Math 115 Spring 2015: Quiz 9

Solutions

1. [8 marks] Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. Find a symmetric matrix G that has $\vec{v}_1$ and $\vec{v}_2$ as eigenvectors, with corresponding eigenvalues 5 and 10, respectively.

   Hint: Note that $\vec{v}_1 \cdot \vec{v}_2 = 0$, but $|\vec{v}_1| \neq 1$ and $|\vec{v}_2| \neq 1$.

   Solution: If we construct a matrix $Q = [\vec{v}_1 \ \ \vec{v}_2]$, then we get the diagonalization $Q^{-1}AQ = \operatorname{diag}(5, 10)$. We can then construct $A = Q \operatorname{diag}(5, 10) Q^{-1}$. However, in this case, we want A to be symmetric. Therefore, we want Q to be an orthogonal matrix, i.e. $\{\vec{v}_1, \vec{v}_2\}$ must be orthonormal. The vectors $\vec{v}_1$ and $\vec{v}_2$ are already orthogonal, so we just need to make their lengths be one. We obtain

$$\frac{\vec{v}_1}{|\vec{v}_1|} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \text{and} \quad \frac{\vec{v}_2}{|\vec{v}_2|} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}.$$

   We construct

$$Q = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

   which is an orthogonal matrix and get

$$\begin{aligned} G &= Q \operatorname{diag}(5, 10) Q^{-1} \\ &= \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} 3 & -8 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{41}{5} & -\frac{12}{5} \\ -\frac{12}{5} & \frac{34}{5} \end{bmatrix} \end{aligned}$$

2. [6 marks] Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Given

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   such that $P^{-1}AP = D$, compute A^{10}. (**Note:** $2^{10} = 1024$).

Solution: As seen in the tutorial,

$$\begin{aligned}
A^{10} &= PD^{10}P^{-1} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1024 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1024 & 1024 & 1 \\ 0 & 1024 & 0 \\ 0 & -1024 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1024 & -1023 & -1023 \\ 0 & 1024 & 0 \\ 0 & -1023 & 1 \end{bmatrix}
\end{aligned}$$

3. [6 marks] Prove that if A is orthogonally diagonalizable, then A^2 is orthogonally diagonalizable.

Solution: By the Principal Axis Theorem, A is orthogonally diagonalizable if and only if it is symmetric, so A is symmetric. Then,

$$A^2 = AA = A^T A^T = (AA)^T = (A^2)^T$$

so A^2 is symmetric as well. Therefore, A^2 is orthogonally diagonalizable.

Solution 2: Since A is orthogonally diagonalizable, there exist an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$. As seen in class, $A^2 = PD^2P^T$ where D^2 is a diagonal matrix. Therefore, A^2 is orthogonally diagonalizable.

Solution 3: Since A is orthogonally diagonalizable, there exist an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$, so $A = PDP^T$. By squaring both sides of the latter equation, we obtain

$$A^2 = (PDP^T)^2 = (PDP^T) \cdot (PDP^T) = PDP^TPDP^T.$$

Since P is orthogonal,

$$A^2 = PDDP^T = PD^2P^T$$

where D^2 is a diagonal matrix. Therefore, A^2 is orthogonally diagonalizable.