On the dichotomic collective behaviors of large populations of pulse-coupled firing oscillators

PhD defense

A. Mauroy
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We want to link the **macroscopic** collective behaviors to the **microscopic** properties of a single agent.
A number of phenomena correspond to networks of pulse-coupled firing oscillators (impulsive coupling).

- **firing neurons**
- **pacemaker cells of the heart**
- **flashing fireflies**
- **earthquakes**

Olami et al., PRL, vol 68, 8
We have studied the collective behavior of several models of pulse-coupled oscillators.

<table>
<thead>
<tr>
<th>Microscopic Level</th>
<th>Macroscopic Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leaky Integrate-and-Fire (LIF)</td>
<td>Synchronization (Mirollo &amp; Strogatz, 1990)</td>
</tr>
<tr>
<td>Other models of oscillators</td>
<td>??</td>
</tr>
</tbody>
</table>
Pulse-coupled oscillators frequently exhibit a remarkable **dichotomic behavior**

We observe two opposite behaviors:

- **synchronization**
- **anti-synchronization**

The dichotomic behavior results from strong stability properties of the networks
Outline

Pulse-coupled firing oscillators and dichotomic behavior

Large populations of monotone oscillators: finite and infinite cases

Beyond monotone oscillators: around the QIF conjecture
Multidimensional firing oscillators are approximated by integrate-and-fire oscillators

Multidimensional firing oscillator (e.g. Hodgkin-Huxley neuron)

\[ \dot{x} = F(x) \quad x \in \mathbb{R}^n \]

One-dimensional integrate-and-fire oscillator

\[ \dot{x} = F(x) \quad x \in [\underline{x}, \overline{x}] \]
We have investigated various models of integrate-and-fire oscillators \( \dot{x} = F(x) \)

Leaky integrate-and-fire (LIF) oscillators

\[
F(x) = S - \gamma x
\]
We have investigated various models of integrate-and-fire oscillators \( \dot{x} = F(x) \)

Monotone oscillators

\[
\frac{dF}{dx} > 0 \quad \text{OR} \quad \frac{dF}{dx} < 0
\]

Quadratic (QIF) oscillators

\[
F(x) = S + x^2
\]
When the oscillator $i$ fires, it emits a pulse that increments the state $x_j$ of any oscillator $j \neq i$ by a constant $\epsilon$.

Peskin, 1975
Integrate-and-fire oscillators are phase oscillators on the circle

In the state space
\[ \dot{x} = F(x) \]

In the phase space
\[ \dot{\theta} = \omega \]

\[ X \in \mathbb{R}^n \]

\[ \theta \in S^1 \]

Phase reduction
Integrate-and-fire oscillators are phase oscillators on the circle

In the state space:

\[ x \in [\underline{x}, \overline{x}] \]

\[ x = \underline{x} \]
\[ x = \overline{x} \]
\[ \dot{x} = F(x) \]

In the phase space:

\[ \theta \in S^1 \]

\[ \theta = 0 \]
\[ \theta = 2\pi \]
\[ \dot{\theta} = \omega \]

change of variables
We have observed that the oscillators either gather into phase-locked clusters or synchronize.

\[ \theta_1^*, \theta_2^*, \theta_{N_g-1}^* \]

(Anti-synchronized) phase-locked clustering

Mauroy & Sepulchre, 2008

Synchronization

Mirollo & Strogatz, 1990
Heterogeneous populations: the clusters are robust against discrepancies between the oscillators

\[ \dot{x}_i = S_i + \gamma x \quad S_i > 0 \text{ randomly distributed} \]

Clusters are still observed

Some oscillators travel between the clusters:
Monotone oscillators exhibit a dichotomic behavior

The dichotomic behavior is determined by the monotonicity property of the oscillators

\[ dF/dx > 0 \]

(Anti-synchronized) phase-locked clustering

\[ dF/dx < 0 \]

Synchronization
The (non-monotone) QIF oscillators \( \dot{x} = S + x^2 \) also exhibit a dichotomotic behavior

The dichotomotic behavior is determined by an “average monotonicity property” of the oscillators

\[ F(x) < F(\bar{x}) \]  
\[ F(x) > F(\bar{x}) \]

(Anti-synchronized) phase-locked clustering

Synchronization
The discrete-time firing map $\mathbf{H}$ gives the configuration $\Theta$ of the network at each firing.

