Stabilizing an infinite dimensional quantum stochastic system

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Outline

Quantum dynamical systems from a control theory perspective

- A reformulation of classical dynamical systems.
- Quantum (i.e. non-commutative) probability spaces.
- Evolution in quantum systems:
  - Deterministic, reversible evolution: Schrödinger's equation.
  - Non-deterministic, irreversible evolution: von Neumann measurement postulates.

Stabilizing the quantum microwave cavity

- Stabilizing the cavity near a photon number state.

A reformulation of classical systems

The closest classical analogue of an isolated quantum system is:

- A **deterministic** dynamical system on some set Ω, i.e.

$$\dot{\omega} = f(\omega), \qquad \omega(t) \in \Omega,$$

  with **unknown/random** initial value $\omega(0)$.

- The probability measure, μ_0 of $\omega(0)$ is known.

What is the state space of this system?

Ignore the dynamics for the time being and examine how to describe the state of this system.

- In classical theories, the set Ω is usually called the state space (or sample space in stochastic systems theory).
- There is no (obvious) quantum mechanical analogue of the set Ω!
- We look at this system from a different perspective to see the analogy with quantum mechanics.

Observables and probability measures

- Let (Ω, Σ) be a measure space[2].
- $\mathcal{P}$ - set of probability measures on (Ω, Σ).
- $\mathcal{B}$ - set of complex-valued, measurable functions on (Ω, Σ).

The spaces $\mathcal{B}$ and $\mathcal{P}$ are dual to each other through the relation[3]:

$$\langle X, \mu \rangle \triangleq \mathbb{E}_\mu[X] \triangleq \int X(\omega) d\mu(\omega) \text{ for all } X \in \mathcal{B}, \mu \in \mathcal{P}$$

This relation has a direct generalization to quantum systems.

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A reformulation of classical systems

(Most) statements concerning classical systems can be expressed in terms of this duality relation. e.g

1. If $S \subset \Omega$ is a measurable set and ξ_S is the characteristic function of this set, then

$$\omega \in S \text{ at } t = 0 \iff \mathbb{E}_{\mu_0}[\xi_S] = \langle \xi_S, \mu_0 \rangle = 1.$$

2. If $X \in \mathcal{B}$ is some random variable then at $t = 0$,

$$\mathbf{P}[X \in E] = \mathbb{E}_{\mu_0}[\xi_{X^{-1}(E)}] = \left\langle \xi_{X^{-1}(E)}, \mu_0 \right\rangle.$$

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A reformulation of classical systems

- We call $\mu_0 \in \mathcal{S}$ the **state of the system** at time $t = 0$.
- We call a real-valued random variable $X \in \mathcal{B} = \mathcal{B}(\Omega)$ an **observable** that can be measured by a physical experiment.
- Each state defines a **linear functional** $\mu \mapsto \mathbb{E}_\mu[\cdot]$ on the set of random varaibles.
- We can call a pair $(\mathcal{B}, \mu)$ a classical probability space.
  - Note that we no longer make a reference to the sample space Ω.

Generalization to quantum systems

Consider a finite sample space $\Omega = \{1, \ldots, n\}$ and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{P}$ and $X \in \mathcal{B}$. Then,

$$\begin{aligned}
&\mathbb{E}_\mu[X] \\
&\quad = \sum_j \mu_j X(j) \\
&\quad = \mathrm{Tr}\left\{ \begin{pmatrix} \mu_1 & 0 & \ldots & 0 \\ 0 & \mu_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_n \end{pmatrix} \begin{pmatrix} X(1) & 0 & \ldots & 0 \\ 0 & X(2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & X(n) \end{pmatrix} \right\}
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A quantum probability space

Definition

A quantum probability space consists of a pair $(\mathcal{B}(\mathcal{H}), \mathbb{E}_\mu)$ where

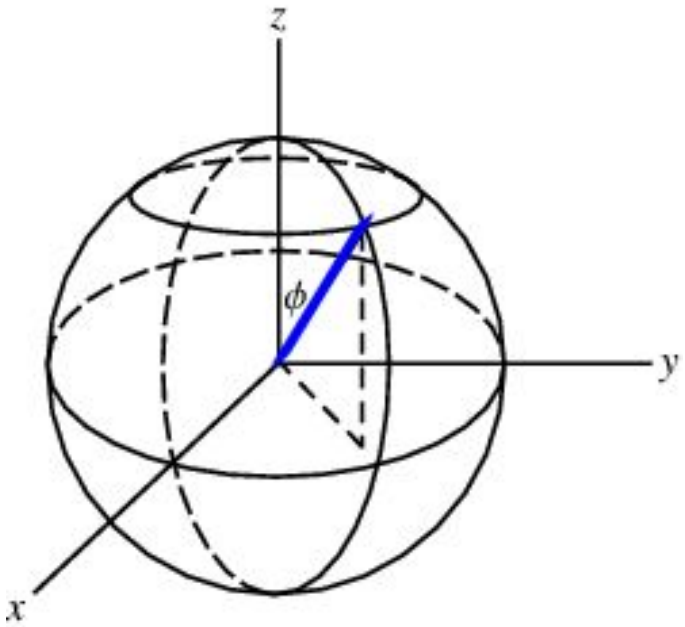
1. $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators (matrices) on a Hilbert space $\mathcal{H}$.
2. $\mathbb{E}_\mu : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is a positive linear functional (+some technical conditions) and we can write

