On the dichotomic collective behaviors of large populations of pulse-coupled firing oscillators

PhD defense

A. Mauroy

Université de Liège

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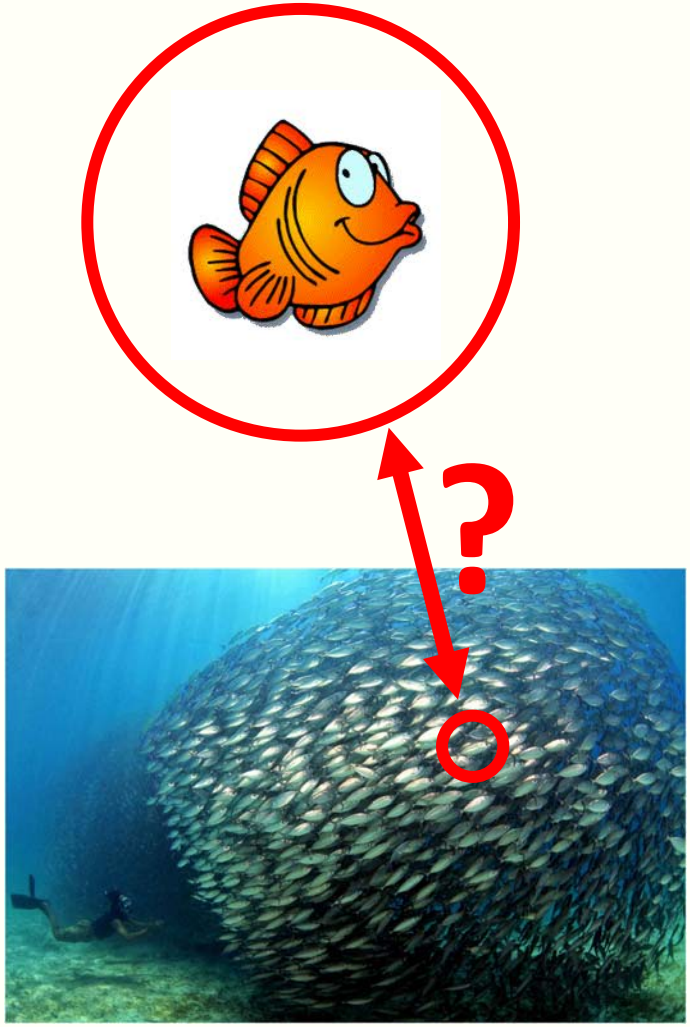
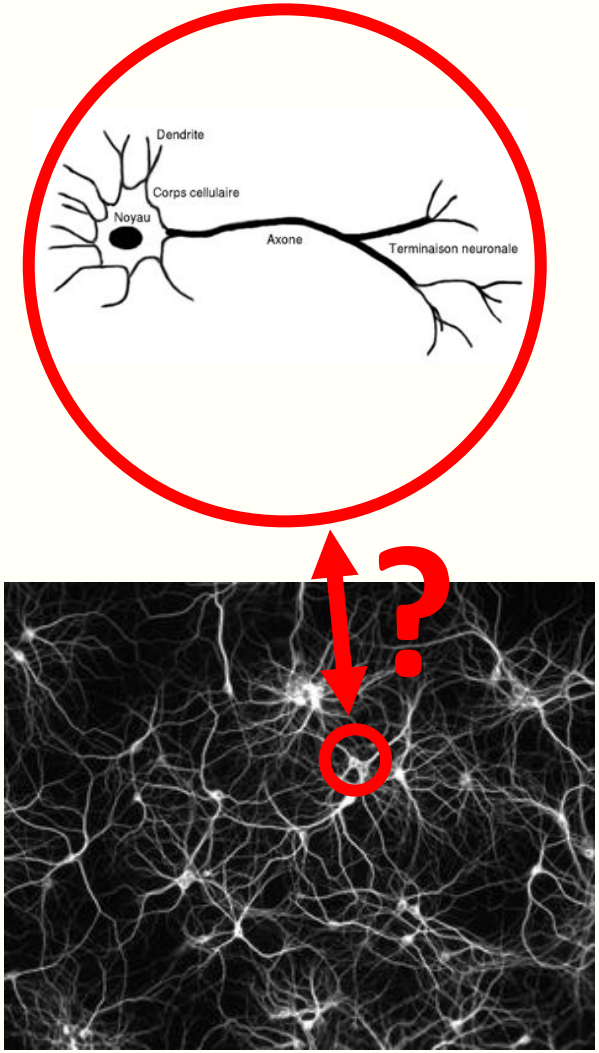
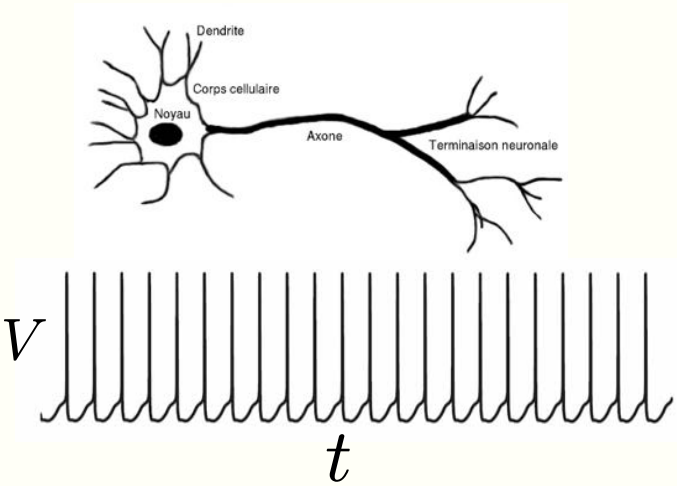


