
\vec{x} \cdot \vec{w} &= (s\vec{u} + t\vec{v}) \cdot (-t\vec{u} + s\vec{v}) \\
&= -st\vec{u}^2 + s^2\vec{u}\vec{v} - t^2\vec{v}\vec{u} + ts\vec{v}^2 \\
&= 0
\end{aligned}$$

Note: Alternatively, one could also notice that $\vec{0} \in \text{span}\{\vec{u}, \vec{v}\}$. Therefore, setting $\vec{w} = \vec{0}$, $\vec{x} \cdot \vec{w}$ is always zero. The hypothesis of the statement is never fullfilled, so the statement is trivially true.