$$\mathbb{E}_\mu[X] \triangleq \mathrm{Tr}\left\{\rho_\mu X\right\} \text{ for all } X \in \mathcal{B}(\mathcal{H})$$

for some positive matrix ρ_μ with trace 1 which is called the density matrix

Example: Stern Gerlach experiment

- The Stern-Gerlach experiment was designed to measure the orientation of a (microscopic) magnet.
- An experiment is set up to measure the deflection angle ϕ of a magnet with respect to some axis



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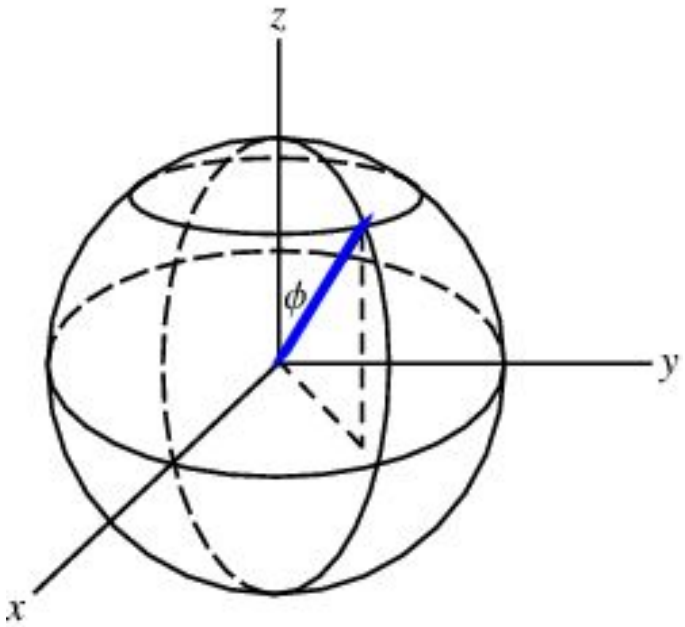

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- So the outcome of measuring the angle ϕ for a fixed axis can be compared to the outcome of tossing a coin.

- If we want a classical description of this system we could use:

  Sample space:

  $$\Omega_z = \{|\downarrow_z\rangle, |\uparrow_z\rangle\}.$$

  The **observable** being measured:

  $$\sigma_z(|\uparrow_z\rangle) = 1 \text{ and } \sigma_z(|\downarrow_z\rangle) = -1$$

  The **system state** given by the probability measure:

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$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \rho_\mu = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_{-1} \end{pmatrix}$$

so that

$$\mathbb{E}_\mu[\sigma_z] = \mathrm{Tr}\left\{\rho_\mu \sigma_z\right\} = 1 \cdot \mu_1 + (-1) \cdot \mu_{-1}.$$

- In particular, the state

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Example: Stern Gerlach experiment

- But there is nothing special about the z-axis.
- If the experiment is setup to measure the deflection about the x-axis instead then we will again only two outcomes (parallel and anti-parallel to the x-axis).

- Classically, to model this situation, we assume that the sample space of the system is

$$\Omega_x \times \Omega_y = \{(|\uparrow_x\rangle, |\uparrow_z\rangle), (|\uparrow_x\rangle, |\downarrow_z\rangle), (|\downarrow_x\rangle, |\uparrow_z\rangle), (|\downarrow_x\rangle, |\downarrow_z\rangle)\}.$$

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- You can not build a classical sample space on which the two random variables σ_x and σ_z are well defined.

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Example: Stern Gerlach experiment

- Quantum mechanically, this situation is modeled by assuming that σ_x corresponds to an operator that is not diagonal in the same basis as σ_z.

- i.e. in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$, we have

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Summary: State of a quantum system

- Associated with any quantum system is a Hilbert space $\mathcal{H}$.
- Any experimentally measurable quantity (**observable**) is specified by a self-adjoint operator X on $\mathcal{H}$.
- The **state of a quantum system** is a density matrix ρ.
- The state ρ determines a **linear functional**

$$X \mapsto \mathbb{E}_\rho(X) = \mathrm{Tr}\,\{\rho X\}$$

which gives the expected (mean) value of measuring the observable X.

