Math 115 Spring 2015: Assignment 2

Solutions

1. Determine whether or not the following sets are subspaces in their respective vector spaces. If so, prove it using the definition of subspaces. If not, provide a counterexample where a rule of subspaces is violated.

   (a) [2 marks] \( S = \{ \vec{x} \in \mathbb{R}^2 \mid x_1 + 2x_2 = 0 \text{ and } x_1 - 3x_2 = 1 \} \).

   **Solution:** \( S \) is not a subspace. For example, \( \begin{bmatrix} 2 \\ 5 \\ -1 \\ 5 \end{bmatrix} \in S \), but \( \vec{0} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). \( \vec{0} \) is the only vector that satisfies both equations of \( S \), i.e. \( S = \left\{ \begin{bmatrix} 2 \\ 5 \\ -1 \\ 5 \end{bmatrix} \right\} \).

   (b) [2 marks] \( T = \{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \vec{v} \geq 0 \}, \) where \( \vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \).

   **Solution:** \( T \) is not a subspace. Let \( \vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \). The product \( \vec{x} \cdot \vec{v} = 2.2 + 2.2 + (-1)(-1) = 9 \geq 0 \) so \( \vec{x} \in T \). However, \( ((-1)\vec{x}) \cdot \vec{v} = (-2).2 + (-2).2 + (1)(-1) = -9 \nless 0 \), so \( (-1)\vec{x} \notin T \). Therefore, the set is not closed under scalar multiplication.

2. Let \( \vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \), \( \vec{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \), \( \vec{w} = \begin{bmatrix} -3 \\ -2 \\ -3 \end{bmatrix} \), and \( \vec{x} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \).

   (a) [3 marks] Show that \( \{ \vec{u}, \vec{v}, \vec{w}, \vec{x} \} \) is linearly dependent.

   **Solution:** We solve the system \( p\vec{u} + q\vec{v} + r\vec{w} + s\vec{x} = \vec{0} \), i.e.

   \[
   \begin{align*}
   p + 2q - 3r &= 0 \\
   2p + 2q - 2r + 2s &= 0 \\
   -p + q - 3r - 3s &= 0
   \end{align*}
   \Rightarrow
   \begin{align*}
   p &= -2q + 3r \\
   2p + 2q - 2r + 2s &= 0 \\
   2q - 3r + q - 3r - 3s &= 0
   \end{align*}
   \Rightarrow
   \begin{align*}
   p &= -2q + 3r \\
   2q - 3r + q - 3r - 3s &= 0 \\
   3q - 6r - 3s &= 0
   \end{align*}
   \Rightarrow
   \begin{align*}
   p &= -2q + 3r \\
   s &= q - 2r \\
   3q - 6r - 3s &= 0
   \end{align*}
   \Rightarrow
   \begin{align*}
   p &= -2q + 3r \\
   s &= q - 2r \\
   3q - 6r - 3s &= 0
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   \end{align*}
   \Rightarrow
   \begin{align*}
   p &= -2q + 3r \\
   s &= q - 2r \\
   3q - 6r - 3s &= 0
   \end{align*}

   If we pick any value for \( q \) and \( r \), and then compute \( p \) and \( s \) according to the above equation, we obtain a solution to the system. Note that we do not pick \( q = r = 0 \), because then \( p = s = 0 \), which does not provide the “not all zero” solution necessary to prove linear dependence.

   For example, we can choose \( q = 0, r = -1 \), yielding \( p = -3, s = 2 \), and verify that

   \[-3\vec{u} - \vec{w} + 2\vec{x} = \vec{0}.\]
3. For each of the following statements, either prove that it is true, or find a counterexample to prove that it is false.

(b) [4 marks] Find a basis for \( \text{span}\{\vec{u}, \vec{v}, \vec{w}\} \), i.e. a set of vectors \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) such that (a) \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) is linearly independent, and (b) \( \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{span}\{\vec{u}, \vec{v}, \vec{x}\} \). (Note: \( k \) will be smaller than 4, so it could be 1, 2 or 3.)

**Solution:** We proved above that \(-3\vec{u} - \vec{w} + 2\vec{x} = 0\). Therefore \(\vec{w} = -3\vec{u} + 0\vec{v} + 2\vec{x}\), thus \(\vec{w} \in \text{span}\{\vec{u}, \vec{v}, \vec{x}\}\), so \(\text{span}\{\vec{u}, \vec{v}, \vec{x}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}\). Let us check that \(\{\vec{u}, \vec{v}\}\) is linearly independent by solving the system

\[
\begin{align*}
 p + 2q &= 0 \\
 2p + 2q + 2s &= 0 \\
 -p + q - 3s &= 0
\end{align*}
\]

Again, for any value of \( q \), we find a solution to the system. In particular, if \( q = -1 \), then \( s = -1, p = 2 \), showing that \(2\vec{u} - \vec{v} - \vec{x} = 0\), i.e. the system is linearly dependent. Rearranging the previous equation, we see that \(\vec{x} = 2\vec{u} - \vec{v}\), so \(\vec{x} \in \text{span}\{\vec{u}, \vec{v}\}\), implying that \(\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{v}, \vec{x}\}\).