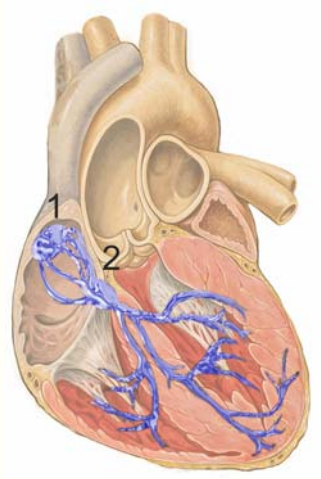


We want to link the **macroscopic** collective behaviors to the **microscopic** properties of a single agent

A number of phenomena correspond to networks of pulse-coupled firing oscillators (impulsive coupling)

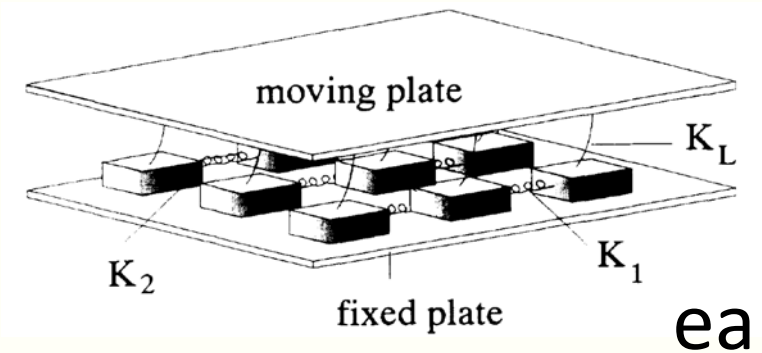


firing neurons



pacemaker cells of the heart

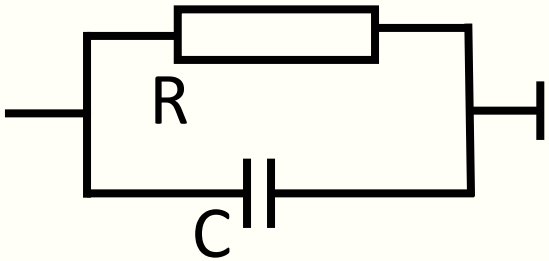
flashing fireflies



Olami et al., PRL, vol 68, 8

earthquakes

We have studied the collective behavior of several models of pulse-coupled oscillators

microscopic level	**macroscopic level**
leaky integrate-and-fire (LIF) 	synchronization *(Mirollo & Strogatz, 1990)*
other models of oscillators	??

Pulse-coupled oscillators frequently exhibit a remarkable **dichotomic behavior**

We observe two opposite behaviors:

synchronization

anti-synchronization

The dichotomic behavior results from strong stability properties of the networks

Outline

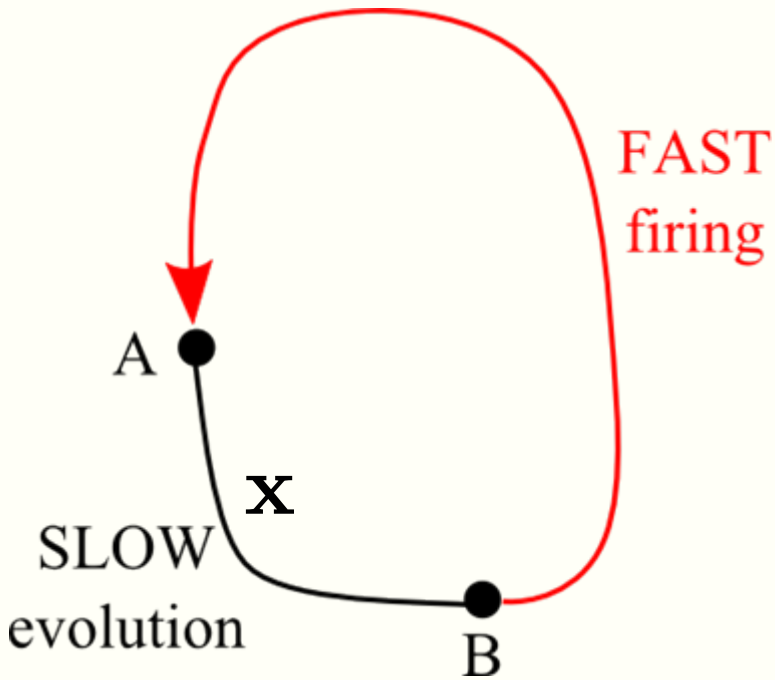
Pulse-coupled firing oscillators and dichotomic behavior

