Math 115 Spring 2015: Quiz 4

Solutions

1. [5 marks] Let $C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Compute $CD^T\vec{y}$.

   Solution: There are two valid ways of doing this. Either

$$CD^T\vec{y} = \left(\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}$$

   or

$$CD^T\vec{y} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}.$$

2. [5 marks] Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$, for all $\vec{x} \in \mathbb{R}^3$.

   Solution: The vector equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

   corresponds to three scalar equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= x_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= x_1. \end{aligned}$$

   We identify the coefficients in both sides of the equations to find that $a_{13} = a_{22} = a_{31} = 1$, and all others are zero. Therefore,

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

3. [5 marks] Given a vector $\vec{v} \in \mathbb{R}^2$, find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $A \cdot \vec{x} = \text{perp}_{\vec{v}}\, \vec{x}$, for all $x \in \mathbb{R}^2$. **Note 1:** The matrix must be the same for all $\vec{x}$, but $\vec{v}$ is constant, so v_1 and v_2 may appear in A. **Note 2:** Recall that $\text{perp}_{\vec{v}}\, \vec{x} = \vec{x} - \frac{\vec{v}\cdot\vec{x}}{||\vec{v}||^2}\vec{v}$.

Solution: In the vector equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x} - \frac{\vec{v} \cdot \vec{x}}{||\vec{v}||^2} \vec{v},$$

we rewrite the vectors on the right-hand side in term of their components:

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \frac{\vec{v} \cdot \vec{x}}{||\vec{v}||^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - \frac{\vec{v} \cdot \vec{x}}{||\vec{v}||^2} v_1 \\ x_2 - \frac{\vec{v} \cdot \vec{x}}{||\vec{v}||^2} v_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 - \frac{v_1 x_1 + v_2 x_2}{v_1^2 + v_2^2} v_1 \\ x_2 - \frac{v_1 x_1 + v_2 x_2}{v_1^2 + v_2^2} v_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(1 - \frac{v_1 v_1}{v_1^2 + v_2^2}\right) x_1 - \frac{v_1 v_2}{v_1^2 + v_2^2} x_2 \\ -\frac{v_1 v_2}{v_1^2 + v_2^2} x_1 + \left(1 - \frac{v_2 v_2}{v_1^2 + v_2^2}\right) x_2 \end{bmatrix}. \end{aligned}$$

We identify the coefficients and obtain

$$\begin{aligned} a_{11} &= 1 - \frac{v_1^2}{v_1^2 + v_2^2} = \frac{v_1^2 + v_2^2 - v_1^2}{v_1^2 + v_2^2} = \frac{v_2^2}{v_1^2 + v_2^2} \\ a_{12} &= -\frac{v_1 v_2}{v_1^2 + v_2^2} \\ a_{21} &= -\frac{v_1 v_2}{v_1^2 + v_2^2} \\ a_{22} &= 1 - \frac{v_2^2}{v_1^2 + v_2^2} = \frac{v_1^2 + v_2^2 - v_2^2}{v_1^2 + v_2^2} = \frac{v_1^2}{v_1^2 + v_2^2}. \end{aligned}$$

The matrix we are looking for is thus

$$A = \begin{bmatrix} \frac{v_2^2}{v_1^2 + v_2^2} & -\frac{v_1 v_2}{v_1^2 + v_2^2} \\ -\frac{v_1 v_2}{v_1^2 + v_2^2} & \frac{v_1^2}{v_1^2 + v_2^2} \end{bmatrix}$$

4. [5 marks] Prove that if $B \in \mathbb{R}^{m \times n}$, then $(BB^T)_{ii} \geq 0$ for all $i = 1, 2, \ldots, m$. **Hints:** Use the definitions of the product and transpose of matrices to write the value of $(BB^T)_{ii}$ in terms of elements of B. Then, remark that for any $\lambda \in \mathbb{R}$, $\lambda^2 \geq 0$.

Solution: By the definition of matrix product,

$$(BB^T)_{ij} = (B)_{i1}(B^T)_{1j} + (B)_{i2}(B^T)_{2j} + (B)_{i3}(B^T)_{3j} + \cdots + (B)_{ip}(B^T)_{pj}.$$

In particular, we are interested in elements B_{ij} of B such that $j = i$. Their value is

$$(BB^T)_{ii} = (B)_{i1}(B^T)_{1i} + (B)_{i2}(B^T)_{2i} + (B)_{i3}(B^T)_{3i} + \cdots + (B)_{ip}(B^T)_{pi}.$$

Then, by the definition of the transpose,

$$\begin{aligned} (BB^T)_{ii} &= (B)_{i1}(B)_{i1} + (B)_{i2}(B)_{i1} + (B)_{i3}(B)_{i3} + \cdots + (B)_{ip}(B)_{ip} \\ &= (B)_{i1}^2 + (B)_{i2}^2 + (B)_{i3}^2 + \cdots + (B)_{ip}^2. \end{aligned}$$

Since $(B)^2_{i1}, (B)^2_{i2}, (B)^2_{i3}, \ldots, (B)^2_{ip}$ are squared scalars, they are all nonnegative (i.e. greater than or equal to zero). Therefore, $(B)^2_{i1} + (B)^2_{i2} + (B)^2_{i3} + \cdots + (B)^2_{ip} \geq 0$, so $(BB^T)_{ii} \geq 0$.