Spectral theorem and sample space of an observable

The spectral decomposition theorem: If X is self-adjoint, then

$$X = \sum_{\lambda \in \mathrm{spec}(X)} \lambda P_\lambda.$$

Here,

$\mathrm{spec}(X)$ is the spectrum of X and

P_λ is the projection corresponding to $\lambda \in \mathrm{spec}(X)$.

Spectral theorem and sample space of an observable

- Given any observable X, we can associate with it a classical sample space

$$\Omega_X = \mathrm{spec}(X)$$

and given a state ρ we can get a probability measure on this sample space

$$\mu_\rho[E] = \mathrm{Tr}\left\{\rho \sum_{\lambda \in E} P_\lambda\right\} \text{ for all } E \subset \Omega_X.$$

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Measurements in quantum mechanics

- Consider a classical probability space $(\mathcal{B}, \mathbb{E}_\mu)$.
- How does the probability measure μ change if we get more information about the system through measurements?
- After measuring an observable $X \in \mathcal{B}$, the probability measure μ changes to μ' according to:

$$\mathbb{E}_{\mu'}[\cdot] = \mathbb{E}_\mu[\cdot|X].$$

- The quantum equivalent of this is the von Neumann measurement postulate.

von Neuman Measurement postulate

1. Measuring X gives (randomly) a result $\lambda \in \mathrm{spec}(X)$.

2. Probability that we measure $\lambda \in E \subset \mathrm{spec}(X)$ given state ρ is [4]:

$$\mathbf{P}[X \in E] = \mu_\rho[E] = \mathbb{E}_\rho\left[\sum_{\lambda \in E} P_\lambda\right] = \mathrm{Tr}\left\{\rho \sum_{\lambda \in E} P_\lambda\right\}.$$

3. After knowing measurement result $\bar{\lambda}$, the state **collapses** to[5]

$$\rho \to \bar{\rho} = \frac{P_{\bar{\lambda}} \rho P_{\bar{\lambda}}}{\mathrm{Tr}\left\{P_{\bar{\lambda}} \rho P_{\bar{\lambda}}\right\}} \quad \left(= P_{\bar{\lambda}} \text{ if } \bar{\lambda} \text{ is degenerate.} \right)$$

[4] c.f. classical situation $\mathbf{P}[X \in E] = \mathbb{E}_{\mu_0}[\xi_{X^{-1}(E)}] = \left\langle \xi_{X^{-1}(E)}, \mu_0 \right\rangle$.

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Pure states

- The set of all states is a closed, convex set and the extreme points of this set are called pure states.
- The set of all pure states are in one-to-one correspondence with the set of all unit vectors (modulo constant multiples) in $\mathcal{H}$.
- The density matrix corresponding to unit vector $|v\rangle$ is [6]

$$\rho_v = |v\rangle \langle v| .$$

[6] $\langle v|$ is the row vector formed by taking the complex conjugate transpose of column vector $|v\rangle$.

Probabilities in a pure state

- Consider a Hilbert space H and a self-adjoint observable with a discrete spectrum

$$X = \sum_{\lambda_n \in \mathrm{spec}(X)} \lambda_n P_{\lambda_n} = \sum_{\lambda_n \in \mathrm{spec}(X)} \lambda_n \left| n \right\rangle \left\langle n \right| .$$

- In any pure state $|v\rangle$, we have

$$\mathbf{P}_{|v\rangle}[X = \lambda_n] = \mathrm{Tr}\left\{\rho_v P_{\lambda_n}\right\} = |\langle n|v\rangle|^2.$$

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Variance

- The variance of an observable X in state ρ is defined

$$\mathrm{Var}_\rho[X] = \mathbb{E}_\rho[X^2] - (\mathbb{E}_\rho[X])^2.$$

- Unlike the classical situation, in any state ρ there will be observables with non-zero variance.

- e.g. If in the Stern-Gerlach experiment, the state is $|\uparrow_z\rangle$, then

$$\mathrm{Var}_{|\uparrow_z\rangle}(\sigma_z) = 0 \text{ and } \mathrm{Var}_{|\uparrow_z\rangle}(\sigma_x) = 1/4.$$

Deterministic and reversible evolution

Suppose our classical system dynamics are deterministic and reversible. i.e. the equation

$$\dot{\omega} = f(\omega), \qquad \omega(t) \in \Omega.$$

satisfies sufficient (smoothness) conditions such that $x(0) = x_0$ determines the entire trajectory

$$x(t) = \Gamma_t x_0, t \in \mathbb{R}.$$

Deterministic and reversible evolution

- In our new language this imples,

$$\text{if } \mu_0 = \xi_{\{x_0\}} \text{ then } \mu_t = \xi_{\{\Gamma_t x_0\}}.$$

- That is, the system evolution is given by a one-parameter group $U(t)$, of automorphisms on the set of extreme points of $\mathcal{S}$:

$$U(t) : \xi_{\{x_0\}} \mapsto \xi_{\{\Gamma_t x_0\}}.$$

Schrödinger's equation

- Schrodinger's equation governs the reversible, deterministic evolution of (closed) quantum systems.
- The group of automorphisms $U(t)$ map pure state $|\psi_0\rangle \in \mathcal{H}$ to pure state $|\psi_t\rangle = U(t)\,|\psi_0\rangle$, $t \in \mathbb{R}$.
- A theorem of Wigner tells us that $U(t)$ must be a unitary group.
- Stone-von Neumann theorem tells us that[7] the generator of $U(t)$ must be skew-adjoint. i.e.

$$\frac{d\,|\psi_t\rangle}{dt} = -iH\,|\psi_t\rangle$$

with H self-adjoint, called the system Hamiltonian.