Large populations of monotone oscillators:
finite and infinite cases

Beyond monotone oscillators: around the QIF conjecture

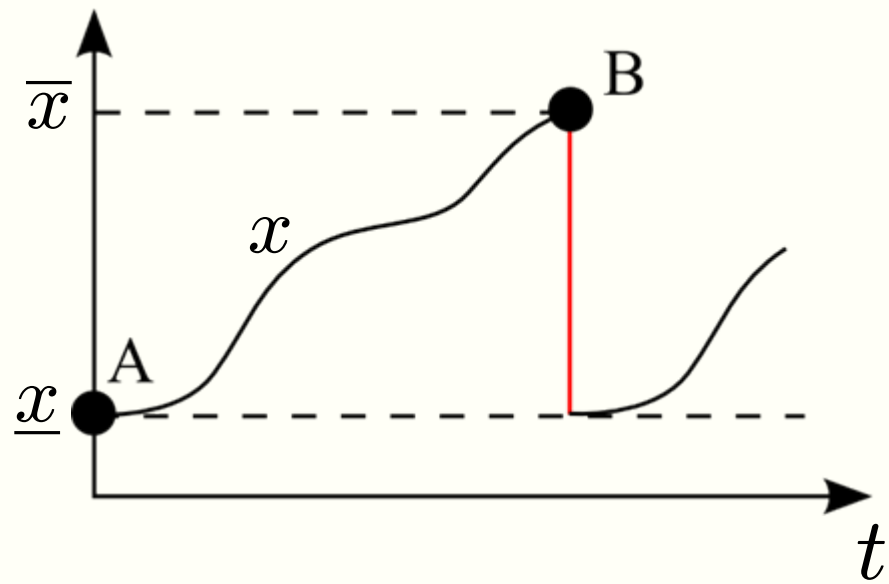
Multidimensional firing oscillators are approximated by integrate-and-fire oscillators

Multidimensional firing oscillator (e.g. Hodgkin-Huxley neuron)



$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^n$$

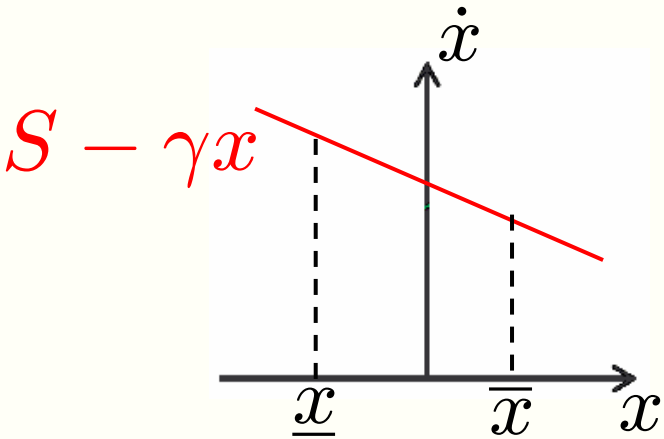
One-dimensional integrate-and-fire oscillator

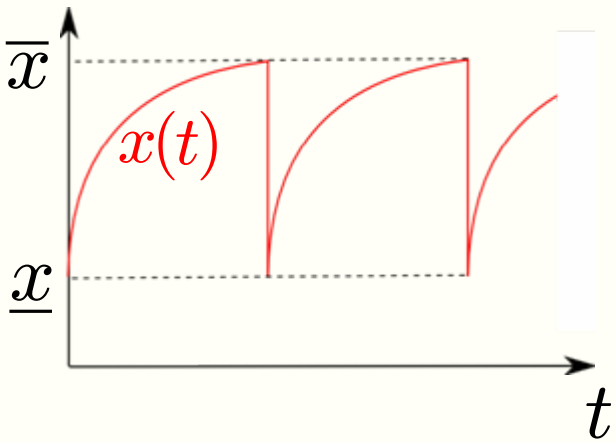


$$\dot{x} = F(x) \quad x \in [\underline{x}, \overline{x}]$$

We have investigated various models of integrate-and-fire oscillators $\dot{x} = F(x)$

Leaky integrate-and-fire (LIF) oscillators $F(x) = S - \gamma x$

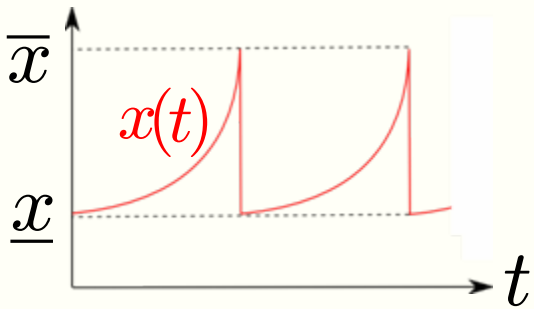


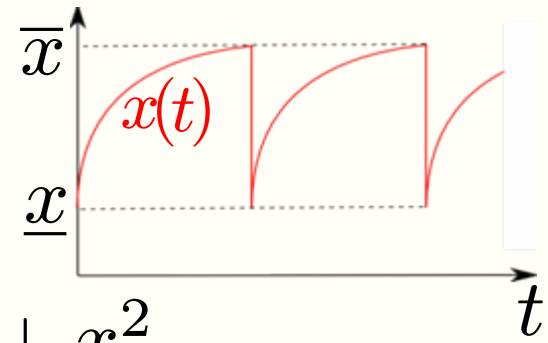


We have investigated various models of integrate-and-fire oscillators $\dot{x} = F(x)$

