Stabilizing an infinite dimensional quantum stochastic system

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Joint work with Prof. Pierre Rouchon, Mines ParisTech and Dr. Mazyar Mirrahimi, INRIA
Quantum dynamical systems from a control theory perspective

- A reformulation of classical dynamical systems.
- Quantum (i.e. non-commutative) probability spaces.
- Evolution in quantum systems:
  - Deterministic, reversible evolution: Schrödinger’s equation.
  - Non-deterministic, irreversible evolution: von Neumann measurement postulates.

Stabilizing the quantum microwave cavity

- Stabilizing the cavity near a photon number state.
A reformulation of classical systems

The closest classical analogue of an isolated quantum system is:

- A **deterministic** dynamical system on some set $\Omega$, i.e.

  $$\dot{\omega} = f(\omega), \quad \omega(t) \in \Omega,$$

  with **unknown/random** initial value $\omega(0)$.

- The probability measure, $\mu_0$ of $\omega(0)$ is known.

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What is the state space of this system?

Ignore the dynamics for the time being and examine how to describe the state of this system.

- In classical theories, the set $\Omega$ is usually called the state space (or sample space in stochastic systems theory).

- There is no (obvious) quantum mechanical analogue of the set $\Omega$!

- We look at this system from a different perspective to see the analogy with quantum mechanics.
Let $(\Omega, \Sigma)$ be a measure space$^2$.

- $\mathcal{P}$ - set of probability measures on $(\Omega, \Sigma)$.
- $\mathcal{B}$ - set of complex-valued, measurable functions on $(\Omega, \Sigma)$.

The spaces $\mathcal{B}$ and $\mathcal{P}$ are dual to each other through the relation$^3$:

$$\langle X, \mu \rangle \triangleq \mathbb{E}_\mu[X] \triangleq \int X(\omega) d\mu(\omega) \text{ for all } X \in \mathcal{B}, \mu \in \mathcal{P}$$

This relation has a direct generalization to quantum systems.

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$^2$ i.e. $\Sigma$ is a $\sigma$-algebra on $\Omega$

$^3$c.f. Reisz representation theorem
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\[
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\]

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A reformulation of classical systems

(Most) statements concerning classical systems can be expressed in terms of this duality relation. e.g

1. If $S \subset \Omega$ is a measurable set and $\xi_S$ is the characteristic function of this set, then

$$\omega \in S \text{ at } t = 0 \iff \mathbb{E}_{\mu_0}[\xi_S] = \langle \xi_S, \mu_0 \rangle = 1.$$ 

2. If $X \in \mathcal{B}$ is some random variable then at $t = 0$, 

$$\mathbb{P}[X \in E] = \mathbb{E}_{\mu_0}[\xi_{X^{-1}(E)}] = \langle \xi_{X^{-1}(E)}, \mu_0 \rangle.$$
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$$\mathbb{P}[X \in E] = \mathbb{E}_{\mu_0}[\xi_{X^{-1}(E)}] = \langle \xi_{X^{-1}(E)}, \mu_0 \rangle.$$
We call $\mu_0 \in S$ the **state of the system** at time $t = 0$.

We call a real-valued random variable $X \in \mathcal{B} = \mathcal{B}(\Omega)$ an **observable** that can be measured by a physical experiment.

Each state defines a **linear functional** $\mu \mapsto \mathbb{E}_\mu[\cdot]$ on the set of random variables.

We can call a pair $(\mathcal{B}, \mu)$ a classical probability space.
- Note that we no longer make a reference to the sample space $\Omega$. 

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Consider a finite sample space $\Omega = \{1, \ldots, n\}$ and let
$\mu = (\mu_1, \ldots, \mu_n) \in \mathcal{P}$ and $X \in \mathcal{B}$. Then,

$$
\mathbb{E}_\mu[X] = \sum_j \mu_j X(j)
$$

$$
= \text{Tr} \left\{ \begin{pmatrix}
\mu_1 & 0 & \cdots & 0 \\
0 & \mu_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu_n
\end{pmatrix}
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The generalization to quantum mechanics comes from assuming that the off diagonal elements of $X$ and $\mu$ maybe non-zero.
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The generalization to quantum mechanics comes from assuming that the off diagonal elements of $X$ and $\mu$ maybe non-zero.
A quantum probability space