[7] Assuming certain smoothness conditions

Summary: Evolutions in quantum systems

- Associated with any quantum system is a Hilbert space $\mathcal{H}$.
- The state of a quantum system is a quantum probability ρ that maps observables X to $\mathbb{E}_\rho[X]$.
- Measuring an observable $X \in \mathcal{B}(\mathcal{H})$ changes the state ρ (non-deterministically and irreversibly) von Neumann measurement postulate.
- If no measurement is performed, the state evolves deterministically and reversibly according to Schrödinger's equation.

Experimental Setup

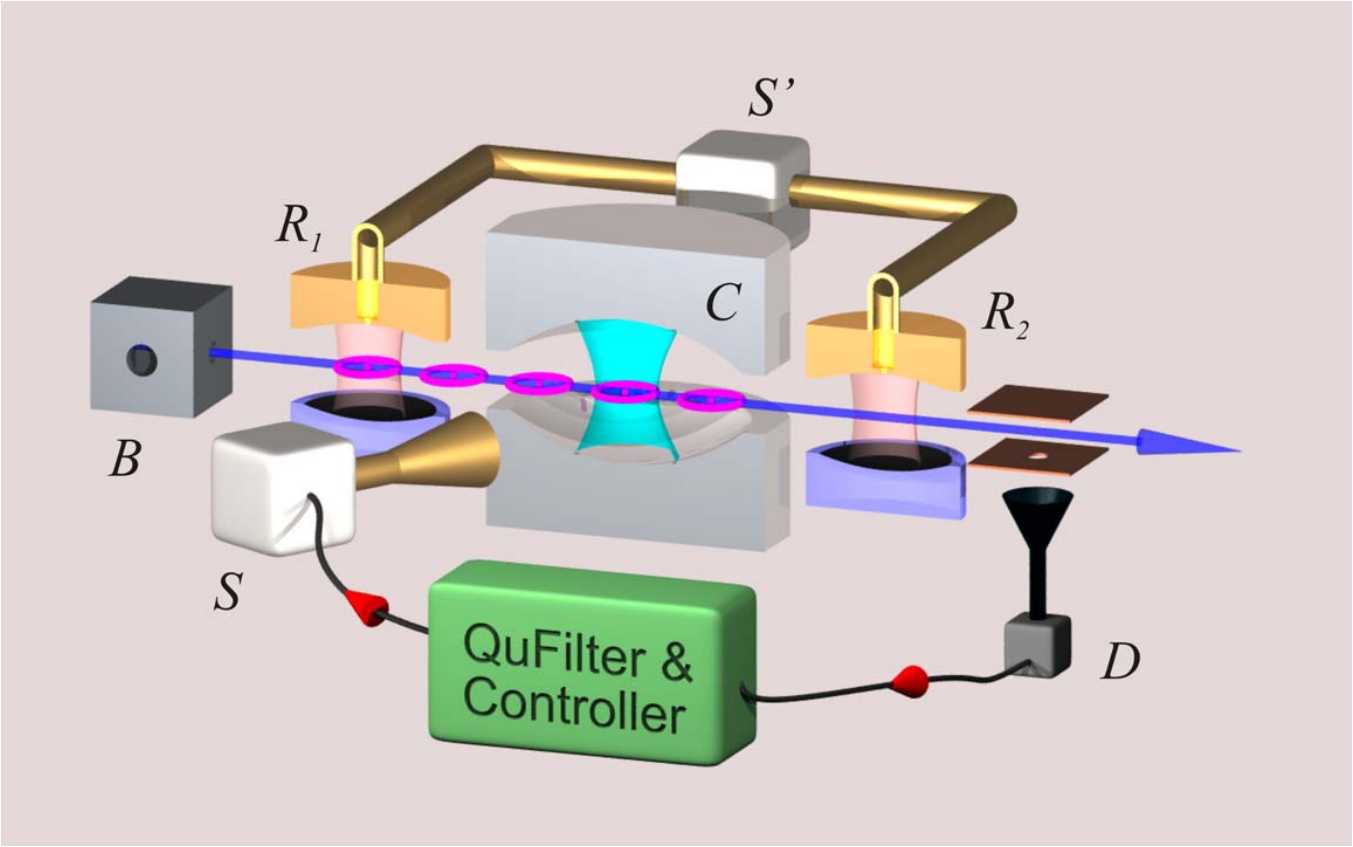

Figure: The microwave cavity QED setup with its feedback scheme (in green) (Fig. Courtesy Igor Dotsenko).

