Math 115 Spring 2015: Assignment 4

Solutions

1. [5 marks] Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$. Compute AA^T and A^TA.

   Solution:

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$$

2. [5 marks] Given scalars $t_1, t_2 \in \mathbb{R}$, find a matrix $A \in \mathbb{R}^{3\times3}$ such that $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} (x_1 + t_1) \\ (x_2 + t_2) \\ 1 \end{bmatrix}$, for all $x_1, x_2 \in \mathbb{R}$. **Note:** The matrix must be the same for all values of x_1 and x_2, but the scalars t_1 and t_2 are constant, so they may appear in A. **Hint:** Expand the product $A\vec{x}$ in function of the scalar elements of A and $\vec{x}$.

   Solution: Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

   Then,

$$A \cdot \vec{x} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 + c \\ dx_1 + ex_2 + f \\ gx_1 + hx_2 + i \end{bmatrix}, \text{ which we want equal to } \begin{bmatrix} x_1 + t_1 \\ x_2 + t_2 \\ 1 \end{bmatrix}.$$

   In the first row, $ax_1 + bx_2 + c = x_1 + t_1$ can be obtained easily by setting $a = 1$, $b = 0$ and $c = t_1$. Similarly we obtain $d = 0$, $e = 1$ and $f = t_2$. And finally $g = 0$, $h = 0$ and $i = 1$. We verify that

$$\text{if } A = \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} (x_1 + t_1) \\ (x_2 + t_2) \\ 1 \end{bmatrix}$$

3. [5 marks] Given a vector $\vec{v} \in \mathbb{R}^2$, find a matrix $A \in \mathbb{R}^{2\times2}$ such that $A \cdot \vec{x} = \text{proj}_{\vec{v}}\,\vec{x}$, for all $x \in \mathbb{R}^2$. **Note:** The matrix must be the same for all $\vec{x}$, but $\vec{v}$ is constant, so v_1 and v_2 may appear in A. **Hint:** Expand $\text{proj}_{\vec{v}}\,\vec{x}$ and the product $A\vec{x}$.

   Solution: We proceed as in the previous question, letting

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Here however, we want to obtain the vector $\text{proj}_{\vec{v}}\,\vec{x}$, i.e.

$$\frac{\vec{v}\cdot\vec{x}}{||\vec{v}||^2}\vec{v} = \frac{v_1x_1+v_2x_2}{\left(\sqrt{v_1^2+v_2^2}\right)^2}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{v_1x_1+v_2x_2}{v_1^2+v_2^2}v_1 \\ \frac{v_1x_1+v_2x_2}{v_1^2+v_2^2}v_2 \end{bmatrix} = \begin{bmatrix} \frac{v_1^2x_1+v_1v_2x_2}{v_1^2+v_2^2} \\ \frac{v_1v_2x_1+v_2^2x_2}{v_1^2+v_2^2} \end{bmatrix} = \begin{bmatrix} \frac{v_1^2}{v_1^2+v_2^2}x_1 + \frac{v_1v_2}{v_1^2+v_2^2}x_2 \\ \frac{v_1v_2}{v_1^2+v_2^2}x_1 + \frac{v_2^2}{v_1^2+v_2^2}x_2 \end{bmatrix}.$$

Again,

$$A\cdot\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1+bx_2 \\ cx_1+dx_2 \end{bmatrix}, \text{ which we want equal to } \begin{bmatrix} \frac{v_1^2}{v_1^2+v_2^2}x_1 + \frac{v_1v_2}{v_1^2+v_2^2}x_2 \\ \frac{v_1v_2}{v_1^2+v_2^2}x_1 + \frac{v_2^2}{v_1^2+v_2^2}x_2 \end{bmatrix}.$$

The matrix we are looking for is thus

$$A = \begin{bmatrix} \frac{v_1^2}{v_1^2+v_2^2} & \frac{v_1v_2}{v_1^2+v_2^2} \\ \frac{v_1v_2}{v_1^2+v_2^2} & \frac{v_2^2}{v_1^2+v_2^2} \end{bmatrix}$$

4. [5 marks] A matrix B is *symmetric* if $(B)_{ij} = (B)_{ji}$ for all i and j. Show that, for any matrix $A \in \mathbb{R}^{m\times n}$, the product (A^TA) is (a) defined, (b) a square matrix, and (c) a symmetric matrix.

Solution:

(a) The product of $B \in \mathbb{R}^{a\times b}$ and $C \in \mathbb{R}^{c\times d}$ is defined if $b = c$. In our case, the first matrix is $A^T \in \mathbb{R}^{n\times m}$ because it is the transpose of A. The second matrix is $A \in \mathbb{R}^{m\times n}$. The first matrix has m columns and the second has m rows, so the product is defined.

(b) The product has as many rows as A^T, i.e. n, and as many columns as A, i.e. n. So it has n rows and columns, i.e. it is square.

(c) By the definition of the product of two matrices,

$$\begin{aligned} (A^TA)_{ij} &= (A^T)_{i1}A_{1j} + (A^T)_{i2}A_{2j} + \cdots + (A^T)_{im}A_{mj} \\ &= A_{1i}A_{1j} + A_{2i}A_{2j} + \cdots + A_{mi}A_{mj}. \end{aligned}$$

On the other hand, using the same definition, $(A^TA)_{ji}$ is given by

$$\begin{aligned} (A^TA)_{ji} &= (A^T)_{j1}A_{1i} + (A^T)_{j2}A_{2i} + \cdots + (A^T)_{jm}A_{mi} \\ &= A_{1j}A_{1i} + A_{2j}A_{2i} + \cdots + A_{mj}A_{mi} \\ &= A_{1i}A_{1j} + A_{2i}A_{2j} + \cdots + A_{mi}A_{mj}, \end{aligned}$$

showing that $(A^TA)_{ij} = (A^TA)_{ji}$.