**Definition**

A quantum probability space consists of a pair \((\mathcal{B}(\mathcal{H}), \mathbb{E}_\mu)\) where

1. \(\mathcal{B}(\mathcal{H})\) is the set of bounded linear operators (matrices) on a Hilbert space \(\mathcal{H}\).
2. \(\mathbb{E}_\mu : \mathcal{B}(\mathcal{H}) \to \mathbb{C}\) is a positive linear functional (+some technical conditions) and we can write

\[
\mathbb{E}_\mu[X] \triangleq \text{Tr}\{\rho_\mu X\} \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{H})
\]

for some positive matrix \(\rho_\mu\) with trace 1 which is called the density matrix.
The Stern-Gerlach experiment was designed to measure the orientation of a (microscopic) magnet. An experiment is set up to measure the deflection angle $\phi$ of a magnet with respect to some axis.

It was found that the measurement outcome is always $\phi = 0$ or $\phi = \pi$. I.e. the orientation is measured to be parallel or anti-parallel to the $z$-axis.
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Example: Stern Gerlach experiment

- So the outcome of measuring the angle $\phi$ for a fixed axis can be compared to the outcome of tossing a coin.

- If we want a classical description of this system we could use:

  \[
  \text{Sample space:} \quad \Omega_z = \{|\downarrow_z\rangle, |\uparrow_z\rangle\}.
  \]

  The \textit{observable} being measured:

  \[
  \sigma_z(|\uparrow_z\rangle) = 1 \quad \text{and} \quad \sigma_z(|\downarrow_z\rangle) = -1
  \]

  The \textit{system state} given by the probability measure:

  \[
  \mu = (\mu_1, \mu_{-1})
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Equivalently, in the matrix notation:

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho_\mu = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_{-1} \end{pmatrix}
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so that

\[
\mathbb{E}_\mu[\sigma_z] = \text{Tr} \{ \rho_\mu \sigma_z \} = 1 \cdot \mu_1 + (-1) \cdot \mu_{-1}.
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In particular, the state

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corresponds to the case where the magnet is pointing parallel to the z-axis.
Example: Stern Gerlach experiment

- But there is nothing special about the z-axis.
- If the experiment is setup to measure the deflection about the x-axis instead then we will again only two outcomes (parallel and anti-parallel to the x-axis).

- Classically, to model this situation, we assume that the sample space of the system is

\[ \Omega_x \times \Omega_y = \{ (|\uparrow_x\rangle, |\uparrow_z\rangle), (|\uparrow_x\rangle, |\downarrow_z\rangle), (|\downarrow_x\rangle, |\uparrow_z\rangle), (|\downarrow_x\rangle, |\downarrow_z\rangle) \} \]

- This sample space does not make sense!

- You can not build a classical sample space on which the two random variables \( \sigma_x \) and \( \sigma_z \) are well defined.
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This sample space does not make sense!

You can not build a classical sample space on which the two random variables $\sigma_x$ and $\sigma_z$ are well defined.
Quantum mechanically, this situation is modeled by assuming that $\sigma_x$ corresponds to an operator that is not diagonal in the same basis as $\sigma_z$.

i.e. in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$, we have

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\times = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
Associated with any quantum system is a Hilbert space $\mathcal{H}$.

Any experimentally measurable quantity (observable) is specified by a self-adjoint operator $X$ on $\mathcal{H}$.

The state of a quantum system is a density matrix $\rho$.

The state $\rho$ determines a linear functional

$$X \mapsto \mathbb{E}_\rho(X) = \text{Tr} \{ \rho X \}$$

which gives the expected (mean) value of measuring the observable $X$. 
The spectral decomposition theorem: If $X$ is self-adjoint, then

$$X = \sum_{\lambda \in \text{spec}(X)} \lambda P_{\lambda}.$$ 

Here,

- $\text{spec}(X)$ is the spectrum of $X$ and
- $P_{\lambda}$ is the projection corresponding to $\lambda \in \text{spec}(X)$. 
Given any observable $X$, we can associate with it a classical sample space

$$\Omega_X = \text{spec}(X)$$

and given a state $\rho$ we can get a probability measure on this sample space

$$\mu_\rho[E] = \text{Tr} \left\{ \rho \sum_{\lambda \in E} P_\lambda \right\} \text{ for all } E \subset \Omega_X.$$
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Associated to every observable is a sample space and only two commuting observables can correspond to random variables on the same sample space!
Consider a classical probability space \((\mathcal{B}, \mathbb{E}_\mu)\).

