Math 115 Spring 2015: Assignment 8

Solutions

1. Let

$$A = \begin{bmatrix} 1 & 6 & -3 \\ 0 & 4 & 0 \\ -3 & 6 & 1 \end{bmatrix}.$$

(a) [5 marks] Find the eigenvalues of A. For each eigenvalue, give the algebraic multiplicity, determine a basis of the corresponding eigenspace, and give the geometric multiplicity.

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 6 & -3 \\ 0 & 4-\lambda & 0 \\ -3 & 6 & 1-\lambda \end{bmatrix} \\ &= (4-\lambda).(-1)^{2+2}.\det \begin{bmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{bmatrix} \\ &= (4-\lambda).\left((1-\lambda).(1-\lambda) - (-3).(-3)\right) \\ &= (4-\lambda).\left(1+\lambda^2 - 2\lambda - 9\right) \\ &= (4-\lambda).\left(\lambda^2 - 2\lambda - 8\right) \end{aligned}$$

We need to find the roots of $\lambda^2 - 2\lambda - 8$. The discriminant is given by

$$\delta = 2^2 - 4.1.(-8) = 4 + 32 = 36$$

and the roots are

$$\lambda = \frac{-(-2) \pm \sqrt{36}}{2.1} = \frac{2 \pm 6}{2}$$

so 4 and -2. Therefore,

$$\det(A - \lambda I) = (4-\lambda)(4-\lambda)(2+\lambda).$$

First eigenvalue: $\lambda_1 = 4$, algebraic multiplicity 2. The system

$$\begin{bmatrix} -3 & 6 & -3 \\ 0 & 0 & 0 \\ -3 & 6 & -3 \end{bmatrix} \vec{v} = \vec{0}$$

has the augmented matrix

$$\left[\begin{array}{ccc|c} -3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so

$$\vec{v}_1 = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

And a basis of the eigenspace associated to eigenvalue 4 is given by $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. The geometric multiplicity of this eigenvalue is 2.

Second eigenvalue: $\lambda_2 = -2$, algebraic multiplicity 2. The system

$$\begin{bmatrix} 3 & 6 & -3 \\ 0 & 6 & 0 \\ -3 & 6 & 3 \end{bmatrix}\vec{v} = \vec{0}$$

has the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 6 & -3 & 0 \\ 0 & 6 & 0 & 0 \\ -3 & 6 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so

$$\vec{v}_2 = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

And a basis of the eigenspace associated to eigenvalue -2 is given by $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$. The geometric multiplicity of this eigenvalue is 1.

(b) [1 mark] Diagonalize A by finding an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$

Solution: We collect the three basis vectors for the eigenspaces of all eigenvalues of A found above as columns of P. We construct D as a matrix whose diagonal elements are the corresponding eigenvalues.

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \text{diag}(4, 4, -2) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

2. [5 marks] Determine a matrix $B \in \mathbb{R}^{2\times 2}$ for which $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are eigenvectors, with corresponding eigenvalue -4 and 2, respectively.

Solution:

$$\begin{aligned}
P^{-1}BP &= \operatorname{diag}(\lambda_1, \lambda_2) \\
B &= P\operatorname{diag}(\lambda_1, \lambda_2)P^{-1} \\
B &= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \operatorname{diag}(-4, 2) \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}^{-1} \\
B &= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}^{-1}
\end{aligned}$$

We compute the inverse of $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$:

$$\left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

so

$$\begin{aligned}
B &= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -12 & -16 \\ 4 & 6 \end{bmatrix} \\
&= \begin{bmatrix} -52 & -72 \\ 36 & 50 \end{bmatrix}
\end{aligned}$$

3. [4 marks] Let $C \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix that has n distinct eigenvalues. Prove that $\det(C)$ is the product of the n eigenvalues of C.
Solution: Since C is diagonalizable, there exists an invertible matrix $R \in \mathbb{R}^{n\times n}$ such that

$$R^{-1}CR = \operatorname{diag}(\nu_1, \ldots, \nu_n)$$

where $\nu_1, \ldots, \nu_n$ are the eigenvalues of C. Then, $C = R\operatorname{diag}(\nu_1, \ldots, \nu_n)R^{-1}$ and

$$\begin{aligned}
\det(C) &= \det(R)\det(\operatorname{diag}(\nu_1, \ldots, \nu_n))\det(R^{-1}) \\
&= \det(R)\det(\operatorname{diag}(\nu_1, \ldots, \nu_n))\frac{1}{\det(R)} \\
&= \det(\operatorname{diag}(\nu_1, \ldots, \nu_n)) \\
&= \nu_1 \cdot \cdots \cdot \nu_n
\end{aligned}$$

4. [5 marks] Let $G \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix such that $Q^{-1}GQ = \operatorname{diag}(\mu_1, \ldots, \mu_n)$. Find an expression of G^4 that does not involve G (it may involve, Q and $\mu_1, \ldots, \mu_n$). Give the eigenvalues of G^4.
Solution:

$$G = Q\operatorname{diag}(\mu_1, \ldots, \mu_n)Q^{-1}$$

so

$$\begin{aligned}
G^k &= (Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1})(Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1})(Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1})(Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1}) \\
&= Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1}Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1}Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1}Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1} \\
&= Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)\,\mathrm{diag}(\mu_1,\ldots,\mu_n)\,\mathrm{diag}(\mu_1,\ldots,\mu_n)\,\mathrm{diag}(\mu_1,\ldots,\mu_n)Q^{-1} \\
&= Q\,\mathrm{diag}(\mu_1,\ldots,\mu_n)^4 Q^{-1} \\
&= Q\,\mathrm{diag}(\mu_1^4,\ldots,\mu_n^4)Q^{-1}
\end{aligned}$$

The eigenvalues of G^4 are $\mu_1^4,\ldots,\mu_n^4$.