Notation: Cavity state

- Let $\mathcal{H}$ be the Hilbert space of the cavity and denote by B_1 the unit ball in $\mathcal{H}$.
- An orthonormal basis for this Hilbert space is the set

$$\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\}.$$

- Physically, $|n\rangle \in B_1$ denotes a cavity state with exactly n photons.

Notation: Some operators on $\mathcal{H}$

- The annihilation a, creation $a^\dagger$ and number N operators are

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,,\ a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,,\ N\,|n\rangle = n\,|n\rangle\,.$$

- The displacement operator

$$D_\alpha = \exp(\alpha^* a - \alpha a^\dagger)$$

injects light of amplitude α into the cavity.

- The measurement operators are defined

$$M_g\,|n\rangle = e^{i(\phi+\pi)/2}\sin(n\Phi_0 - \phi/2)\,|n\rangle\,,$$

$$M_e\,|n\rangle = -e^{-i\phi/2}\cos(n\Phi_0 - \phi/2)\,|n\rangle\,.$$

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System dynamics

We split our system dynamics into two parts:

1. Non-deterministic and irreversible collapse due to measurements of the atomic state.
2. Deterministic unitary evolution of the collapsed cavity state until measurement of the next atom.

System dynamics: Non-deterministic part

- Let $|\psi_k\rangle \in B_1$ be the cavity state before interacting with the kth atom.
- After the kth atom interacts with the cavity it is measured in either the ground (g) or excited (e) state.
- After this collapse due to measurement, the cavity state is

$$|\psi_{k+1/2}\rangle = \begin{cases} \mathcal{M}_g(|\psi_k\rangle) \triangleq \frac{M_g|\psi_k\rangle}{\|M_g|\psi_k\rangle\|}, & \text{if atom detected in } g, \\ \mathcal{M}_e(|\psi_k\rangle) \triangleq \frac{M_e|\psi_k\rangle}{\|M_e|\psi_k\rangle\|}, & \text{if atom detected in } e \end{cases}$$

and

$$\mathbf{P}_g(|\psi_k\rangle) = \|M_g\,|\psi_k\rangle\,\|^2 \text{ and } \mathbf{P}_e(|\psi_k\rangle) = \|M_e\,|\psi_k\rangle\,\|^2.$$

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System dynamics: Deterministic part

- After knowing outcome g or e the controller can set the parameter α, so that

$$|\psi_{k+1}\rangle = D_{\alpha_k} |\psi_{k+1/2}\rangle .$$

Markov chain and control goal

- The sequence $|\psi_1\rangle, |\psi_2\rangle, \ldots,$ is a (classical) Markov chain:

$$|\psi_{k+1}\rangle = D_{\alpha_k}\mathcal{M}_{s_k}(|\psi_k\rangle)$$

  where $s_k = g$ or e with probability $\mathbf{P}_g(|\psi_k\rangle)$ and $\mathbf{P}_e(|\psi_k\rangle)$.

- Note that the state space of this Markov chain is the set of pure states and each pure state is an extreme point in the set of quantum probabilities.

Control Goal

Control Goal: The control goal is to stabilize the cavity state near the state $|\bar{n}\rangle$. In this state

$$\mathbb{E}_{|\bar{n}\rangle}[N] = \bar{n} \text{ and } \mathrm{Var}_{|\bar{n}\rangle}[N] = 0.$$

How?

- Use repeated measurements and (non-deterministically) change the state of the system through collapse.
- Use the deterministic part by injecting light of amplitude $|\alpha|$ that is in phase or out of phase with the light in the cavity.

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Stochastic Stability theorem

Theorem (Kushner)

If $V \geq 0$ is a continuous function on B_1 such that

$$\mathbb{E}\left[V(|\psi_{k+1}\rangle)\middle| |\psi_k\rangle\right] \leq V(|\psi_k\rangle)$$

and if the limit set

$$\Omega_\infty = \{|\psi\rangle : |\psi_{k_n}\rangle \to |\psi\rangle\}$$

then for all $|\psi\rangle \in \Omega_\infty$ we have

$$K(|\psi\rangle) \triangleq \mathbb{E}\left[V(|\psi_{k+1}\rangle)\middle| |\psi_k\rangle = |\psi\rangle\right] - V(|\psi\rangle) = 0.$$

How do we choose the feedback α_k

The controller design comes down to choosing the right V and feedback α_k such that:

- The Lyapunov function V is a super-martingale.
- The trajectories $|\psi_k\rangle, k = 1, 2, \ldots$ stay in some compact subset of B_1 with high probability.
  - This ensures the limit set Ω_∞ is non-empty.
- The condition $K(|\psi\rangle) = 0$ implies $|\psi\rangle = |\bar{n}\rangle$.