How does the probability measure \(\mu\) change if we get more information about the system through measurements?

After measuring an observable \(X \in \mathcal{B}\), the probability measure \(\mu\) changes to \(\mu'\) according to:

\[
\mathbb{E}_{\mu'}[\cdot] = \mathbb{E}_{\mu}[\cdot | X].
\]

The quantum equivalent of this is the von Neumann measurement postulate.
von Neuman Measurement postulate

1. Measuring $X$ gives (randomly) a result $\lambda \in \text{spec}(X)$.

2. Probability that we measure $\lambda \in E \subset \text{spec}(X)$ given state $\rho$ is $^4$:

$$P[X \in E] = \mu_\rho[E] = \mathbb{E}_\rho \left[ \sum_{\lambda \in E} P_\lambda \right] = \text{Tr} \left\{ \rho \sum_{\lambda \in E} P_\lambda \right\}.$$

3. After knowing measurement result $\bar{\lambda}$, the state collapses to $^5$

$$\rho \to \bar{\rho} = \frac{P_{\bar{\lambda}} \rho P_{\bar{\lambda}}}{\text{Tr} \left\{ P_{\bar{\lambda}} \rho P_{\bar{\lambda}} \right\}} \quad \left( = P_{\bar{\lambda}} \text{ if } \bar{\lambda} \text{ is degenerate.} \right)$$

$^4$ c.f. classical situation $P[X \in E] = \mathbb{E}_{\mu_0} [\xi_{X^{-1}(E)}] = \langle \xi_{X^{-1}(E)}, \mu_0 \rangle$.

$^5$ This is equivalent to the classical Bayes’ rule.
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Pure states

The set of all states is a closed, convex set and the extreme points of this set are called pure states.

The set of all pure states are in one-to-one correspondence with the set of all unit vectors (modulo constant multiples) in $\mathcal{H}$.

The density matrix corresponding to unit vector $|v\rangle$ is

$$\rho_v = |v\rangle \langle v|.$$ 

$\langle v|$ is the row vector formed by taking the complex conjugate transpose of column vector $|v\rangle$. 

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Consider a Hilbert space $H$ and a self-adjoint observable with a discrete spectrum

$$X = \sum_{\lambda_n \in \text{spec}(X)} \lambda_n P_{\lambda_n} = \sum_{\lambda_n \in \text{spec}(X)} \lambda_n |n\rangle \langle n|.$$

In any pure state $|\nu\rangle$, we have

$$P_{|\nu\rangle}[X = \lambda_n] = \text{Tr} \{ \rho_{\nu} P_{\lambda_n} \} = |\langle n|\nu\rangle|^2.$$
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In any pure state $|v\rangle$, we have

$$P_{|v\rangle}[X = \lambda_n] = \text{Tr} \{ \rho_v P_{\lambda_n} \} = |\langle n|v\rangle|^2.$$
The variance of an observable $X$ in state $\rho$ is defined

$$\text{Var}_\rho[X] = \mathbb{E}_\rho[X^2] - (\mathbb{E}_\rho[X])^2.$$ 

Unlike the classical situation, in any state $\rho$ there will be observables with non-zero variance.

e.g. If in the Stern-Gerlach experiment, the state is $|\uparrow_z\rangle$, then

$$\text{Var}_{|\uparrow_z\rangle}(\sigma_z) = 0 \text{ and } \text{Var}_{|\uparrow_z\rangle}(\sigma_x) = 1/4.$$
Suppose our classical system dynamics are deterministic and reversible. i.e. the equation

\[ \dot{\omega} = f(\omega), \quad \omega(t) \in \Omega. \]

satisfies sufficient (smoothness) conditions such that \( x(0) = x_0 \) determines the entire trajectory

\[ x(t) = \Gamma_t x_0, \quad t \in \mathbb{R}. \]
In our new language this implies:

\[ \text{if } \mu_0 = \xi\{x_0\} \text{ then } \mu_t = \xi\{\Gamma_t x_0\}. \]

That is, the system evolution is given by a one-parameter group \( U(t) \), of automorphisms on the set of extreme points of \( S \):

\[ U(t) : \xi\{x_0\} \mapsto \xi\{\Gamma_t x_0\}. \]
Schrödinger’s equation

- Schrodinger’s equation governs the reversible, deterministic evolution of (closed) quantum systems.