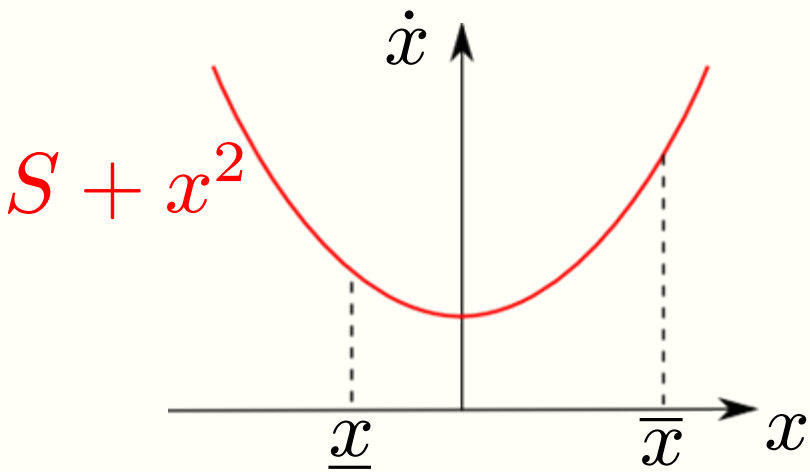
Monotone oscillators

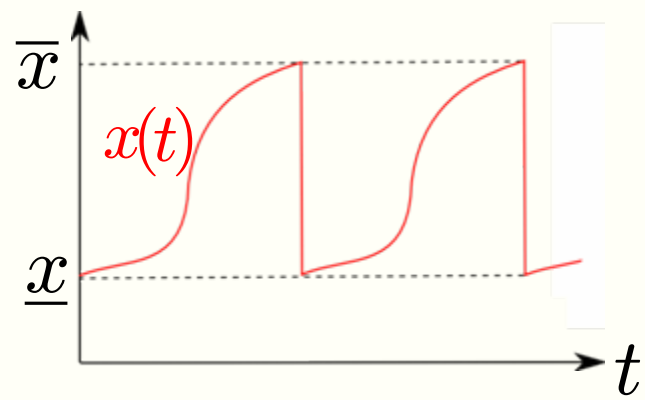
$dF/dx > 0$ OR $dF/dx < 0$



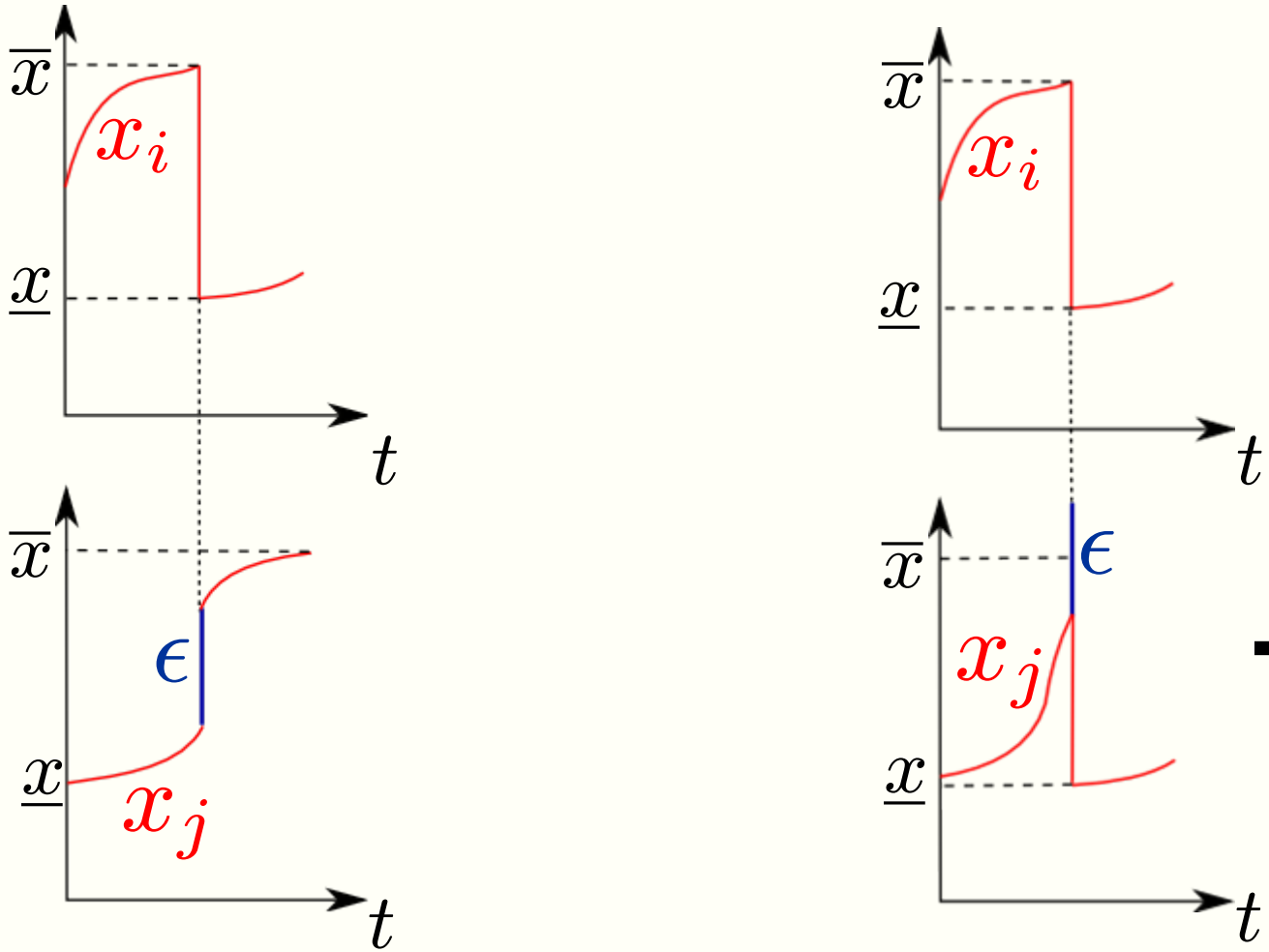


Quadratic (QIF) oscillators $F(x) = S + x^2$