Existence of Open-loop Martingales

- Recall that in state $|\psi\rangle$

$$\mathbf{P}_{|\psi\rangle}[N = n] = \mathbb{E}_{|\psi\rangle}[P_n] = |\langle n|\psi\rangle|^2$$

is the probability that there are n photons in the cavity.

- The fact that M_g, M_e and N are co-diagonal[8] implies the existence of open-loop martingales.

- That is, for any n the function

$$|\psi\rangle \mapsto |\langle n|\psi\rangle|^2$$

is a martingale and satisfies

$$\mathbb{E}\left[|\langle n|\psi_{k+1}\rangle|^2 \middle| |\psi_k\rangle, \alpha_k = 0 \right] = |\langle n|\psi_k\rangle|^2.$$

[8] This condition is referred to as the quantum non-demolition condition.

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Central idea of the controller

- The Lyapunov function V we choose is a concave function of the open-loop martingales.
- Therefore $V(|\psi_k\rangle)$ is an open-loop supermartingale (when $\alpha_k = 0$).
- Choose feedback α_k to minimize $V(|\psi_{k+1}\rangle)$ for a given $|\psi_k\rangle$:

$$\alpha_k = \mathrm{argmin}_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(|\psi_{k+1}\rangle) \big| \, |\psi_k\rangle].$$

- With this feedback V is a closed-loop supermartingale: because $0 \in [-\bar{\alpha}, \bar{\alpha}]$ the chosen α_k with always perform at least as good (on average) as choosing $\alpha_k = 0$.

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General form of Lyapunov function

- The form of the Lyapunov function we choose is

$$V(|\psi\rangle) = \delta V_F(|\psi\rangle) + \sum_n \sigma_n |\langle n|\psi\rangle|^2.$$

- Here, δ is some small constant and

$$V_F(|\psi\rangle) = 1 - \sum_n |\langle n|\psi\rangle|^4.$$

- The V_F term ensures convergence to a Fock state.
- The σ_n are a sequence of non-negative numbers that ensure:
  - The trajectories remain in a compact set with high probability ($\sigma_n \to \infty$).
  - The system does not converge to open loop attractors $|n\rangle$ for $n \neq \bar{n}$.

Main result

Theorem

Given any constants $C, \epsilon > 0$, there exist constants $\bar{\alpha}$, δ and $\sigma_0, \sigma_1, \ldots$ such that for all $|\psi_1\rangle$ satisfying $V(|\psi_1\rangle) \leq C$ we have

$$|\psi_k\rangle \to |\bar{n}\rangle \text{ with probability greater than } 1 - \epsilon$$

if we use the feedback

$$\alpha_k = \mathrm{argmin}_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(|\psi_{k+1}\rangle) \big| \, |\psi_k\rangle]$$

where,

$$V(|\psi\rangle) = \delta V_F(|\psi\rangle) + \sum_n \sigma_n |\langle n|\psi\rangle|^2.$$

Why does this feedback work?

- Firstly note that if there is no feedback and if $|\psi_k\rangle = |n\rangle$ then
$$|\psi_{k+1}\rangle = |\psi_k\rangle = |n\rangle \,,$$
because
$$M_g\,|n\rangle = e^{i(\phi+\pi)/2}\sin(n\Phi_0 - \phi/2)\,|n\rangle \,,$$
$$M_e\,|n\rangle = -e^{-i\phi/2}\cos(n\Phi_0 - \phi/2)\,|n\rangle \,.$$
- So $|n\rangle$ is a stationary state of this dynamical process.

Why does this feedback work?

- Also, for $|\psi_k\rangle = |n'\rangle$,

$$\mathbf{P}[s_k = g] = \|M_g \left|n'\right\rangle\|^2 = sin^2(n'\Phi_0 - \phi/2) \text{ and}$$

$$\mathbf{P}[s_k = e] = \|M_e \left|n'\right\rangle\|^2 = cos^2(n'\Phi_0 - \phi/2).$$

- So by repeatedly getting atoms to interact with the cavity we can estimate $\mathbf{P}[s_k = g]$ and $\mathbf{P}[s_k = e]$.

- Using this information, we can calculate n' if 2π is not a multiple of Φ_0.

- That is, repeated measurements of atomic state is equivalent to a measurement of the number operator N of the cavity.

Why does this feedback work?

- Now even if the state is not $|n'\rangle$ for some n', repeated measurements of the atomic state is still equivalent to a measurement of N.
- Therefore, the probability measure of the system *collapses* gradually to a state that is an eigenstate $|n'\rangle$ of N.
- We simply use the weight factors σ_n and feedback α_k to ensure that we move towards the correct eigenstate $|\bar{n}\rangle$.