- The group of automorphisms $U(t)$ map pure state $|\psi_0\rangle \in \mathcal{H}$ to pure state $|\psi_t\rangle = U(t) |\psi_0\rangle$, $t \in \mathbb{R}$.

- A theorem of Wigner tells us that $U(t)$ must be a unitary group.

- Stone-von Neumann theorem tells us that $U(t)$ must be skew-adjoint. i.e.

  \[ \frac{d}{dt} |\psi_t\rangle = -iH |\psi_t\rangle \]

  with $H$ self-adjoint, called the system Hamiltonian.

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\textsuperscript{7}Assuming certain smoothness conditions
Associated with any quantum system is a Hilbert space $\mathcal{H}$.

The state of a quantum system is a quantum probability $\rho$ that maps observables $X$ to $\mathbb{E}_\rho[X]$.

Measuring an observable $X \in \mathcal{B}(\mathcal{H})$ changes the state $\rho$ (non-deterministically and irreversibly) von Neumann measurement postulate.

If no measurement is performed, the state evolves deterministically and reversibly according to Schrödinger’s equation.
Experimental Setup

**Figure:** The microwave cavity QED setup with its feedback scheme (in green) (Fig. Courtesy Igor Dotsenko).

**Figure:** The microwave cavity QED setup with its feedback scheme (in green) (Fig. Courtesy Igor Dotsenko).
Let $\mathcal{H}$ be the Hilbert space of the cavity and denote by $B_1$ the unit ball in $\mathcal{H}$.

An orthonormal basis for this Hilbert space is the set

$$\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\}.$$ 

Physically, $|n\rangle \in B_1$ denotes a cavity state with exactly $n$ photons.
Notation: Some operators on $\mathcal{H}$

- The annihilation $a$, creation $a^\dagger$ and number $N$ operators are
  \[ a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle. \]

- The displacement operator
  \[ D_\alpha = \exp(\alpha^*a - \alpha a^\dagger) \]
  injects light of amplitude $\alpha$ into the cavity.

- The measurement operators are defined
  \[ M_g|n\rangle = e^{i(\phi + \pi)/2} \sin(n\Phi_0 - \phi/2)|n\rangle, \]
  \[ M_e|n\rangle = -e^{-i\phi/2} \cos(n\Phi_0 - \phi/2)|n\rangle. \]

Note that $M_g$ and $M_e$ are diagonal on the number basis $\{|n\rangle\}$. 
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Note that $M_g$ and $M_e$ are diagonal on the number basis $\{|n\rangle\}$. 
We split our system dynamics into two parts:

1. Non-deterministic and irreversible collapse due to measurements of the atomic state.
2. Deterministic unitary evolution of the collapsed cavity state until measurement of the next atom.
Let $|\psi_k\rangle \in B_1$ be the cavity state before interacting with the $k^{th}$ atom.

After the $k^{th}$ atom interacts with the cavity it is measured in either the ground ($g$) or excited ($e$) state.

After this collapse due to measurement, the cavity state is

$$|\psi_{k+1/2}\rangle = \begin{cases} 
\mathcal{M}_g(|\psi_k\rangle) \triangleq \frac{M_g|\psi_k\rangle}{\|M_g|\psi_k\rangle\|}, & \text{if atom detected in } g, \\
\mathcal{M}_e(|\psi_k\rangle) \triangleq \frac{M_e|\psi_k\rangle}{\|M_e|\psi_k\rangle\|}, & \text{if atom detected in } e
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and

$$P_g(|\psi_k\rangle) = \|M_g|\psi_k\rangle\|^2 \text{ and } P_e(|\psi_k\rangle) = \|M_e|\psi_k\rangle\|^2.$$
System dynamics: Non-deterministic part

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and

$$P_g(|\psi_k\rangle) = \|M_g|\psi_k\rangle\|^2 \text{ and } P_e(|\psi_k\rangle) = \|M_e|\psi_k\rangle\|^2.$$
After knowing outcome $g$ or $e$ the controller can set the parameter $\alpha$, so that

$$|\psi_{k+1}\rangle = D_{\alpha_k} |\psi_{k+1/2}\rangle.$$
The sequence $|\psi_1\rangle, |\psi_2\rangle, \ldots$, is a (classical) Markov chain:

$$|\psi_{k+1}\rangle = D_{\alpha_k} M_{s_k}(|\psi_k\rangle)$$

where $s_k = g$ or $e$ with probability $P_g(|\psi_k\rangle)$ and $P_e(|\psi_k\rangle)$.