When the oscillator i fires, it emits a pulse that increments the state x_j of any oscillator $j \neq i$ by a constant ϵ

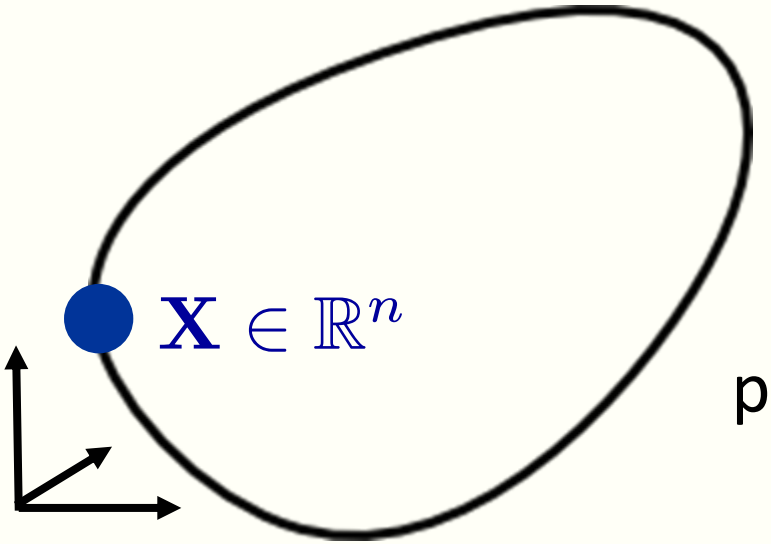


➔ absorption

Peskin, 1975

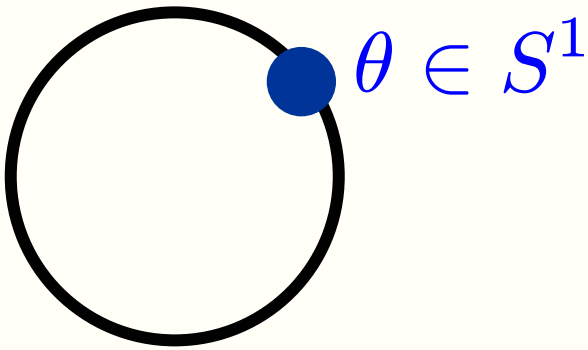
Integrate-and-fire oscillators are phase oscillators on the circle