Proof outline

We show:

1. $V(|\psi_k\rangle)$ is a supermartingale.
2. This implies that the trajectories are in a sequentially compact set with high probability.
3. The trajectories only converge to a Fock state.
4. The trajectories converge to the correct Fock state $|\bar{n}\rangle$ with high probability.

Step 1: V is a supermartingale

Some simple algebra shows that

$$
\begin{aligned}
&\mathbb{E}[V(|\psi_{k+1}\rangle)|\,|\psi_k\rangle = |\psi\rangle] - V(|\psi\rangle) \\
&= \min_{\alpha\in[-\bar{\alpha},\bar{\alpha}]} \mathbb{E}[V(D_\alpha \mathcal{M}_{s_k}|\psi\rangle)] - \mathbb{E}[V(D_0\mathcal{M}_{s_k}|\psi\rangle)] \\
&\quad + \mathbb{E}[V(D_0\mathcal{M}_{s_k}|\psi\rangle)] - V(|\psi\rangle) \\
&= \min_{\alpha\in[-\bar{\alpha},\bar{\alpha}]} \mathbb{E}[V(D_\alpha \mathcal{M}_{s_k}|\psi\rangle)] - \mathbb{E}[V(D_0\mathcal{M}_{s_k}|\psi\rangle)] \\
&\quad - \delta \frac{\sum_n |\langle n|\psi\rangle|^2 \left(\cos^2(n\phi_0+\phi_1) - \|M_g|\psi\rangle\|^2\right)^2}{\|M_g|\psi\rangle\|^2 \|M_e|\psi\rangle\|^2} \\
&\leq 0.
\end{aligned}
$$

Step 2: Compactness of trajectories

- Because $\mathbb{E}[V(|\psi_k\rangle)] \leq V(|\psi_1\rangle)$, for any $\nu > 0$,

$$\mathbb{P}[V(|\psi_k\rangle) > V(|\psi_1\rangle)/\nu] \leq \nu \cdot V(|\psi_1\rangle).$$

- We choose for large n, $\sigma_n \approx O(\log(n))$.
- Then with probability greater than $1 - C\nu$ the trajectories stay in the set

$$\{|\psi\rangle : V(|\psi\rangle) \leq C/\nu\}$$

which is compact.

Step 3: Convergence to Fock states

- We can now apply Kushner's theorem:

$$|\psi_k\rangle \to \{|\psi\rangle : \mathbb{E}[V(|\psi_{k+1}\rangle)|\,|\psi_k\rangle = |\psi\rangle] - V(|\psi\rangle) = 0\}.$$

- That is $|\psi\rangle$ satisfies

$$\frac{\sum_{n=0}^{n_{max}} |\langle n|\psi\rangle|^2 \left(\cos^2(n\phi_0+\phi_1) - \|M_g|\psi\rangle\|^2\right)^2}{\|M_g|\psi\rangle\|^2 \|M_e|\psi\rangle\|^2} = 0 \qquad (1)$$

$$\min_{\alpha\in[-\bar{\alpha},\bar{\alpha}]} \mathbb{E}[V(D_\alpha \mathcal{M}_{s_k}|\psi\rangle)] - \mathbb{E}[V(\mathcal{M}_{s_k}|\psi\rangle)] = 0 \quad (2)$$

- The first equality is satisfied only for $|\psi\rangle = |n\rangle$ for some n.
- So $|\psi_k\rangle$ converges to some Fock state.

Step 4: Convergence to $|\bar{n}\rangle$

- We choose

$$\sigma_n = \begin{cases} \frac{1}{8} + \sum_{k=1}^{\bar{n}} \frac{1}{k} - \frac{1}{k^2}, & \text{if } n = 0 \\ \sum_{k=n+1}^{\bar{n}} \frac{1}{k} - \frac{1}{k^2}, & \text{if } 1 \leq n < \bar{n} \\ 0, & \text{if } n = \bar{n} \\ \sum_{k=\bar{n}+1}^{n} \frac{1}{k} + \frac{1}{k^2}, & \text{if } n > \bar{n} \end{cases} \tag{3}$$

- There is an algorithmic manner in which the σ_n can be chosen.

Step 4: Convergence to $|\bar{n}\rangle$

- The factors σ_n in the Lyapunov function are chosen so that,

$$\frac{d^2}{d\alpha^2} V(D_\alpha |m\rangle) + O(\delta) < 0.$$

for $m \neq \bar{n}$.

- That is $\alpha = 0$ is not a local minimum for $V(D_\alpha |m\rangle)$ and hence for $|\psi\rangle$ close to $|m\rangle$,

$$\min_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(D_\alpha \mathcal{M}_{s_k} |\psi\rangle)] - \mathbb{E}[V(\mathcal{M}_{s_k} |\psi\rangle)] < 0.$$

- So if $|\psi_k\rangle$ is near $|m\rangle$ we can find a $\alpha_k \in [-\bar{\alpha}, \bar{\alpha}]$ to push the cavity state $|\psi_{k+1}\rangle$ away from $|m\rangle$.
- Therefore $|\psi_k\rangle$ converges to $|\bar{n}\rangle$.