Note that the state space of this Markov chain is the set of pure states and each pure state is an extreme point in the set of quantum probabilities.
Control Goal: The control goal is to stabilize the cavity state near the state $|\bar{n}\rangle$. In this state $E_{|\bar{n}\rangle}[N] = \bar{n}$ and $\text{Var}_{|\bar{n}\rangle}[N] = 0$.

How?

- Use repeated measurements and (non-deterministically) change the state of the system through collapse.
- Use the deterministic part by injecting light of amplitude $|\alpha|$ that is in phase or out of phase with the light in the cavity.
Control Goal: The control goal is to stabilize the cavity state near the state $|\bar{n}\rangle$. In this state

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- Use the deterministic part by injecting light of amplitude $|\alpha|$ that is in phase or out of phase with the light in the cavity.
Theorem (Kushner)

If $V \geq 0$ is a continuous function on $B_1$ such that

$$\mathbb{E} \left[ V(\left| \psi_{k+1} \right\rangle) \left| \psi_k \right\rangle \right] \leq V(\left| \psi_k \right\rangle)$$

and if the limit set

$$\Omega_\infty = \{ |\psi\rangle : |\psi_{k_n}\rangle \to |\psi\rangle \}$$

then for all $|\psi\rangle \in \Omega_\infty$ we have

$$K(\left| \psi \right\rangle) \triangleq \mathbb{E} \left[ V(\left| \psi_{k+1} \right\rangle) \left| \psi_k \right\rangle = |\psi\rangle \right] - V(\left| \psi \right\rangle) = 0.$$
The controller design comes down to choosing the right $V$ and feedback $\alpha_k$ such that:

- The Lyapunov function $V$ is a super-martingale.

- The trajectories $|\psi_k\rangle$, $k = 1, 2, \ldots$ stay in some compact subset of $B_1$ with high probability.
  - This ensures the limit set $\Omega_\infty$ is non-empty.

- The condition $K(|\psi\rangle) = 0$ implies $|\psi\rangle = |\bar{n}\rangle$. 
Existence of Open-loop Martingales

Recall that in state $|\psi\rangle$

$P_{|\psi\rangle}[N = n] = \mathbb{E}_{|\psi\rangle}[P_n] = |\langle n|\psi\rangle|^2$

is the probability that there are $n$ photons in the cavity.

The fact that $M_g$, $M_e$ and $N$ are co-diagonal$^8$ implies the existence of open-loop martingales.

That is, for any $n$ the function

$|\psi\rangle \mapsto |\langle n|\psi\rangle|^2$

is a martingale and satisfies

$\mathbb{E} \left[ |\langle n|\psi_{k+1}\rangle|^2 \bigg| \psi_k, \alpha_k = 0 \right] = |\langle n|\psi_k\rangle|^2.$

$^8$This condition is referred to as the quantum non-demolition condition.

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Existence of Open-loop Martingales

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\mathbb{E}\left[|\langle n|\psi_{k+1}\rangle|^2 \mid \psi_k\right], \alpha_k = 0] = |\langle n|\psi_k\rangle|^2.
\]

\(^8\)This condition is referred to as the quantum non-demolition condition.
Central idea of the controller

- The Lyapunov function $V$ we choose is a concave function of the open-loop martingales.

- Therefore $V(|\psi_k\rangle)$ is an open-loop supermartingale (when $\alpha_k = 0$).

- Choose feedback $\alpha_k$ to minimize $V(|\psi_{k+1}\rangle)$ for a given $|\psi_k\rangle$:

$$\alpha_k = \arg\min_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(|\psi_{k+1}\rangle)| |\psi_k\rangle].$$

- With this feedback $V$ is a closed-loop supermartingale: because $0 \in [-\bar{\alpha}, \bar{\alpha}]$ the chosen $\alpha_k$ with always perform at least as good (on average) as choosing $\alpha_k = 0$. 