In the state space



phase reduction

In the phase space

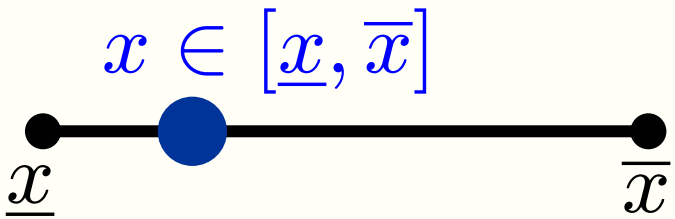


$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

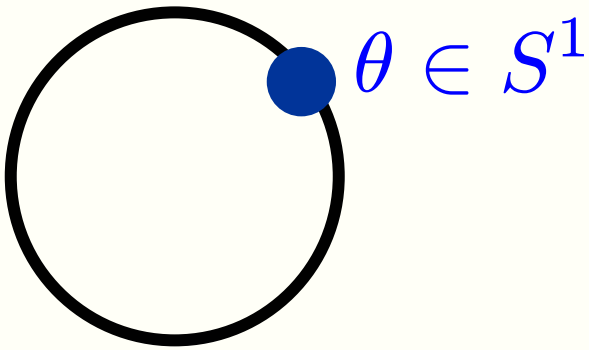
$$\dot{\theta} = \omega$$

Integrate-and-fire oscillators are phase oscillators on the circle

In the state space



In the phase space



change of variables

$$x = \underline{x}$$

$$x = \overline{x}$$

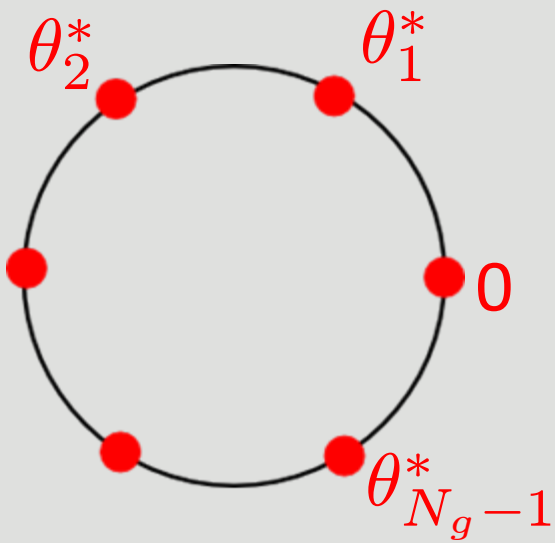
$$\dot{x} = F(x)$$

$$\theta = 0$$

$$\theta = 2\pi$$

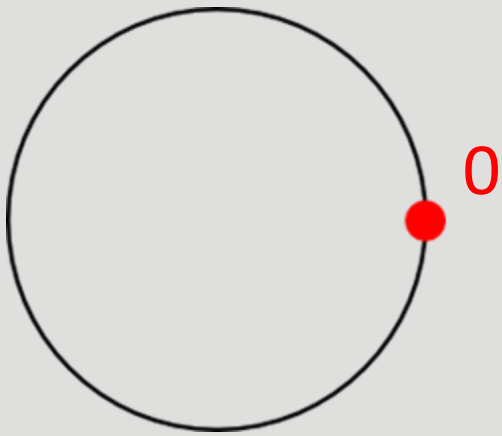
$$\dot{\theta} = \omega$$

We have observed that the oscillators either gather into phase-locked clusters or synchronize