Experimental Results

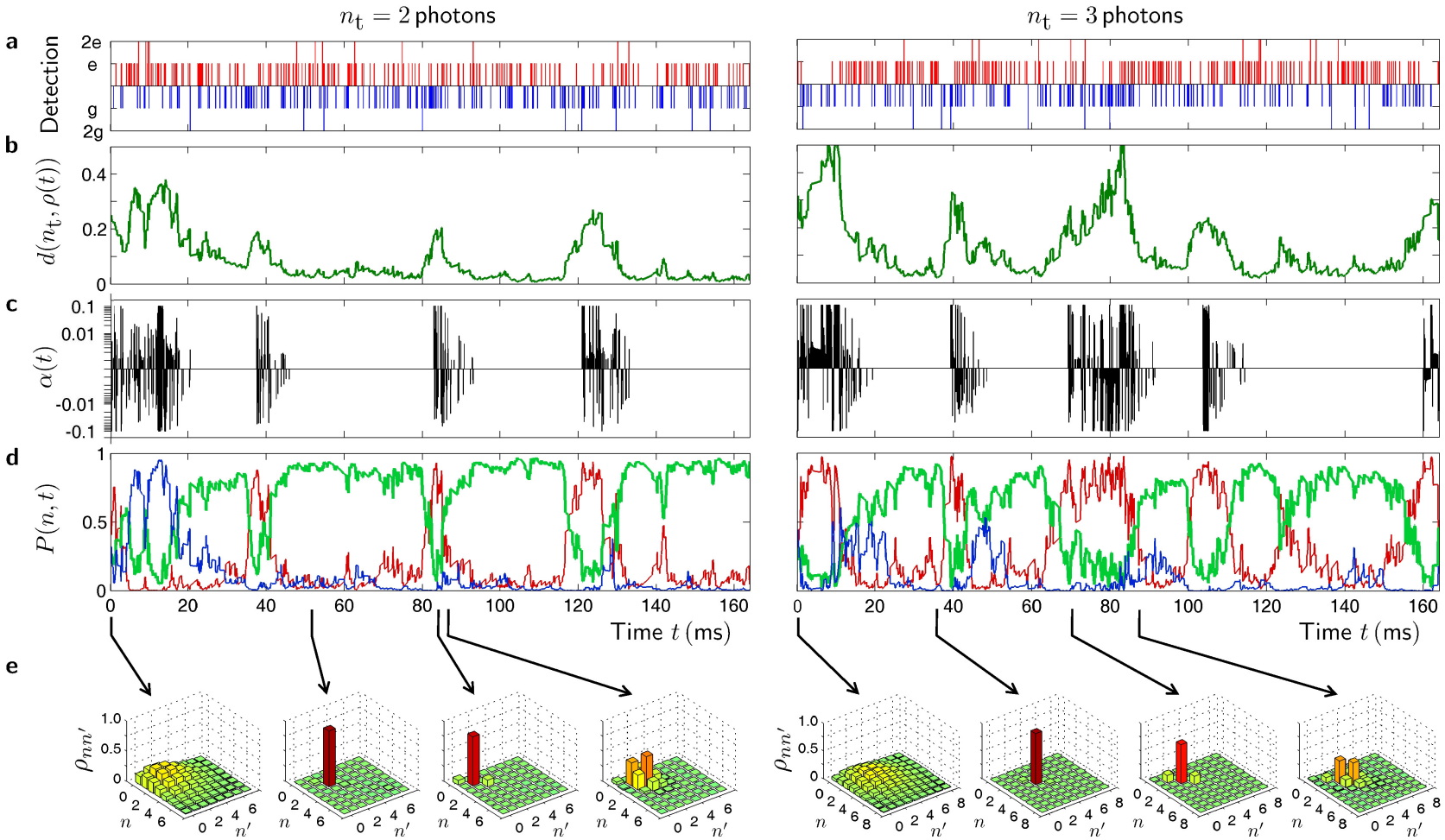

Figure: Cavity state as a function of time

Conclusion

- QND measurements imply the existence of open-loop martingales.
- Use concave functions of these open-loop martingales to construct closed-loop supermartingales.
- Increasingly penalize high photon number states ($\sigma_n \to \infty$) to ensure compactness of trajectories.

Extensions

- In this talk I consider the control of a cavity using an ideal sequence of measurements with no delays in the feedback loop.
- This approach can be generalized to realistic situations with faulty detections and feedback delays using predictive filters.
- Also, we can consider a more general situation, where the measurement operators are not fixed but are functions of the control:

$$M_{s_k}^{\alpha_k}$$

is the measurement operator.

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