Let us now check that \(\{\vec{u}, \vec{v}\}\) is linearly independent. The system

\[
\begin{align*}
 p + 2q &= 0 \\
 2p + 2q &= 0 \\
 -p + q &= 0
\end{align*}
\]

has a solution only if \(-2q = q\). This happens only if \(p = q = 0\). Therefore, \(\{\vec{u}, \vec{v}\}\) is linearly independent.

To summarize, we showed that so \(\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{v}, \vec{w}\}\) and \(\{\vec{u}, \vec{v}\}\) is linearly independent. Thus, \(\{\vec{u}, \vec{v}\}\) is a basis of \(\text{span}\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}\).

3. For each of the following statements, either prove that it is true, or find a counterexample to prove that it is false.

(a) [3 marks] Let \(\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n\). If \(\{\vec{u}, \vec{v}, \vec{w}\}\) is linearly dependent, then \(\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}\).

**Solution:** False. For example, take \(\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\) and \(\vec{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\). Clearly, \(0\vec{u} + \vec{v} - \frac{1}{2}\vec{w} = 0\), so they are linearly dependent. But \(\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}\).

**Note:** If \(\{\vec{u}, \vec{v}, \vec{w}\}\) are linearly dependent, then we know that \(at least one\) of the following statements is true:

1. \(\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}\).  
2. \(\vec{v} \in \text{span}\{\vec{u}, \vec{w}\}\).  
3. \(\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}\).

However, they are not necessarily \(all\) true. In the above example, (2) and (3) hold true, but (1) does not.

As a consequence, \(\text{span}\{\vec{u}, \vec{v}\} = \text{span}\{\vec{u}, \vec{w}\} = \text{span}\{\vec{v}, \vec{w}\}\), but \(\text{span}\{\vec{v}, \vec{w}\} \neq \text{span}\{\vec{u}, \vec{v}, \vec{w}\}\). Indeed \(\text{span}\{\vec{v}, \vec{w}\}\) is the line \(\vec{x} \in \mathbb{R}^2 \mid \vec{x} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\), while \(\text{span}\{\vec{u}, \vec{v}, \vec{w}\}\) is the entire plane \(\mathbb{R}^2\).

(b) [3 marks] Let \(\vec{u}, \vec{v} \in \mathbb{R}^n\) be two nonzero vectors (i.e. \(\vec{u} \neq \vec{0}\) and \(\vec{v} \neq \vec{0}\)). If \(\vec{u}\) and \(\vec{v}\) are orthogonal, then \(\{\vec{u}, \vec{v}\}\) is linearly independent.
Solution: True. We prove that if $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ were such that $\{\vec{u}, \vec{v}\}$ is linearly dependent, then $\vec{u}$ and $\vec{v}$ could not be orthogonal.

Assume $\{\vec{u}, \vec{v}\}$ is linearly dependent. Then, there exist $s, t$ not both zero such that $s\vec{u} + t\vec{v} = \vec{0}$. Furthermore, we know that both $s \neq 0$ and $t \neq 0$. Indeed, if $s = 0$ and $t \neq 0$, then $t\vec{v} = \vec{0}$ which contradicts $\vec{v} \neq \vec{0}$, and if $s \neq 0$ and $t = 0$, then $s\vec{u} = \vec{0}$ which contradicts $\vec{u} \neq \vec{0}$. Therefore, we can write $\vec{u} = -\frac{t}{s}\vec{v}$. So the product $\vec{u}.\vec{v} = -\frac{t}{s}\vec{v}.\vec{v} = -\frac{t}{s}|\vec{v}|^2 \neq 0$, showing that the vectors are not orthogonal.

(c) [3 marks] Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ be three nonzero vectors. If (a) $\vec{u}$ is orthogonal to $\vec{v}$, and (b) $\vec{u}$ is orthogonal to $\vec{w}$, then $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution: False. Take any $\vec{u}$ orthogonal to $\vec{v}$ and $\vec{w} = \vec{v}$. Then $\vec{u}$ is also orthogonal to $\vec{w}$, but $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.

For example, let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\vec{u}.\vec{v} = \vec{u}.\vec{w} = 0$, but $0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent.