Math 115 Spring 2015: Assignment 6

Solutions

1. [5 marks] Let \( G = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 5 & 3 & 0 \end{bmatrix} \). Compute \( G^{-1} \).

Solution: We compute the RREF of \( \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 5 & 3 & 0 \end{bmatrix} \) and obtain \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). so

\[ G^{-1} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 0 & 2 \\ -2 & 1 & 0 \end{bmatrix}. \]

2. [4 marks] Let

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 \\ 2 & 3 & 5 & 1 & 2 \\ 4 & 1 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 \end{bmatrix}. \]

Compute \( \det(A) \). Hint: choose carefully the columns or rows to expand in order to reduce your work.

Solution: We notice that the fourth column of \( A \) has only one element different from zero. It thus seems like an good choice for expansion: \( \det(A) = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44} + a_{54}C_{54} = 1.C_{34}. \) The cofactor \( C_{34} \) evaluates to \((-1)^{3+4} \det(A(3,4)) \) where \( A(3,4) \) is the matrix obtained from \( A \) by removing the third row and the fourth column. Thus,

\[ \det(A) = C_{34} = -1. \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 \\ 4 & 1 & 1 & 0 \\ 5 & 0 & 1 & 0 \end{bmatrix}. \]

The first row of \( A(3,4) \) only has one nonzero, so we compute the determinant of \( A(3,4) \) using that row. We denote by \( C_{ij}' \) the cofactors of \( A(3,4) \), and get

\[ \det(A) = -1. (1.C'_{11} + 0.C'_{12} + 0.C'_{13} + 0.C'_{14}) = -1.C'_{11} = -1.(-1)^{1+1}. \det \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \]

Again, we only have one cofactor to compute. This time, we use the third column for determinant expansion.

\[ \det(A) = -1.1.(-1)^{1+3}. \det \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \]
Since the determinant of \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\] is \((ad - bc)\), we get

\[
\det(A) = -1.(1.1 - 0.1) = -1
\]

3. [5 marks] Let

\[
B = \begin{bmatrix}
1 & 4 & 5 & 3 \\
0 & 2 & 3 & 3 \\
0 & 0 & 3 & 7 \\
0 & 0 & 0 & 4
\end{bmatrix}.
\]

Notice that \(B\) is upper-triangular (i.e. all elements below the diagonal are zero). Use the cofactor expansion of determinants to show (on this example) that the \(\det(B)\) is simply the product of the diagonal elements of \(B\).

**Solution:** If we always use the first column for expansion, we get

\[
\det(B) = b_{11}C_{11} + b_{21}C_{21} + b_{31}C_{31} + b_{41}C_{41} = b_{11}C_{11} = b_{11}(-1)^{1+1}\det(B(1,1)).
\]

Let \(B' = B(1,1)\). Notice that \(B'\) is upper-triangular as well, so

\[
\det(B') = \det\left(\begin{bmatrix}
2 & 3 & 3 \\
0 & 3 & 7 \\
0 & 0 & 4
\end{bmatrix}\right) = b'_{11}C'_{11} + b'_{21}C'_{21} + b'_{31}C'_{31} = b'_{11}C'_{11} = b_{22}(-1)^{1+1}\det(B'(1,1)).
\]

Finally, let \(B'' = B'(1,1)\). Again, \(B''\) is upper-triangular, so

\[
\det(B'') = \det\left(\begin{bmatrix}
3 & 7 \\
0 & 4
\end{bmatrix}\right) = b''_{11}b''_{22} - 0.b''_{12} = b_{33}b_{44}.
\]

Collecting all our result, we see that

\[
\det(B) = b_{11}b_{22}b_{33}b_{44} = 1.2.3.4 = 24.
\]

4. For each of the following statements, either prove that it is true, or find a counterexample to prove that it is false.

(a) [3 marks] If \(A\) and \(B\) are \(n \times n\) invertible matrices, then \(A + B\) is also invertible.

**Solution:** Take, for example

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}.
\]

The determinants of those matrices are both not zero (\(\det(A) = 1, \det(B) = -1\)), so they are invertible.
But

\[
A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

is not invertible because \(\text{det}(A + B) = 0\).

(b) [3 marks] If \(A\) and \(B\) are \(n \times n\) invertible matrices and \((AB)^2 = A^2B^2\), then \(AB = BA\).

**Solution:** First, we note that \((AB)^2 = (AB)(AB) = ABAB\), and \(A^2B^2 = AABB\), so we know that

\[
ABAB = AABB.
\]

Since \(A\) and \(B\) are invertible, they have inverses \(A^{-1}\) and \(B^{-1}\). If we premultiply both side of the above equation by \(A^{-1}\), we obtain

\[
A^{-1}ABAB = A^{-1}AABB,
\]

i.e.

\[
BAB = ABB.
\]

Now, we postmultiply both sides by \(B^{-1}\) and get

\[
BABB^{-1} = ABBB^{-1},
\]

i.e.

\[
BA = AB.
\]