1. (a) [2 marks] Find a matrix $A \in \mathbb{R}^{2\times2}$ such that $\vec{y} = A\vec{x}$, where $\vec{y}$ is $\vec{x}$ rotated by an angle of $\frac{2}{3}\pi$ (counterclockwise around the origin), for any $\vec{x} \in \mathbb{R}^2$.

(b) [2 marks] Find a matrix $B \in \mathbb{R}^{2\times2}$ such that $\vec{z} = B\vec{y}$, where $z_1$ is $y_1$ scaled by a factor 3 and $z_2$ is $y_2$ scaled by a factor 2, for any $\vec{y} \in \mathbb{R}^2$.

(c) [2 marks] Find a matrix $C \in \mathbb{R}^{2\times2}$ such that $\vec{w} = C\vec{z}$, where $\vec{w}$ is $\vec{z}$ rotated by an angle of $-\frac{2}{3}\pi$ (counterclockwise around the origin, i.e. $\frac{2}{3}\pi$ clockwise), for any $\vec{z} \in \mathbb{R}^2$.

(d) [2 marks] Find a matrix $G \in \mathbb{R}^{2\times2}$ such that $\vec{w} = G\vec{x}$, where $\vec{w}$ is $\vec{x}$ that is rotated by $\frac{2}{3}\pi$ and then scaled with factors 3 and 2, and then rotated by $-\frac{2}{3}\pi$. Note that this amounts to performing on $\vec{x}$ all three transformations found in points (a), (b) and (c), successively.

Solution:

$$A = \begin{bmatrix} \cos \left(\frac{2}{3}\pi\right) & -\sin \left(\frac{2}{3}\pi\right) \\ \sin \left(\frac{2}{3}\pi\right) & \cos \left(\frac{2}{3}\pi\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} \cos \left(-\frac{2}{3}\pi\right) & -\sin \left(-\frac{2}{3}\pi\right) \\ \sin \left(-\frac{2}{3}\pi\right) & \cos \left(-\frac{2}{3}\pi\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$G = C \cdot B \cdot A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{3\sqrt{3}}{4} \\ -\frac{3\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix}$$

2. [3 marks] Find a matrix $A \in \mathbb{R}^{2\times2}$ that has no zero elements, such that $\vec{x} = A^k\vec{x}$. Note that $A^k = A \cdot A \cdots A$ where there are $k$ factors $A$. For example $A^3 = A \cdot A \cdot A$. **Hint:** Think about a geometric transformation that, when applied $k$ times on the vector $\vec{x}$, gives back the vector $\vec{x}$ itself. $k$ may appear in some form in the matrix.

Solution: Applying $k$ rotations of $\frac{2\pi}{k}$ radians on a vector yields the same vector. So $A$ is given by

$$A = \begin{bmatrix} \cos \left(\frac{2\pi}{k}\right) & -\sin \left(\frac{2\pi}{k}\right) \\ \sin \left(\frac{2\pi}{k}\right) & \cos \left(\frac{2\pi}{k}\right) \end{bmatrix}.$$  

This matrix has no zero elements for $k \geq 3$.

**Bonus points:** For $k = 2$,

$$A = \begin{bmatrix} \cos (\pi) & -\sin (\pi) \\ \sin (\pi) & \cos (\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
which has zero elements. But in this case, we only need $A^2 = AA = I$. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

In other words, we have the four equations

$$\begin{cases} a^2 + bc = 1 \\ ab + bd = 0 \\ ca + dc = 0 \\ cb + d^2 = 1 \end{cases}$$

The last equation gives $bc = 1 - d^2$. Using that in the first equation gives $a^2 + 1 - d^2 = 1$ so $a^2 = d^2$, i.e. $|a| = |d|$. The second equation is $b(a + d) = 0$ and the third is $c(a + d) = 0$. We want $a, b, c, d \neq 0$ so we need $a = -d$ and $bc = 1 - a^2$. For example, we can choose $a = 2$, $b = -1$, $c = 3$ and $d = -2$:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$$  

3. The matrix $A \in \mathbb{R}^{4 \times 7}$ and its reduced row echelon form $B$ are given as follows:

$$A = \begin{bmatrix} 1 & -3 & 0 & 1 & 4 & 1 & -5 \\ 0 & 0 & -1 & 5 & -9 & -1 & 4 \\ 3 & -9 & -1 & 8 & 3 & 1 & -5 \\ -1 & 3 & 1 & -6 & 5 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 1 & -5 & 9 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [2 marks] Determine a basis for the columnspace of $A$.

**Solution:** One basis of $\text{col}(A)$ is given by the columns of $A$ that correspond to columns of $B$ having a leading one. Therefore,

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is a basis of $\text{col}(A)$.

(b) [2 marks] Determine a basis for the rowspace of $A$. 

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Solution: One basis of row($A$) is given by the nonzero rows of $B$. Therefore,

\[
\begin{bmatrix}
1 & 0 & 0 \\ 
-3 & 0 & 0 \\ 
1 & 0 & 0 \\ 
4 & 0 & 0 \\ 
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\ 
0 \\ 
-5 \\ 
9 \\ 
2
\end{bmatrix}
\begin{bmatrix}
0 \\ 
0 \\ 
0 \\ 
0 \\ 
-6
\end{bmatrix}
\]

is a basis of row($A$).

(c) [5 marks] The set $S = \{x \in \mathbb{R}^7 | A\vec{x} = \vec{0}\}$ is the set of all solutions to the system $A\vec{x} = \vec{0}$. This set $S$ is a subspace. Determine a basis for $S$. \textbf{Hint:} Find the general solution to $A\vec{x} = \vec{0}$, and write it as a vector equation.

\textbf{Solution:} We write the RREF of the system $A\vec{x} = \vec{0}$. Because the right-hand side is zero in the system, it will be zero in its RREF too. The other coefficients in the RREF will simply be the coefficients in the RREF of $A$:

\[
\begin{bmatrix}
1 & -3 & 0 & 1 & 4 & 0 & 1 & 0 \\ 
0 & 1 & -5 & 9 & 0 & 2 & 0 \\ 
0 & 0 & 0 & 0 & 0 & 1 & -6 & 0 \\ 
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The general solution to this system is

\[
\begin{align*}
 x_1 &= 3x_2 - 1x_4 - 4x_5 - 1x_7 \\
x_3 &= 5x_4 - 9x_5 - 2x_7 \\
x_6 &= 6x_7
\end{align*}
\]

where $x_2, x_4, x_5, x_7$ are free variables. As a vector equation, we can write this subspace as

\[
\begin{bmatrix}
x_1 \\ 
x_2 \\ 
x_3 \\ 
x_4 \\ 
x_5 \\ 
x_6 \\ 
x_7
\end{bmatrix}
= 
\begin{bmatrix}
3 & -1 & -4 & -1 \\ 
1 & 0 & 0 & 0 \\ 
0 & 5 & -9 & -2 \\ 
0 & 1 & 0 & 0 \\ 
0 & 0 & 0 & 6 \\ 
0 & 0 & 0 & 0 \\ 
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
q \\ 
r \\ 
s \\ 
t
\end{bmatrix}
\]

for all $q, r, s, t \in \mathbb{R}$.

It is easy to verify that the only solution for $\vec{x} = 0$ is $q = r = s = t = 0$, so the four vectors above are
linearly independent. Therefore,

\[
\begin{bmatrix}
3 & -1 & -4 & -1 \\
1 & 0 & 0 & 0 \\
0 & 5 & -9 & -2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

is a basis of \( S \).