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The form of the Lyapunov function we choose is

$$V(|\psi\rangle) = \delta V_F(|\psi\rangle) + \sum_{n} \sigma_n |\langle n|\psi \rangle |^2.$$

Here, $\delta$ is some small constant and

$$V_F(|\psi\rangle) = 1 - \sum_{n} |\langle n|\psi \rangle |^4.$$

The $V_F$ term ensures convergence to a Fock state.

The $\sigma_n$ are a sequence of non-negative numbers that ensure:

- The trajectories remain in a compact set with high probability ($\sigma_n \to \infty$).
- The system does not converge to open loop attractors $|n\rangle$ for $n \neq \bar{n}$. 

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Theorem

Given any constants $C, \epsilon > 0$, there exist constants $\bar{\alpha}, \delta$ and $\sigma_0, \sigma_1, \ldots$ such that for all $|\psi_1\rangle$ satisfying $V(|\psi_1\rangle) \leq C$ we have

$$|\psi_k\rangle \rightarrow |\bar{n}\rangle \quad \text{with probability greater than} \quad 1 - \epsilon$$

if we use the feedback

$$\alpha_k = \arg\min_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(|\psi_{k+1}\rangle) | |\psi_k\rangle]$$

where,

$$V(|\psi\rangle) = \delta V_F(|\psi\rangle) + \sum_n \sigma_n |\langle n|\psi \rangle|^2.$$
Firstly note that if there is no feedback and if \( |\psi_k\rangle = |n\rangle \) then
\[
|\psi_{k+1}\rangle = |\psi_k\rangle = |n\rangle,
\]
because
\[
M_g |n\rangle = e^{i(\phi+\pi)/2} \sin(n\Phi_0 - \phi/2) \; |n\rangle,
\]
\[
M_e |n\rangle = -e^{-i\phi/2} \cos(n\Phi_0 - \phi/2) \; |n\rangle.
\]
So \( |n\rangle \) is a stationary state of this dynamical process.
Why does this feedback work?

Also, for $|\psi_k\rangle = |n'\rangle$,

$$P[s_k = g] = \| M_g |n'\rangle \|^2 = \sin^2(n'\Phi_0 - \phi/2)$$ and

$$P[s_k = e] = \| M_e |n'\rangle \|^2 = \cos^2(n'\Phi_0 - \phi/2).$$

So by repeatedly getting atoms to interact with the cavity we can estimate $P[s_k = g]$ and $P[s_k = e]$.

Using this information, we can calculate $n'$ if $2\pi$ is not a multiple of $\Phi_0$.

That is, repeated measurements of atomic state is equivalent to a measurement of the number operator $N$ of the cavity.
Why does this feedback work?

- Now even if the state is not $|n'\rangle$ for some $n'$, repeated measurements of the atomic state is still equivalent to a measurement of $N$.

- Therefore, the probability measure of the system collapses gradually to a state that is an eigenstate $|n'\rangle$ of $N$.

- We simply use the weight factors $\sigma_n$ and feedback $\alpha_k$ to ensure that we move towards the correct eigenstate $|\bar{n}\rangle$. 
Proof outline

We show:

1. $V(|\psi_k\rangle)$ is a supermartingale.
2. This implies that the trajectories are in a sequentially compact set with high probability.
3. The trajectories only converge to a Fock state.
4. The trajectories converge to the correct Fock state $|\bar{n}\rangle$ with high probability.
Step 1: $V$ is a supermartingale

Some simple algebra shows that

$$
\mathbb{E}[V(|\psi_{k+1}\rangle)|\psi_k\rangle = |\psi\rangle] - V(|\psi\rangle)
= \min_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(D_{\alpha}M_{s_k}|\psi\rangle)] - \mathbb{E}[V(D_0M_{s_k}|\psi\rangle)]
+ \mathbb{E}[V(D_0M_{s_k}|\psi\rangle)] - V(|\psi\rangle)
$$

$$
= \min_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(D_{\alpha}M_{s_k}|\psi\rangle)] - \mathbb{E}[V(D_0M_{s_k}|\psi\rangle)]
- \delta \sum_n |\langle n|\psi\rangle|^2 \left( \cos^2(n\phi_0 + \phi_1) - \|M_g|\psi\rangle\|^2 \right)^2
\|M_g|\psi\rangle\|^2 \|M_e|\psi\rangle\|^2
\leq 0.
$$
Step 2: Compactness of trajectories

Because $\mathbb{E}[V(|\psi_k\rangle)] \leq V(|\psi_1\rangle)$, for any $\nu > 0$,

$$\mathbb{P}[V(|\psi_k\rangle) > V(|\psi_1\rangle)/\nu] \leq \nu \cdot V(|\psi_1\rangle).$$

We choose for large $n$, $\sigma_n \approx O(\log(n))$.

Then with probability greater than $1 - C\nu$ the trajectories stay in the set

$$\{|\psi\rangle : V(|\psi\rangle) \leq C/\nu\}$$

which is compact.
Step 3: Convergence to Fock states

- We can now apply Kushner’s theorem:

\[ |\psi_k\rangle \to \{ |\psi\rangle : \mathbb{E}[V(|\psi_{k+1}\rangle)|\psi_k\rangle = |\psi\rangle] - V(|\psi\rangle) = 0 \} . \]

- That is \( |\psi\rangle \) satisfies

\[ \sum_{n=0}^{n_{\text{max}}} |\langle n|\psi\rangle|^2 \left( \cos^2(n\phi_0 + \phi_1) - \|M_g|\psi\rangle\|^2 \right)^2 \frac{\|M_g|\psi\rangle\|^2 \|M_e|\psi\rangle\|^2}{\|M_g|\psi\rangle\|^2 \|M_e|\psi\rangle\|^2} = 0 \]  

(1)

\[ \min_{\alpha \in [-\bar{\alpha},\bar{\alpha}]} \mathbb{E}[V(D\alpha M_{s_k}|\psi\rangle)] - \mathbb{E}[V(M_{s_k}|\psi\rangle)] = 0 \]  

(2)

- The first equality is satisfied only for \( |\psi\rangle = |n\rangle \) for some \( n \).
- So \( |\psi_k\rangle \) converges to some Fock state.
Step 4: Convergence to $|\bar{n}\rangle$

- We choose

$$
\sigma_n = \begin{cases} 
\frac{1}{8} + \sum_{k=1}^{\bar{n}} \frac{1}{k} - \frac{1}{k^2}, & \text{if } n = 0 \\
\sum_{k=n+1}^{\bar{n}} \frac{1}{k} - \frac{1}{k^2}, & \text{if } 1 \leq n < \bar{n} \\
0, & \text{if } n = \bar{n} \\
\sum_{k=\bar{n}+1}^{n} \frac{1}{k} + \frac{1}{k^2}, & \text{if } n > \bar{n}
\end{cases}
$$

(3)

- There is an algorithmic manner in which the $\sigma_n$ can be chosen.
Step 4: Convergence to $|\bar{n}\rangle$

- The factors $\sigma_n$ in the Lyapunov function are chosen so that,

$$\frac{d^2}{d\alpha^2} V(D_\alpha |m\rangle) + O(\delta) < 0.$$

for $m \neq \bar{n}$.

- That is $\alpha = 0$ is not a local minimum for $V(D_\alpha |m\rangle)$ and hence for $|\psi\rangle$ close to $|m\rangle$,

$$\min_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \mathbb{E}[V(D_\alpha M_{s_k} |\psi\rangle)] - \mathbb{E}[V(M_{s_k} |\psi\rangle)] < 0.$$  

- So if $|\psi_k\rangle$ is near $|m\rangle$ we can find a $\alpha_k \in [-\bar{\alpha}, \bar{\alpha}]$ to push the cavity state $|\psi_{k+1}\rangle$ away from $|m\rangle$.

- Therefore $|\psi_k\rangle$ converges to $|\bar{n}\rangle$. 
Experimental Results

Figure: Cavity state as a function of time

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QND measurements imply the existence of open-loop martingales.

Use concave functions of these open-loop martingales to construct closed-loop supermartingales.

Increasingly penalize high photon number states ($\sigma_n \rightarrow \infty$) to ensure compactness of trajectories.
In this talk I consider the control of a cavity using an ideal sequence of measurements with no delays in the feedback loop.

This approach can be generalized to realistic situations with faulty detections and feedback delays using predictive filters.

Also, we can consider a more general situation, where the measurement operators are not fixed but are functions of the control:

\[ M_{\alpha k}^{\alpha k} \]

is the measurement operator.
Experimental setup at Kastler Brossel Laboratory

Controller Design-