At a firing:

\[ \Theta^+ = \mathbf{H}(\Theta) \]

At the next firing:

\[ \Theta^* = [\theta_1^* \cdots \theta_{N_g-1}^*] \]

The fixed point of the firing map $\Theta^*$ corresponds to a phase-locked clustering configuration.
The evolution of 2 oscillators is described by the scalar firing map

$$\theta^+ = h(\theta)$$

Global stability properties of $h$ for

- monotone oscillators: $|h'(\theta)| < 1 \quad \forall \theta$ when $dF/dx > 0$
  $$\Rightarrow$$ contraction property $|h(\theta) - h(\psi)| < |\theta - \psi|$

- QIF oscillators

The fixed point of $h$ is **globally stable**
$$\Rightarrow$$ phase-locking

The fixed point of $h$ is **unstable**
$$\Rightarrow$$ synchronization
Dichotomic behavior of large populations: how do the stability properties extend to higher dimensions?

In higher dimensions: \( \mathbf{H} = \mathbf{N} \circ \mathbf{L} \)

\[
\mathbf{N}(\xi) = [h(\xi_1) \cdots h(\xi_{N_g - 1})]^T
\]

\[
\mathbf{L} = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & \ddots & 0 & \vdots \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

The fixed point of \( \mathbf{H} \) is **globally stable**

\( \Rightarrow \) phase-locked clustering

The fixed point of \( \mathbf{H} \) is **unstable**

\( \Rightarrow \) synchronization
Dichotomic behavior of large populations: how do the stability properties extend to higher dimensions?

In higher dimensions: \( H = N \circ L \)

\[
N(\xi) = [h(\xi_1) \cdots h(\xi_{N_g-1})]^T
\]

\[
L = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & \ddots & 0 & \ddots \\
0 & 0 & -1 & 1 \\
\end{bmatrix}
\]

A fundamental question arises:

Which properties of \( h \) induce the stability of \( H \)?

- Monotone oscillators: has \( H \) a global contraction property?
- QIF oscillators: has \( H \) global stability properties?
Outline

Pulse-coupled firing oscillators and dichotomic behavior

Large populations of monotone oscillators: finite and infinite cases

Beyond monotone oscillators: around the QIF conjecture
We want to find a norm which could extend the contraction property to higher dimensions

2 oscillators: $|h(\theta) - h(\psi)| < |\theta - \psi|$ if $|h'| < 1$

→ the scalar firing map is a contraction

Large population: find a norm $\| \cdot \|$ such that

1. $L$ is an isometry: $\|L\Theta\| = \|\Theta\|

2. $N$ is a contraction:

$$\|N(\Theta) - N(\Psi)\| < \|\Theta - \Psi\|$$ if $|h'| < 1$

→ the multidimensional firing map $H = N \circ L$ is a contraction:

$$\|H(\Theta) - H(\Psi)\| < \|\Theta - \Psi\|$$ if $|h'| < 1$
We have proposed a quadratic norm

\[ \| \Theta - \Psi \| = (\Delta_1)^2 + \sum_{i=1}^{N_g-2} (\Delta_{i+1} - \Delta_i)^2 + (\Delta_{N_g-1})^2 \]

⇒ The firing map is a contraction if

\[ \frac{\max |h'| - \min |h'|}{1 - \min |h'|} < \frac{2\sqrt{N_g}}{N_g+1} \]

and \( h'' \) of constant sign

*Mauroy & Sepulchre, Chaos, 2008*
We finally proposed a 1-norm

\[ \| \Theta - \Psi \| = |\Delta_1| + \sum_{i=1}^{N_g-2} |\Delta_{i+1} - \Delta_i| + |\Delta_{N_g-1}| \]

\[ \Rightarrow \text{The firing map is a contraction if} \]
\[ |h'| < 1 \quad \text{and} \quad h'' \text{ of constant sign} \]
Through the 1-norm, we can prove the global stability of several monotone systems.

Pulse-coupled oscillators

1. Finite populations
2. Weak coupling ("Kuramoto-like" model)
3. Infinite populations (partial differential equation)
A monotone “Kuramoto-like” model is characterized by a contraction property with respect to the 1-norm.

Averaged dynamics of weakly pulse-coupled monotone oscillators

$$\dot{\theta}_i = \sum_{j \neq i} \Gamma(\theta_i - \theta_j)$$

\( \Gamma \) monotone in \((0, 2\pi)\)

\( \theta_i \in S^1 \)

*Kuramoto, Physica D, 1991*
A monotone “Kuramoto-like” model is characterized by a contraction property with respect to the 1-norm

Consensus problem on the real line

\[ \dot{x}_i = \sum_{j \neq i} \Gamma(x_i - x_j) \]

\( x_i \in \mathbb{R} \)

⇒ trivial behavior if \( \Gamma \) is monotone
A monotone “Kuramoto-like” model is characterized by a contraction property with respect to the 1-norm

Averaged dynamics of weakly pulse-coupled monotone oscillators

\[
\dot{\theta}_i = \sum_{j \neq i} \Gamma(\theta_i - \theta_j)
\]

\(\Gamma\) monotone in \((0, 2\pi)\)

\(Kuramoto, \ Physica \ D, \ 1991\)

The dynamics of the phase differences \(\tilde{\theta}_i\) is a contraction provided that \(\Gamma''\) is of constant sign:

\[
\frac{d}{dt} \left\| \tilde{\Theta} - \tilde{\Psi} \right\| < 0 \quad \text{if} \quad \Gamma' < 0
\]

\(\Rightarrow\) dichotomotic behavior (splay state or synchronization)
We consider infinite populations of monotone oscillators

The continuum of oscillators is represented by a density $\rho(\theta, t)$

The flux of oscillators is given by $J(\theta, t) = \rho(\theta, t) \dot{\theta}$

The continuum evolves according to the continuity equation

$$\frac{\partial \rho}{\partial t} = - \frac{\partial J}{\partial \theta}$$

with the boundary condition $J(0, t) = J(2\pi, t) \triangleq J_0(t)$
The impulsive coupling is proportional to the firing rate $J_0(t)$

For a finite population:

$$\dot{x} = F(x) + \epsilon \sum_{k=1}^{N} \delta(t - t_k)$$

But

$$J_0(t) = \frac{1}{N} \sum_{k=1}^{N} \delta(t - t_k) \Rightarrow \dot{x} = F(x) + \epsilon N J_0(t)$$

For infinite populations:

$$N \to \infty \Rightarrow \epsilon N \to K \quad (0 < K < \infty)$$

$$\dot{x} = F(x) + K J_0(t)$$
We obtain a nonlinear partial differential equation with monotonicity properties

\[
\dot{x} = F(x) + K J_0(t) \quad \Rightarrow \quad \dot{\theta} = \omega + K \frac{\omega}{F(x)} J_0(t)
\]
We obtain a nonlinear partial differential equation with monotonicity properties

\[ \dot{x} = F(x) + KJ_0(t) \implies \dot{\theta} = \omega + KZ(\theta)J_0(t) \]

phase response curve

\[ F \text{ monotone} \implies Z \text{ monotone} \]

The continuity equation yields a nonlinear PDE

\[ \frac{\partial \rho(\theta, t)}{\partial t} = -\omega \frac{\partial \rho(\theta, t)}{\partial \theta} - J_0(t) \frac{\partial}{\partial \theta} [KZ(\theta)\rho(\theta, t)] \]

with the stationary solution \( J^*(\theta) = J^* \quad \rho^*(\theta) = J^*/\dot{\theta} \)
According to the stability of the stationary solution, we observe a dichotomic behavior:

**Stable stationary solution**

\[ K \frac{dF}{dx} > 0 \quad K Z' < 0 \]

asynchronous state

\[ J_0(t) \quad J^* \]

\[ t \]

**Unstable stationary solution**

\[ K \frac{dF}{dx} < 0 \quad K Z' > 0 \]

synchronization

\[ J_0(t) \]

\[ t \]
For finite populations, a Lyapunov function induced by our 1-norm is a total variation distance.

Finite populations:

The 1-norm yields the Lyapunov function

\[ V = \| \Theta - \Theta^* \| \]

which corresponds to the total variation distance

\[ V = 2(\Delta_1 + \Delta_2) \]
For infinite populations, a Lyapunov function is the total variation distance **between quantile densities**.

**Infinite populations:**

The 1-norm yields the Lyapunov function

\[ V = \| q - q^* \|_{L^1} \]

which corresponds to the total variation distance

\[ V = 2(\Delta_1 + \Delta_2) \]

**Cumulative density function:**

\[ P(\theta) = \int_0^\theta \rho(s)ds \]

**Quantile function:**

\[ P^{-1}(\varphi) \equiv Q(\varphi) \]

Quantile density \( q = \frac{dQ}{d\varphi} \)
According to our global analysis, infinite populations of monotone oscillators have a dichotomistic behavior

Provided that \( \mathcal{Z}'' \) is of constant sign, \( \mathcal{V} \) evolves according to

\[
J_0 \min (K Z') \mathcal{V} \leq \dot{\mathcal{V}} \leq J_0 \max (K Z') \mathcal{V}
\]

between two values:

\[
\mathcal{V} = 0 \quad \text{(asynchronous state)}
\]

if \( K Z' < 0 \quad \Rightarrow \quad \dot{\mathcal{V}} < 0 \)

\( \Rightarrow \) exponential convergence to the asynchronous state

\[
\mathcal{V} = 4\pi \quad \text{(synchronization)}
\]

if \( K Z' > 0 \quad \Rightarrow \quad \dot{\mathcal{V}} > 0 \)

\( \Rightarrow \) synchronization in finite time

Mauroy & Sepulchre, submitted to IEEE TAC
We have proposed a 1-norm to study the global stability of several monotone systems.

Large populations of pulse-coupled monotone oscillators:
the firing map is a global contraction

Weakly pulse-coupled monotone oscillators:
a “Kuramoto-like” model is a global contraction

Infinite populations of monotone pulse-coupled oscillators:
a nonlinear PDE has global stability properties

➤ All these monotone systems have a dichotomic behavior
Outline

Pulse-coupled firing oscillators and dichotomic behavior

Large populations of monotone oscillators: finite and infinite cases

Beyond monotone oscillators: around the QIF conjecture
Numerical simulations suggest a dichotomic behavior for large populations of QIF oscillators

Numerical simulations: the fixed point of the firing map $H$ is

**globally stable** when

$$F(x) < F(\bar{x}) \quad \bar{x} + x > 0$$

$\Rightarrow$ phase-locked clustering

**unstable** when

$$F(x) > F(\bar{x}) \quad \bar{x} + x < 0$$

$\Rightarrow$ synchronization

When the thresholds are reversed, the stability is reversed:

$$H \quad \begin{bmatrix} x, \bar{x} \end{bmatrix} \rightarrow \begin{bmatrix} -\bar{x}, -x \end{bmatrix} \quad T^{-1}H^{-1}T$$
The global stability problem in large populations of QIF oscillators leads to a standing conjecture

dichotomic behavior of 2 oscillators

global stability of $h$

OK ☺

QIF oscillators are not monotone $\Rightarrow$ the problem is more involved

$$H^n = N \circ L \circ \cdots \circ N \circ L$$

$H = N \circ L$

$N$ contracting “on average”

$L$ mixing: $(-L)^{Ng} = I$

It is difficult to find a global contraction property “on average”
A local analysis suggests great evidence of the dichotomic behavior

Far from the fixed point: the behavior is always compatible with a dichotomic behavior

In the vicinity of the fixed point:
- local stability is proved for three QIF oscillators
- local stability is proved for oscillators satisfying $h''(\theta) < 0 \ \forall \theta$
  - QIF oscillators with low thresholds
  - exponential oscillators $\dot{x} = S e^{x^2}$

Mauroy et al., MTNS, 2010
The local analysis proves that the dichotomic behavior is not a general property.

Three (or more) piecewise linear oscillators \( \dot{x} = S + \gamma |x| \) have not a dichotomic behavior:

- Hopf bifurcation \( \rightarrow \) dense orbit
- no global behavior
- complex patterns

The dichotomy property of 2 oscillators does not always extend to larger networks!
QIF oscillators are a limit case between dichotomic and non-dichotomic oscillators

The length of \( I = \{ \theta | h''(\theta) < 0 \} \) plays a key role to extend the stability of \( h \) to higher dimensions.

**Exponential**

- \(|I| = 2\pi\)
- dichotomic behavior

**QIF**

- \(|I| < 2\pi\)
- dichotomic behavior conjectures (global/local)

**Piecewise linear**

- \(|I| \ll 2\pi\)
- non-dichotomic behavior
In the weak coupling limit, the local stability problem is solved for QIF oscillators.

**Global stability:** the problem is still elusive but can be reformulated ("Kuramoto-like" model).

**Local stability:** the problem is solved.

For finite populations, the fixed point of the firing map is:

- **locally stable** when \( F(x) < F(\bar{x}) \) and \( x + \bar{x} > 0 \)
  - \( x \) phase-locked configuration

- **locally unstable** when \( F(x) > F(\bar{x}) \) and \( x + \bar{x} < 0 \)
  - \( x \) synchronization

*Mauroy & Sepulchre, CDC, 2011*
In the weak coupling limit, the local stability problem is solved for QIF oscillators

**Global stability:** the problem is still elusive but can be reformulated (“Kuramoto-like” model)

**Local stability:** the problem is solved

For infinite populations, the stationary density is

- **locally stable** when $F(x) < F(\bar{x})$ and $x + \bar{x} > 0$
  - asynchronous state

- **locally unstable** when $F(x) > F(\bar{x})$ and $x + \bar{x} < 0$
  - synchronization

*Mauroy & Sepulchre, CDC, 2011*
In the weak coupling limit, we obtain general criteria for the stability of finite and infinite populations

The stability criteria only depend on the phase response curve:

**Finite populations:** local stability if and only if

$$- \sum_k Z'(k \frac{2\pi}{N}) + \sum_k Z'(k \frac{2\pi}{N}) \cos \left( \frac{kn2\pi}{N} \right) \geq 0 \quad \forall n \in \mathbb{N}$$

**Infinite populations:** local stability if and only if

$$K \int_0^{2\pi} Z(\theta) \sin(n\theta) d\theta \geq 0 \quad \forall n \in \mathbb{N}$$

The results apply to general (multidimensional) oscillators (e.g. Hodgkin-Huxley model) and to delayed models
Non-monotone oscillators lead to standing conjectures and non-dichotomic behaviors

The global stability analysis of non-monotone oscillators is an involved problem (QIF conjecture(s))

Local results clearly suggest that the QIF oscillators have a dichotomic behavior

Populations of non-monotone oscillators may exhibit a non-dichotomic behavior
We have linked the (microscopic) dynamics of pulse-coupled oscillators to their (macroscopic) collective behavior.
The present study leads to several perspectives for future research on pulse-coupled oscillators

**QIF conjecture:** a promising lead is to consider statistical time averages of the dynamics (operator theoretic methods)

**Other dynamics:** further research could focus on other application-oriented dynamics (neuroscience, earthquakes, etc.)

**Excitable units:** an interesting extension is to consider excitable units instead of oscillators (or mix both)

**Non-identical oscillators:** considering heterogeneous populations is a challenging problem that is still open
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