AN INTRODUCTION TO THE MOMENT APPROACH FOR OPTIMAL CONTROL

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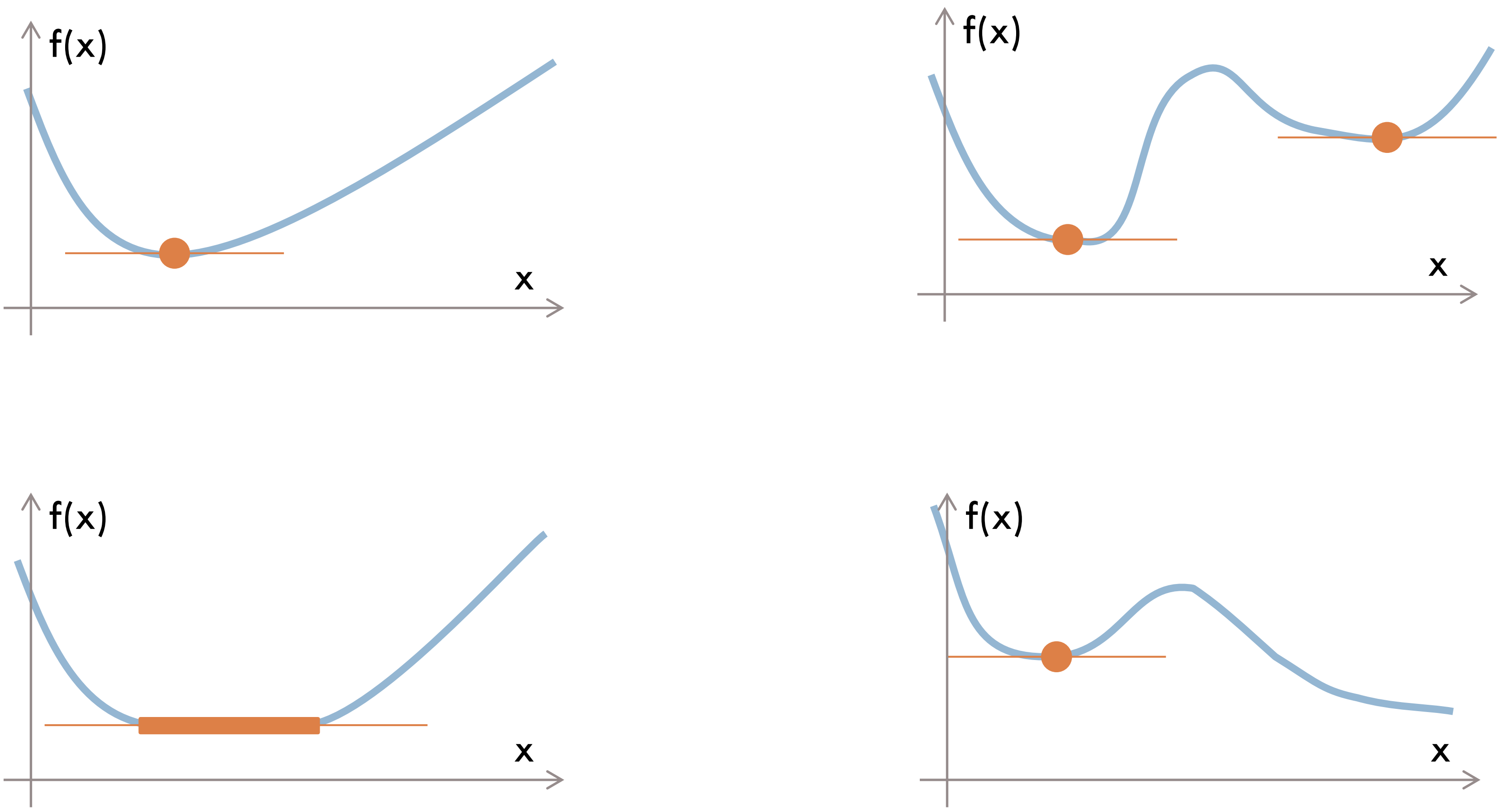
This presentation

- Compute lower bounds on the cost of optimal control problems
  - For a large class of problems
  - Systematically
  - On a computer

- Solve the problem
  - Sometimes only!

The idea



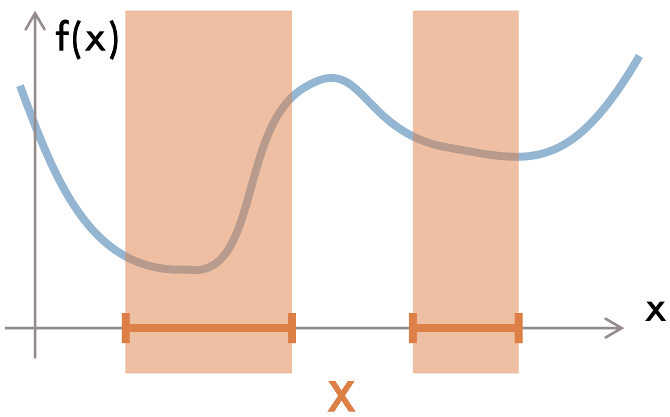
Presentation outline

1. Introduction
2. Static optimization
3. Optimal control: impulsive controls
4. Perspectives

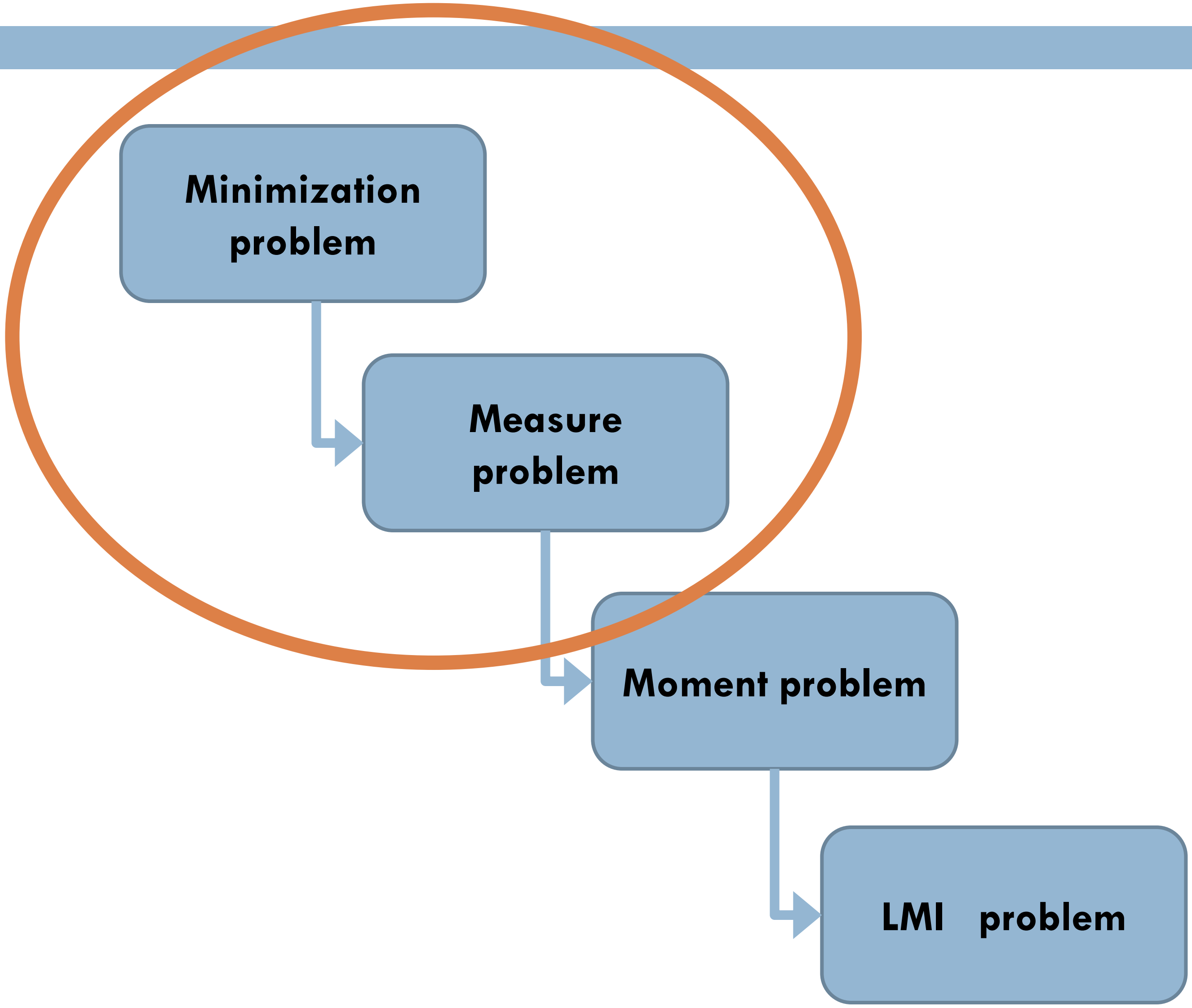
Static optimization

$$J = \inf_x f(x) \quad \text{such that} \quad x \in \mathbf{X} \subset \mathbb{R}^n$$

$$g_1(x) \geq 0$$
$$g_2(x) \geq 0$$
$$\ldots$$



The moment approach



Measures: geometric viewpoint

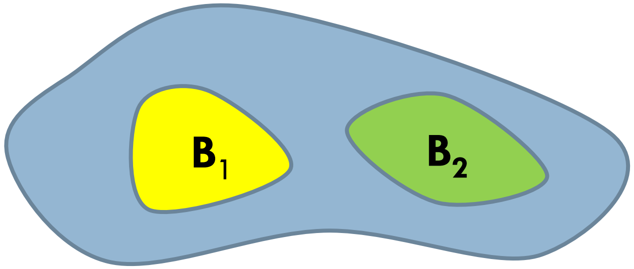
- Measure space $(\mathbf{X}, \Sigma, \mu)$

- Notation $\mu(\mathbf{B})$ or $\int_{\mathbf{B}} 1 \, \mathrm{d}\mu(x)$

- Properties $\mu(\emptyset) = 0$

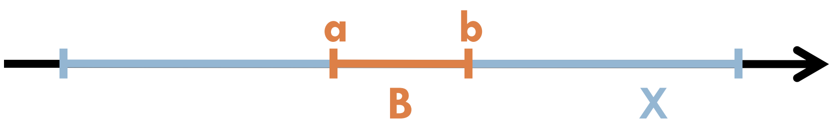
$$\mathbf{B} \in \mathcal{B} \implies \mu(\mathbf{B}) \geq 0$$

$$\mathbf{B}_i \text{ disjoints} \implies \mu(\mathbf{B}_1 \cup \mathbf{B}_2 \cup \ldots) = \mu(\mathbf{B}_1) + \mu(\mathbf{B}_2) + \ldots$$



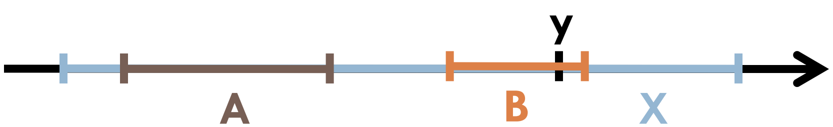
A few measures

- Lebesgue measure (λ, dx)



$$\lambda(\mathbf{B}) = b - a$$

- Dirac measure at y (δ_y)



$$\delta_y(\mathbf{A}) = 0$$

$$\delta_y(\mathbf{B}) = 1$$

- Probality measure

$$\mu(\mathbf{X}) = 1$$

Measures: relation with integration

- The Lebesgue integral $\int_{\mathbf{X}} f(x)\, d\lambda(x)$ is the completion of its Riemann counterpart $\int_{\mathbf{X}} f(x)\, dx$

- Integration w.r.t. Dirac measure:

$$\int_{\mathbf{X}} f(x)\, d\delta_{x^*}(x) = f(x^*)$$

Measures: functional analysis viewpoint

- Let :
  - $C(X)$, bounded continuous functions, $||f|| = \sup_{x \in X} f(x)$
  - $\mathcal{M}(X)$, sufficiently regular measures

- Riesz: then positive linear functionals on $C(X)$ "are" $\mathcal{M}(X)$

- We note $< f, \mu >= \int_{\mathbf{X}} f(x) \, d\mu(x)$

Towards a measure problem

$$J = \inf_x f(x)$$

$$x \in \mathbf{X}$$

$$\Longleftrightarrow$$

$$J_{meas} = \inf_\mu \int_{\mathbf{X}} f(x)\, \mathrm{d}\mu(x)$$

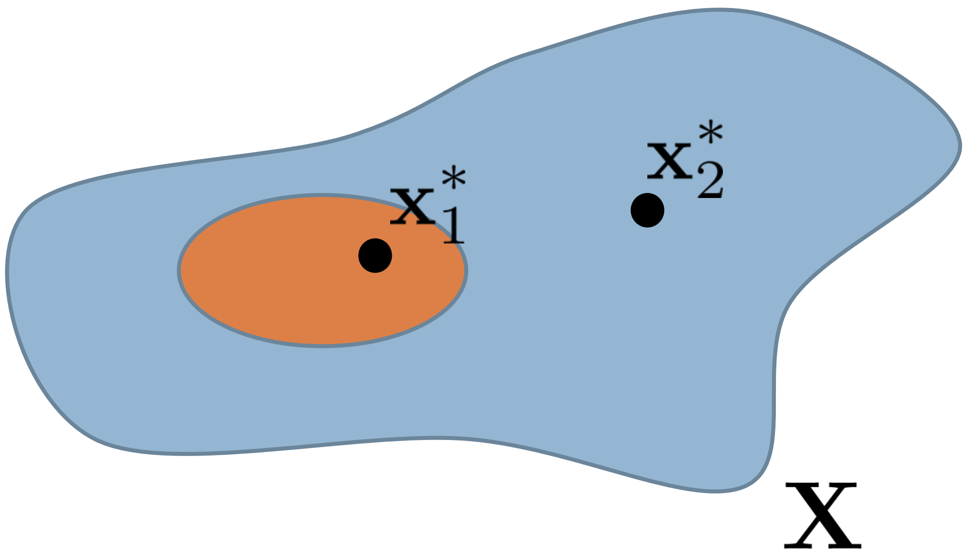
$$\mathrm{supp}(\mu) = \mathbf{X}$$

$$\mu(\mathbf{X}) = 1$$

- $J \geq J_{meas}$: take $\mu = \delta_x$
- $J \leq J_{meas}$: by definition $J \leq f(x), \quad \forall x \in X$

  then $\int_X J\, d\mu(x) \leq \int_X f(x)\, d\mu(x)$

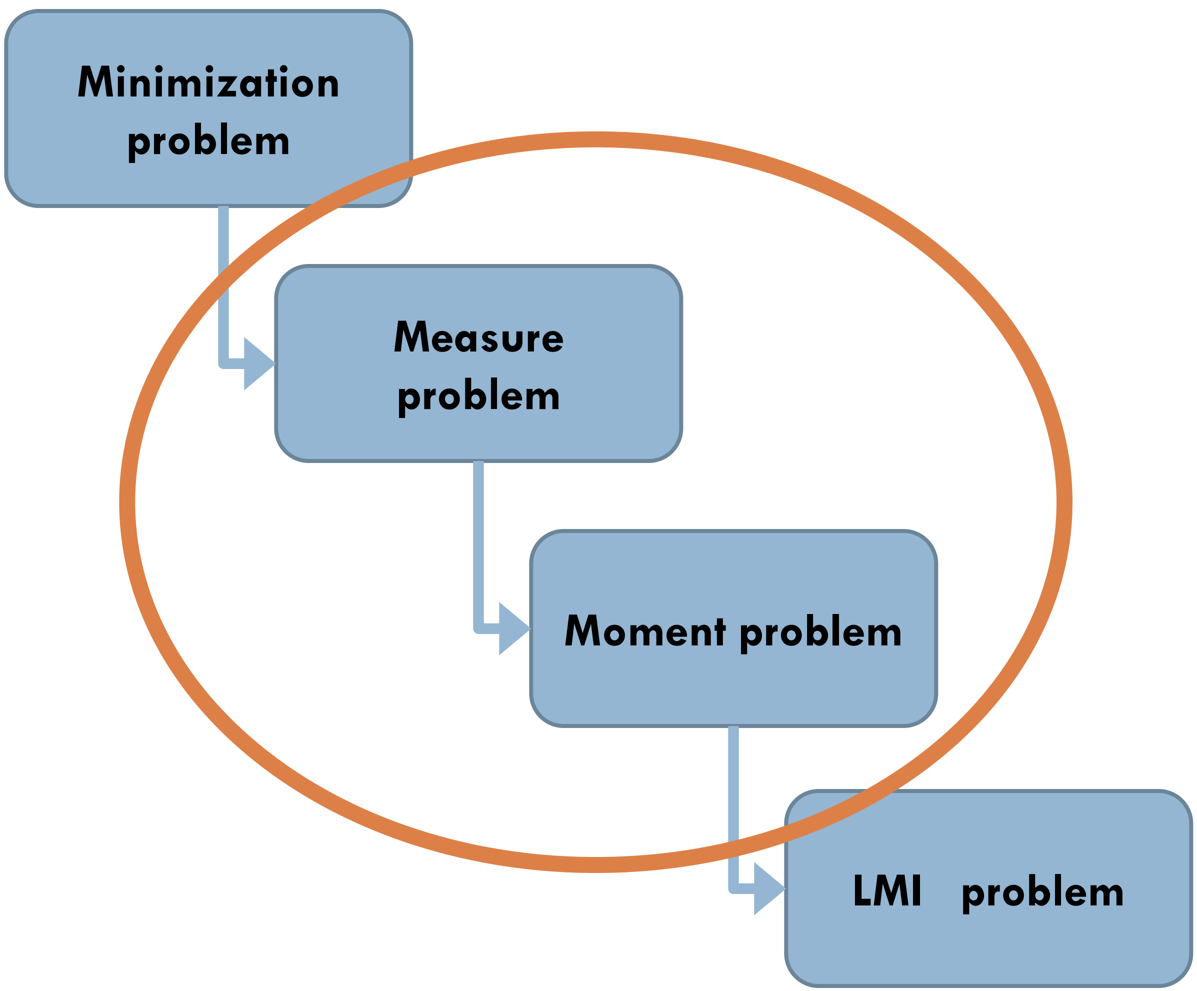
What does it measure?



Properties of the measure problem

- The problem is linear!
  - The set of admissible solutions is convex
  - The set of optimal solutions is convex
  - The infimum is a minimum!

The moment approach



Moments

- Moments $y_\alpha = \int_{\mathbf{X}} x^\alpha \, \mathrm{d}\mu(x)$

- Moment sequence $(y) = (y_0, y_1, y_2, \ldots)$

- Moment matrix $M(y) = \begin{bmatrix} y_0 & y_1 & y_2 & \cdots \\ y_1 & y_2 & y_3 & \\ y_2 & y_3 & y_4 & \\ \vdots & & & \ddots \end{bmatrix} = \int_{\mathbf{X}} \begin{bmatrix} 1 & x & x^2 & \cdots \\ x & x^2 & x^3 & \\ x^2 & x^3 & x^4 & \\ \vdots & & & \ddots \end{bmatrix} \mathrm{d}\mu(x)$

- Localizing matrix at e.g. $4 - x^2$

$$M_{4-x^2}(y) = \int_{\mathbf{X}} \begin{bmatrix} 1 & x & \cdots \\ x & x^2 & \\ \vdots & & \ddots \end{bmatrix} (4 - x^2)\, \mathrm{d}\mu(x) = \begin{bmatrix} 4y_0 - y_2 & 4y_1 - y_3 & \cdots \\ 4y_1 - y_3 & 4y_2 - y_4 & \\ \vdots & & \ddots \end{bmatrix}$$

Properties of moments matrices

- $M(y) \succeq 0$ $M(hy) \succeq 0$

Proof: $$0 \leq \int p^2 d\mu = \int (\sum a_i x^i)^2 d\mu = a' M a$$

- Is the converse true?

The converse

- Let $\mathbf{X} := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \quad i = 1, \ldots, m\}$

μ is supported on $\mathbf{X}$

$$\Longleftrightarrow$$

$$M(y) \succeq 0, \qquad M_{g_i}(y) \succeq 0 \quad \forall i$$

- Proof: Riesz-Haviland + Putinar's Positivstellensatz

From a measure to a moment problem

$$J_{meas} = \inf_{\mu} \int_{\mathbf{X}} f(x)\, d\mu(x)$$

such that

$$\mathrm{supp}(\mu) = \mathbf{X}$$

$$\int_{\mathbf{X}} 1\, d\mu(x) = 1$$

$$\Longleftrightarrow$$

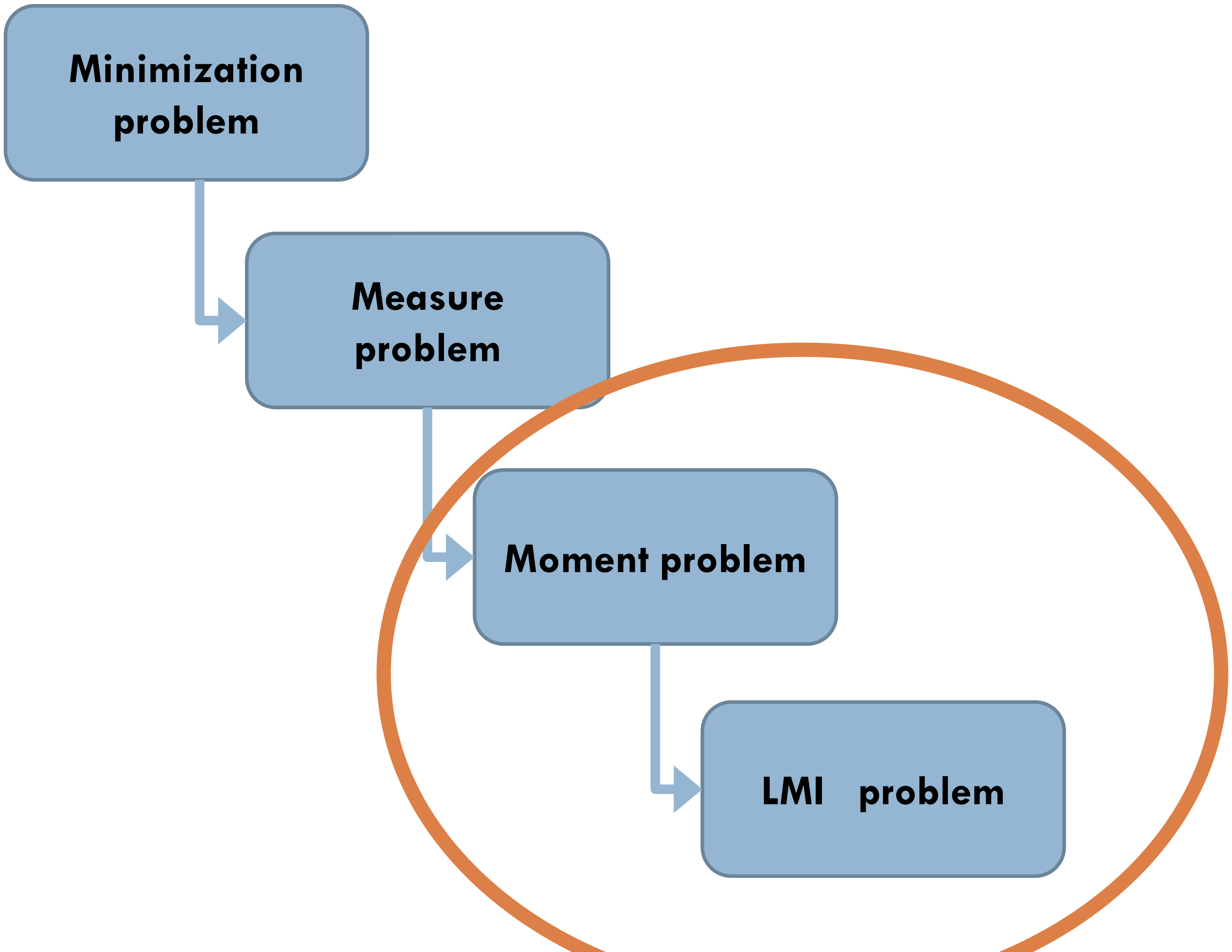
$$J_{mom} = \inf_{(y)} c \cdot y$$

such that

$$M(y) \succeq 0 \quad M(a_i\, y) \succeq 0$$

$$y_0 = 1$$

The moment approach



From a moment to a LMI problem

- (Lasserre) relaxations: optimize only on moments of order up to 2r

- $$J^1_{mom} \leq J^2_{mom} \leq \ldots \leq J^{r\to\infty}_{mom} = J_{mom}$$

- If at some relaxation

$$\mathrm{rank}\left(M_{j-1}\right) = \mathrm{rank}\left(M_j\right) = k$$

Then μ is k-atomic and we can extract its k support points

A simple example

$$J = \inf_x (x-1)^2$$
$$\text{s.t.} \quad 4 - x^2 \geq 0$$

Minimization problem

$$x^* = 1$$

$$J_{meas} = \inf_{\mu} \int_{\mathbf{X}} f(x)\, \mathrm{d}\mu(x)$$
$$\text{s.t.} \quad \mathrm{supp}(\mu) = \mathbf{X}, \quad \mu(\mathbf{X}) = 1$$

Measure problem

$$\mu^*(x) = \delta_1(x)$$

$$J_{mom} = \inf_{(y)} \; y_2 - 2y_1 + y_0$$
$$\text{s.t.} \;\; M(y) \succeq 0, \quad M((4-x^2)\,y) \succeq 0, \quad y_0 = 1$$

Moment problem

$$(y)^* = (1, 1, 1, \dots)$$

$$J_{mom} = \inf_{(y)} \; y_2 - 2y_1 + y_0$$
$$\text{s.t.} \quad \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0, \begin{bmatrix} 4y_0 - y_2 & 4y_1 - y_3 \\ 4y_1 - y_3 & 4y_2 - y_4 \end{bmatrix} \succeq 0, \quad y_0 = 1$$

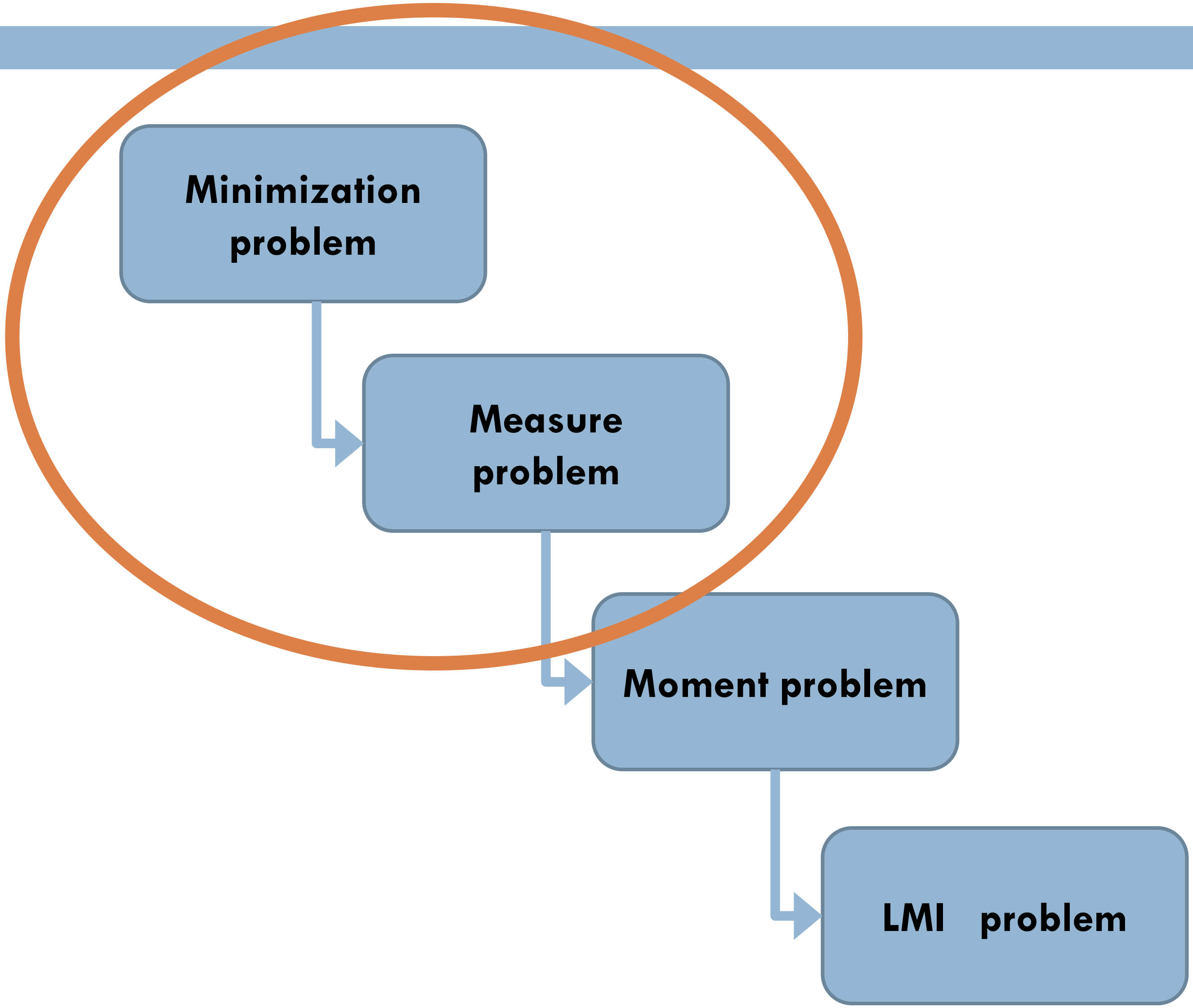
LMI problem

$$M(y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

A GloptiPoly snippet


```
>> mpol x; % define optimization variable
>> P = msdp( min((x-1)^2) , x^2 <= 4, 2); % define problem: criterion, constraint, order
>> [flag, obj] = msol(P) % solve problem

flag =
     1

obj =
  2.8868e-009

>> double(x) % extract solution(s)

ans =
    1.0000

>> double(mmat(meas(1))) % show moment matrix

ans =
    1.0000    1.0000    0.9999
    1.0000    0.9999    0.9999
    0.9999    0.9999    0.9998
```

The moment approach



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Our problem

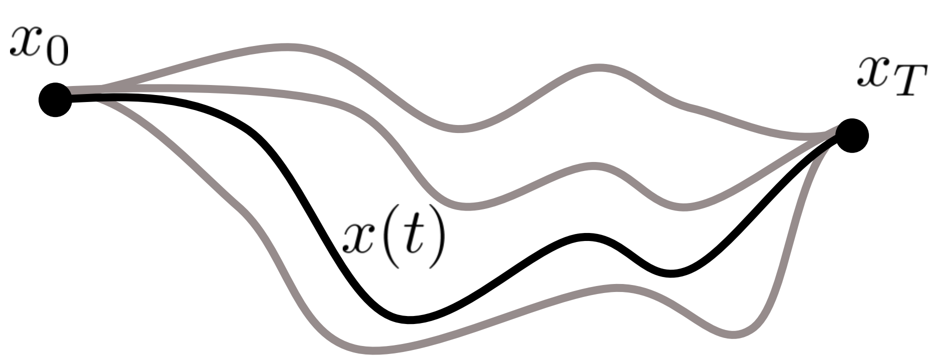
$$J = \inf_{u(t)} \int_0^T \left(h(t, x(t)) + H(t)\, u(t) \right) dt$$

such that :

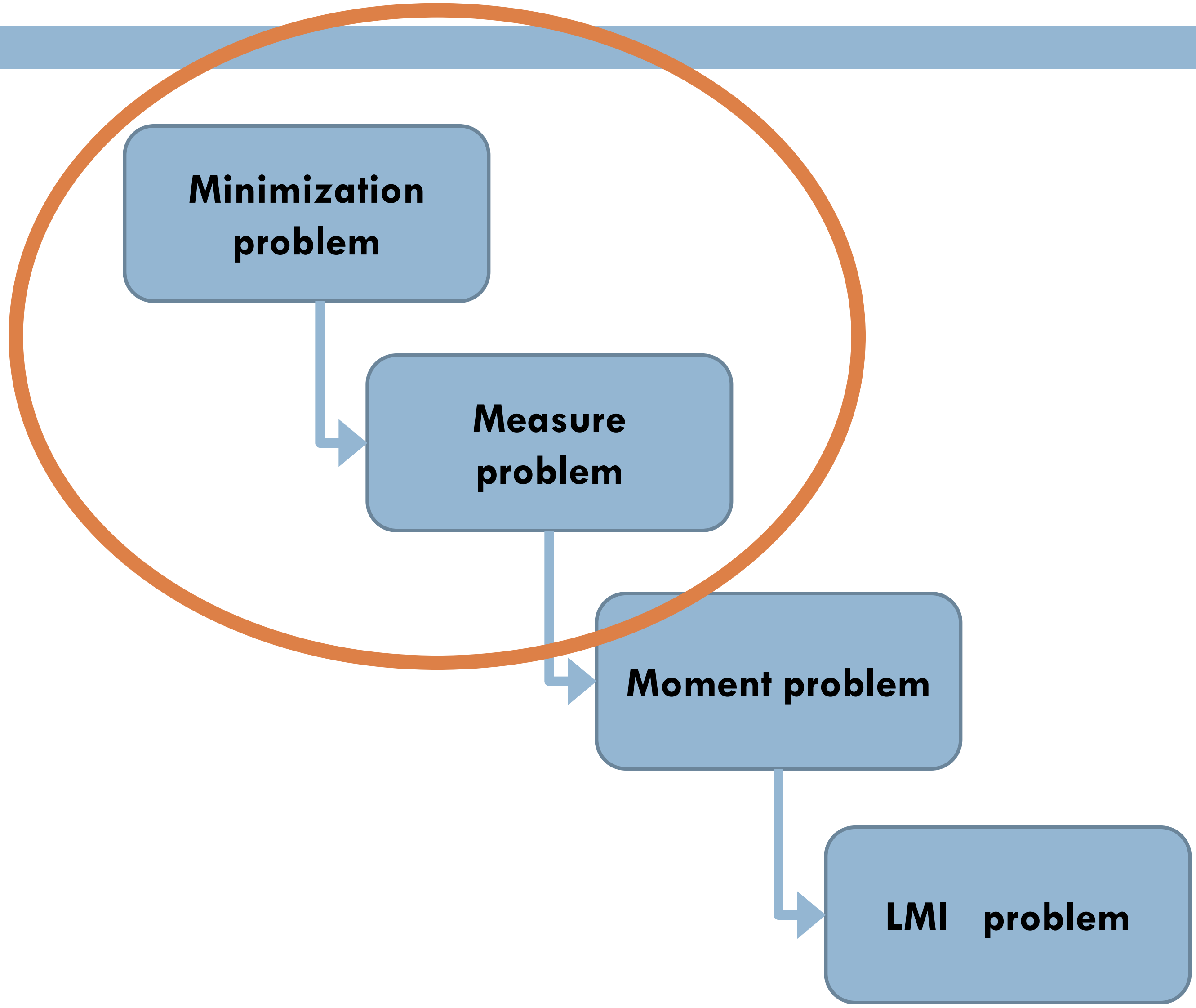
$$\dot{x}(t) = f(t, x(t)) + G(t)\, u(t)$$

$$x(0) = x_0, \quad x(T) = x_T$$

$$x(t) \in \mathbf{X}, \quad u(t) \in \mathbb{R}^m$$



The moment approach



Measure for impulses

- We embed

$$\dot{x}(t) = f(t, x(t)) + G(t)\, u(t)$$

in

$$dx(t) = f(t, x(t))dt + G(t)\, dw(t), \qquad w(t) \in BV([0, T])$$

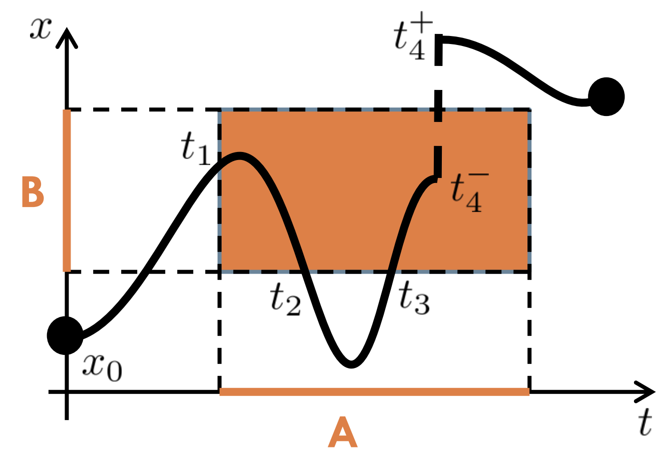
where $w(t)$ can be decomposed as

$$dw(t) = u_{L_1}(t)\, dt + \sum_{j \in J} u_{t_j} \delta_{t_j}(dt)$$

Occupation measures

- Fix $(w(t), x(t))$

- Define $\xi_t(B) = P(x(t) \in B)$



- Occupation measure:

$$\mu_{w(t),x(t)}(\mathbf{A} \times \mathbf{B}) = t_2 - t_1 \; + \; t_4 - t_3$$

- Control measure:

$$\nu_{w(t),x(t)}(\mathbf{A} \times \mathbf{B}) = w(t_2) - w(t_1) \; + \; \frac{w(t_4^+) - w(t_4^-)}{2} - w(t_3)$$

Weak formulation of dynamics

- For each continuous $v(t, x(t))$:

$$v(T, x_T) - v(0, x_0) \left(= \int_0^T dv(t, x(t)) \right) =$$

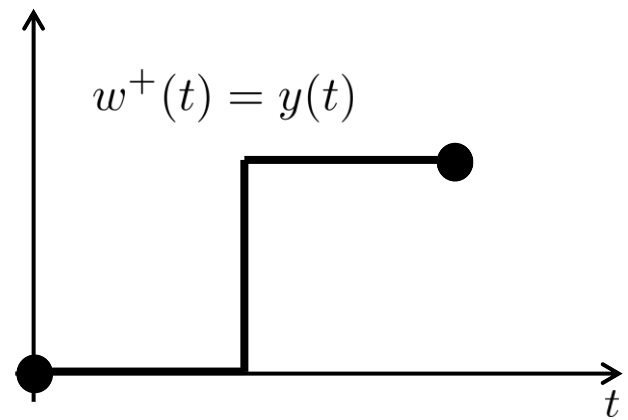
$$\int\limits_{[0,T]\times\mathbf{X}} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot f \right) d\mu_{w(t),x(t)}(t, x) + \int\limits_{[0,T]\times\mathbf{X}} \frac{\partial v}{\partial x} \cdot G \cdot d\nu_{w(t),x(t)}(t, x)$$

This is a relaxation!

$$\dot{x} = (t-1)^2 u$$
$$\dot{y} = u$$

$$u(t) \geq 0$$
$$x(t) = 0$$
$$y(0) = 0$$
$$y(2) = 1$$

$$dx = (t-1)^2 \, dw^+(t)$$
$$dy = dw^+(t)$$



A basic example (1/2)

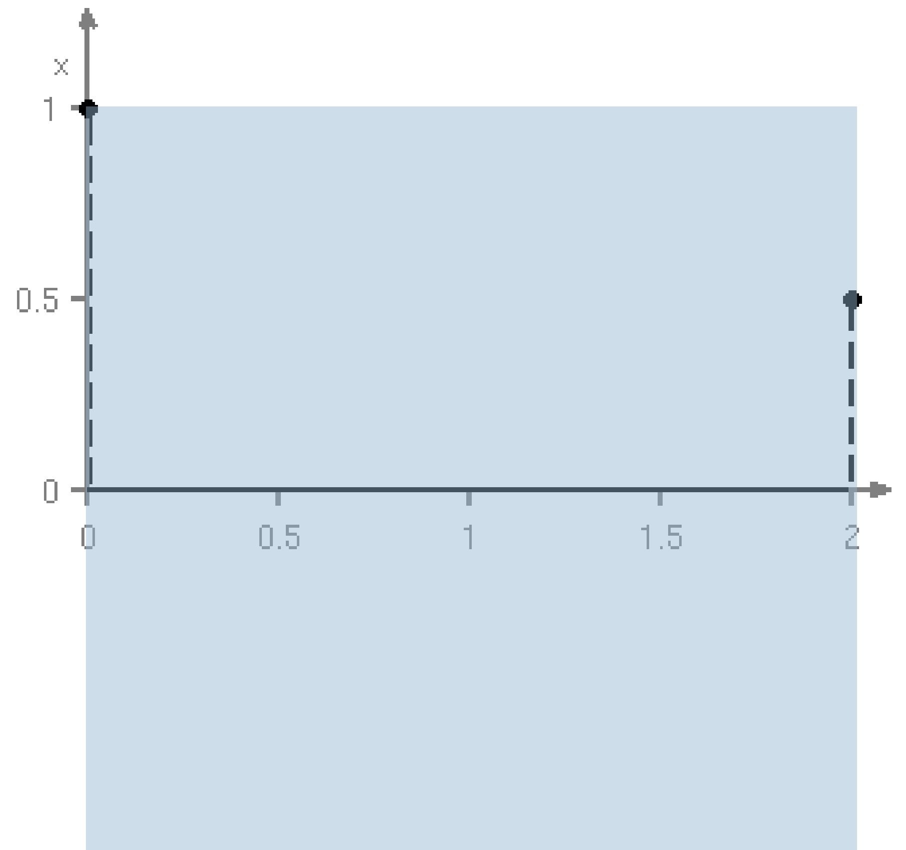
$$V = \inf_{u(t)} \int_0^2 x^2(t)\, dt$$

such that

$$\dot{x}(t) = u(t)$$

$$x(0) = 1, \quad x(2) = \frac{1}{2}$$

$$x^2(t) \leq 1.$$



A basic example (2/2)

$$V = \inf_{u(t)} \int_0^2 x^2(t)\, dt$$

such that

$$\dot{x}(t) = u(t)$$

$$x(0) = 1, \quad x(2) = \frac{1}{2}$$

$$x^2(t) \leq 1.$$

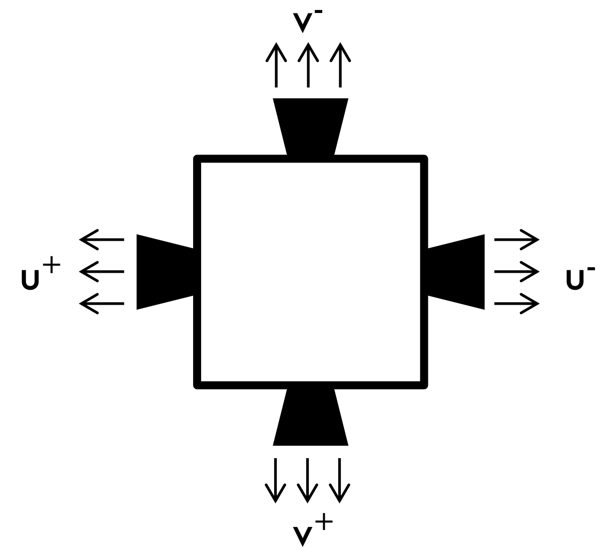
$$V_{meas} = \inf_{\mu,\nu} \int_{[0,T]\times\mathbf{X}} x^2 \,\mathrm{d}\mu$$

such that

$$v(T, x_T) - v(0, x_0) = \int_{[0,T]\times X} \frac{\partial v}{\partial t}\,\mathrm{d}\mu + \int_{[0,T]\times X} \frac{\partial v}{\partial x}\,\mathrm{d}\nu$$

$$X = \left\{x \in \mathbb{R} : 1 - x^2 \geq 0\right\}$$

Orbital rendez-vous, l_1-induced norm



$$V = \inf_{u(t),v(t)} \int_0^T d|u(t)| + \int_0^T d|v(t)|$$

such that

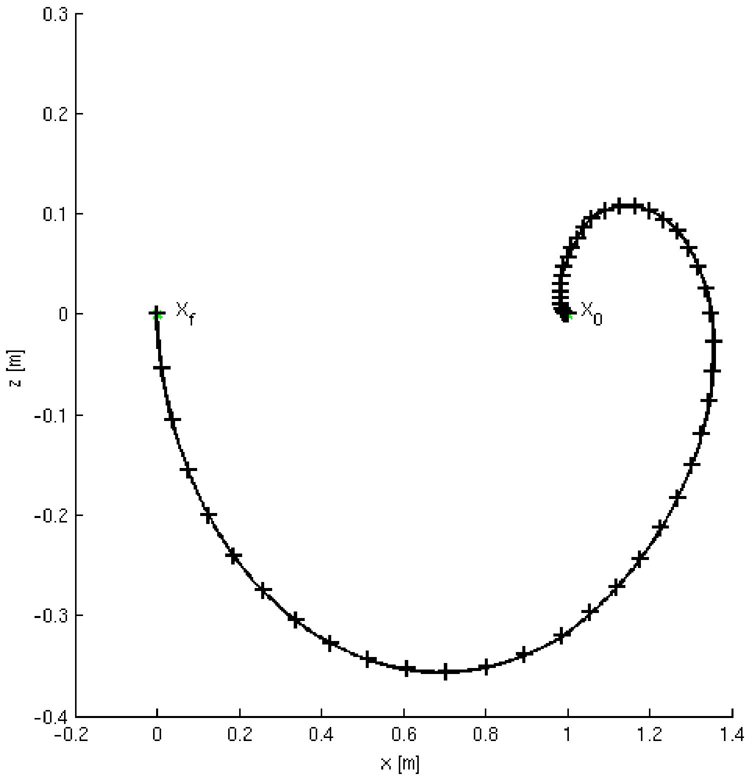
$$dx = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & -2 & 0 \end{bmatrix} x(t)\, dt + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} du(t) \\ dv(t) \end{bmatrix}$$

$$x(0) = x_0, \quad x(T) = x_f$$

On Carter's third example

$$x(0) = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}, \quad x(2\pi) = \begin{bmatrix}0 & 0 & 0 & 0.427\end{bmatrix}$$

d	V_d
1	0.0463
2	0.0680
3	0.2188
4	0.2972



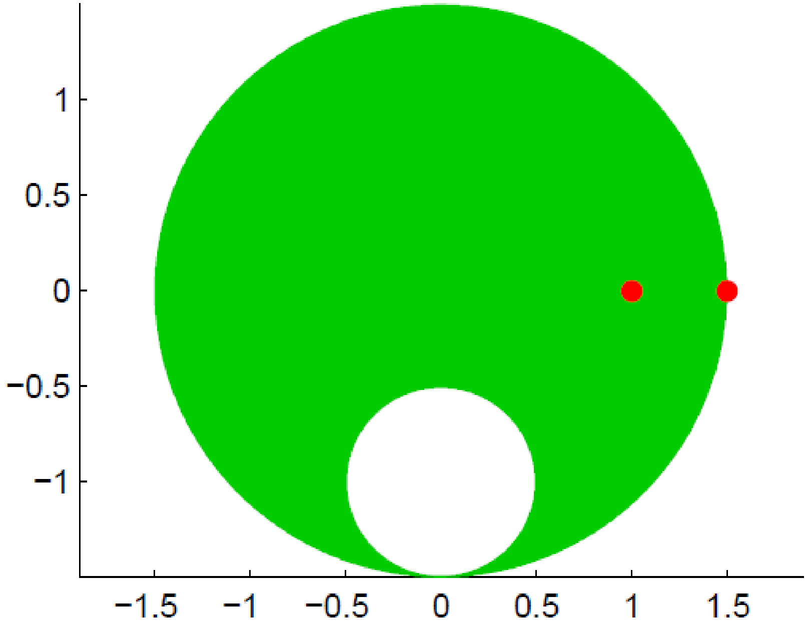
A non-convex example (1/2)

$$V = \inf_{u(t)} \int_0^T d|u(t)|$$

such that

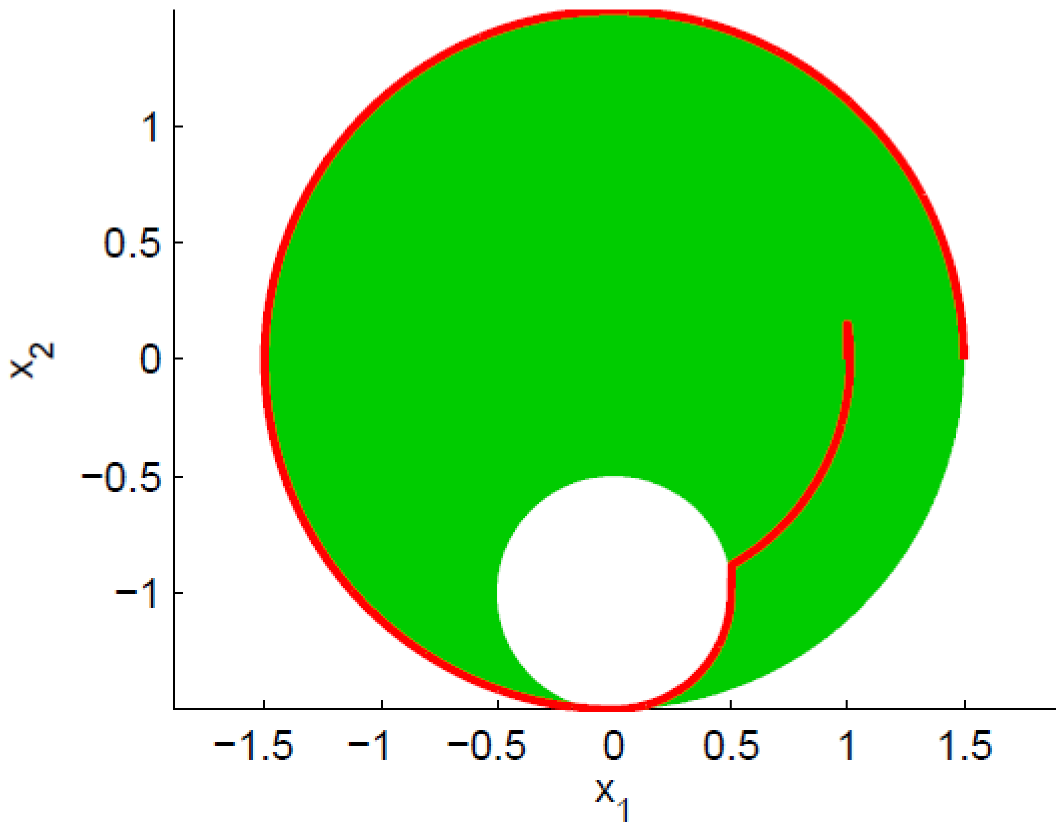
$$dx = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)\,dt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} du(t)$$

$$x(0) = x_0, \quad x(T) = x_f$$



A non-convex example (2/2)

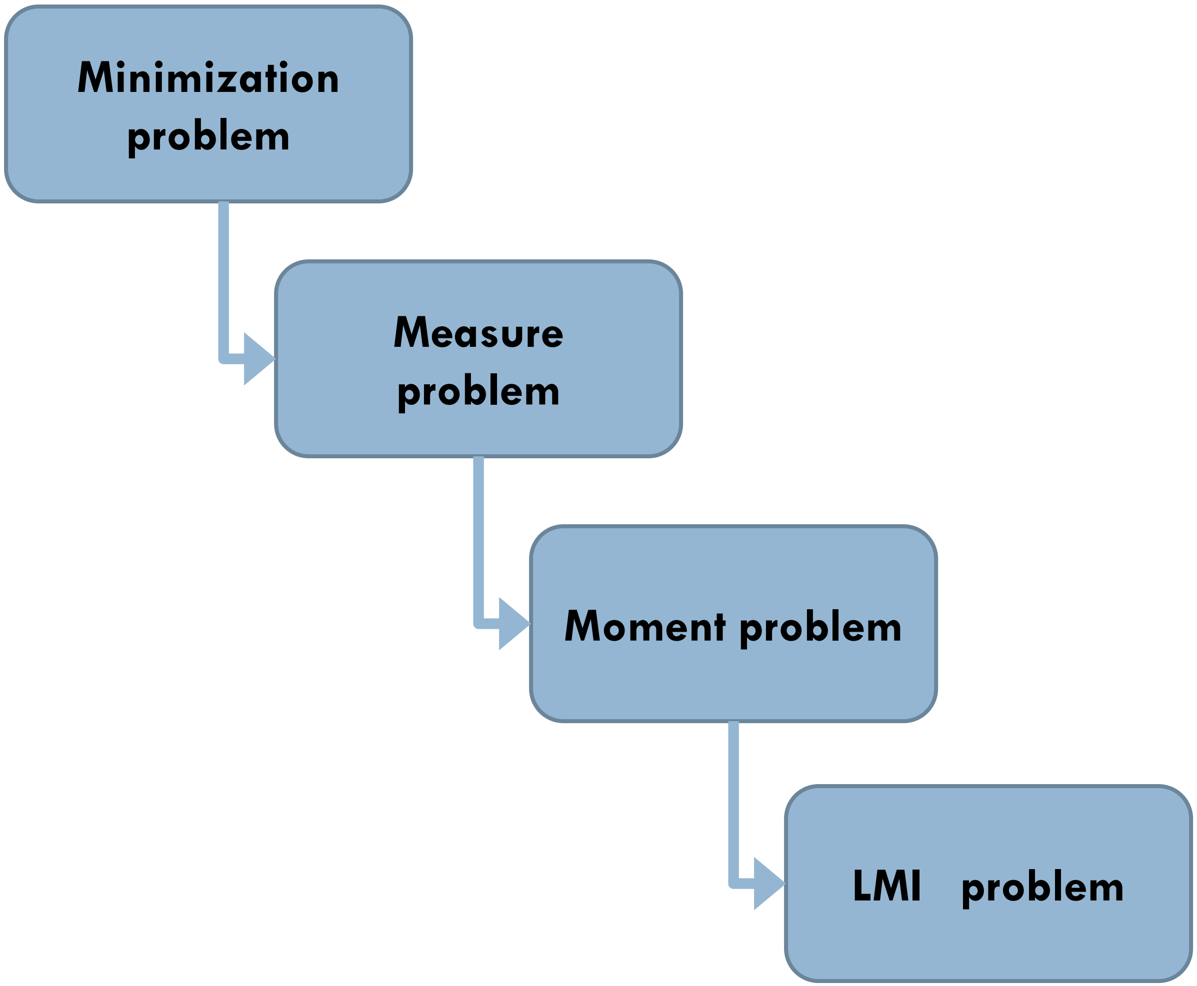
d	V_M^d
1	0.417
2	0.490
3	0.500
4	0.527
5	0.588
6	0.642
7	0.664
8	0.671
9	0.673
10	0.675



Cost = 0.682

Presentation outline

The method, yet again



Critique

+	-
Global optimality	Extraction of solutions
Lower bounds	Finite convergence ?
Certificate of infeasibility	Relies on LMI solvers

The main problem…

- # of monomials of n variables up to degree 2r:

$$\binom{n+2r}{n}$$

- Example with n=6

r	#
0	1
1	28
2	210
3	924
4	3003
5	8008
6	18564

Thank you!

- Presentation available at:

http://homepages.laas.fr/mclaeys