(Anti-synchronized) phase-locked clustering

Mauroy & Sepulchre, 2008



Synchronization

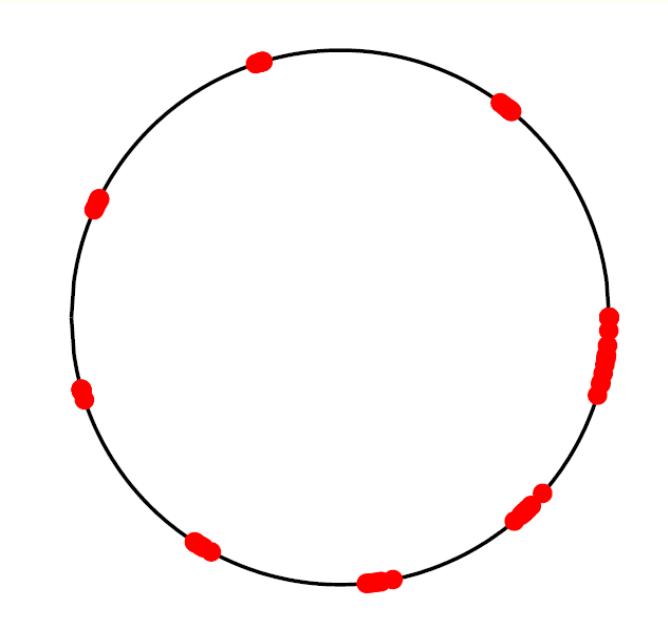
Mirollo & Strogatz, 1990

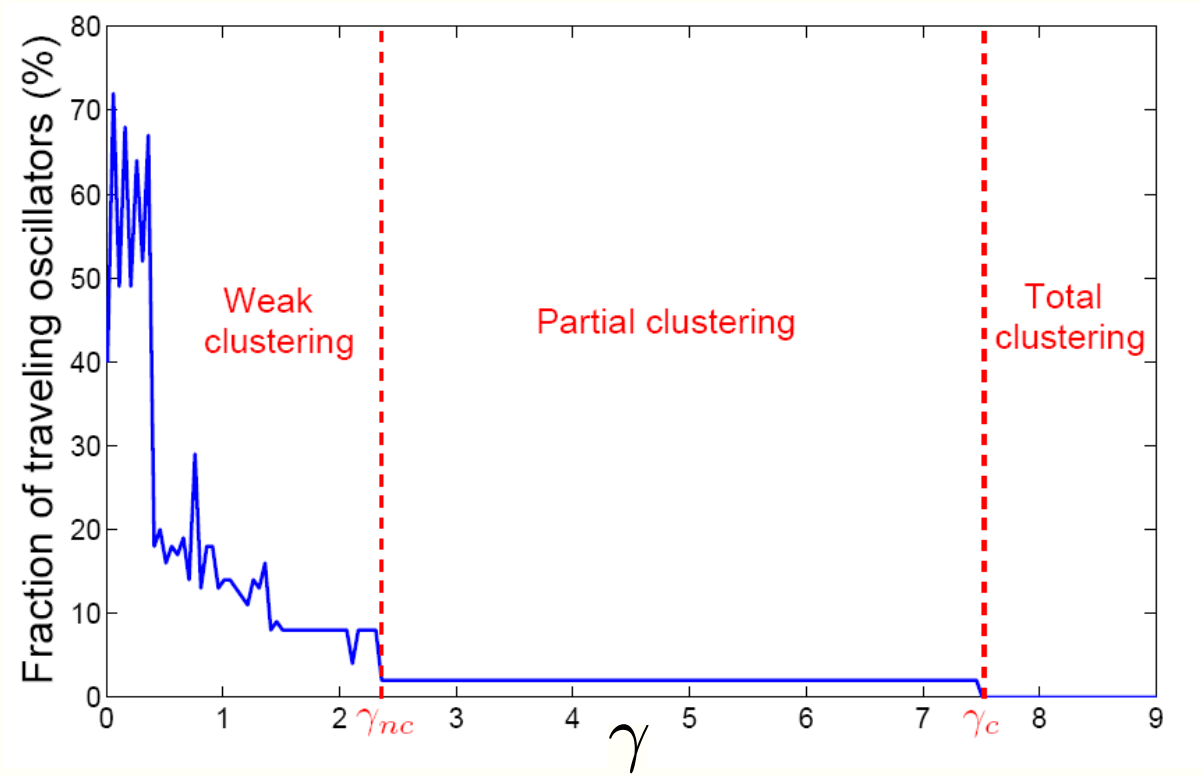
Heterogeneous populations: the clusters are robust against discrepancies between the oscillators

$$\dot{x}_i = S_i + \gamma x \qquad S_i > 0 \text{ randomly distributed}$$

Clusters are still observed

Some oscillators travel between the clusters:

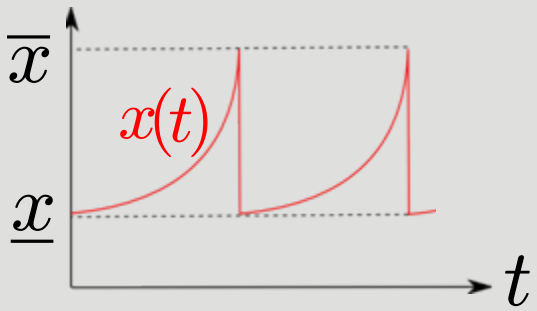




Monotone oscillators exhibit a dichotomic behavior

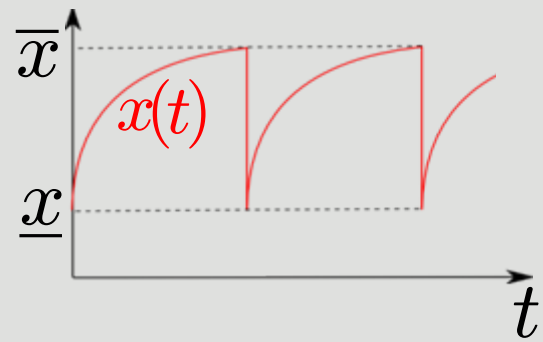
The dichotomic behavior is determined
by the monotonicity property of the oscillators

$$dF/dx > 0$$



(Anti-synchronized)
phase-locked clustering

$$dF/dx < 0$$

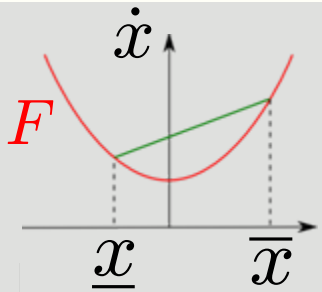


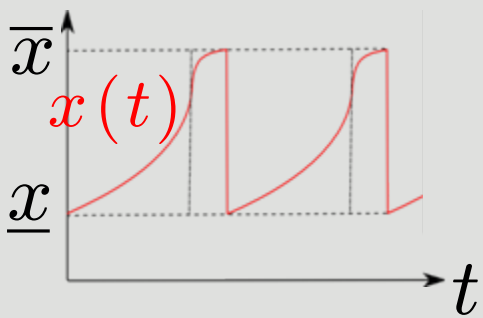
Synchronization

The (non-monotone) QIF oscillators $\dot{x} = S + x^2$ also exhibit a dichotomic behavior

The dichotomic behavior is determined
by an "average monotonicity property" of the oscillators

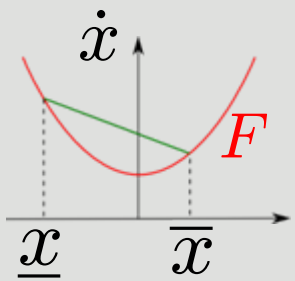
$F(\underline{x}) < F(\overline{x})$

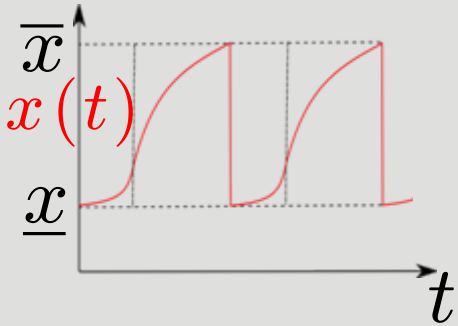




(Anti-synchronized)
phase-locked clustering

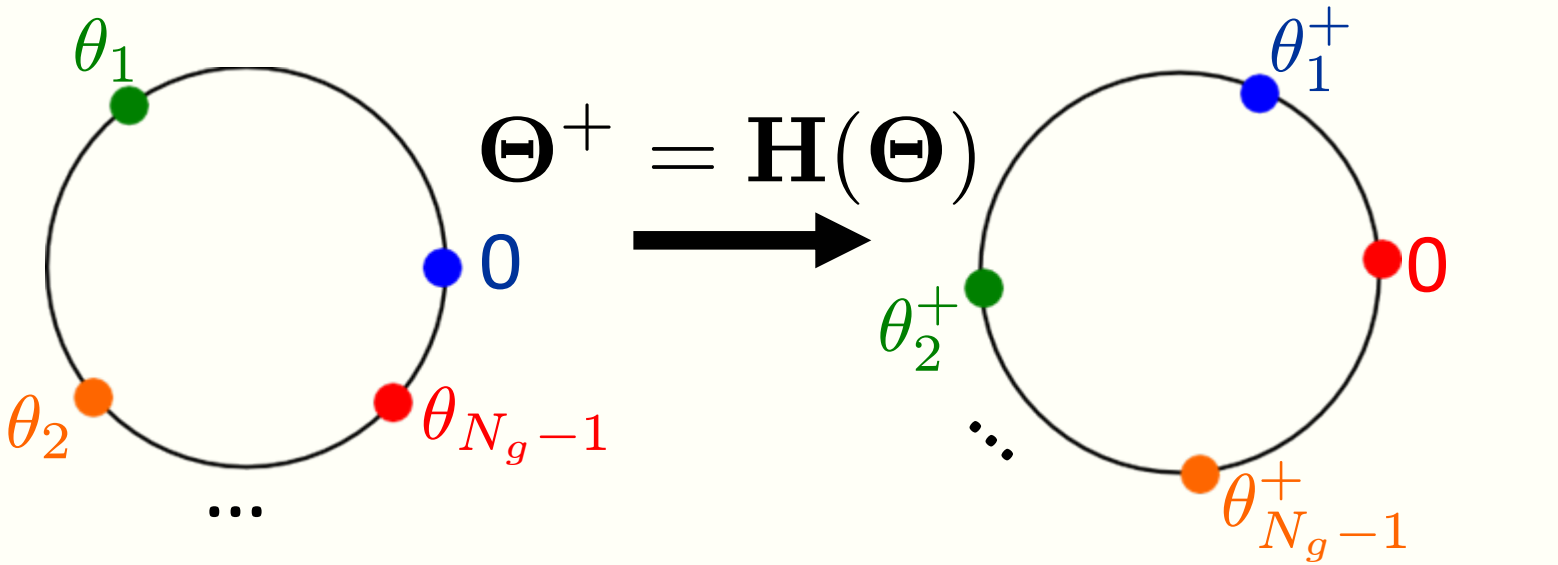
$F(\underline{x}) > F(\overline{x})$





Synchronization

The discrete-time firing map $\mathbf{H}$ gives the configuration $\boldsymbol{\Theta}$ of the network at each firing



at a firing

at the next firing

The fixed point of the firing map $\boldsymbol{\Theta}^* = [\theta_1^* \cdots \theta_{N_g-1}^*]$ corresponds to a phase-locked clustering configuration

The dichotomic behavior of 2 oscillators is induced by strong stability properties of the scalar firing map

The evolution of 2 oscillators is described by the scalar firing map

$$\theta^+ = h(\theta)$$

Global stability properties of h for

- monotone oscillators: $|h'(\theta)| \lessgtr 1 \quad \forall\theta$ when $dF/dx \gtrless 0$

  ➔ contraction property $|h(\theta) - h(\psi)| \lessgtr |\theta - \psi|$

- QIF oscillators

The fixed point of h is **globally stable**

➔ phase-locking

The fixed point of h is **unstable**

➔ synchronization

Dichotomic behavior of large populations: how do the stability properties extend to higher dimensions ?

In higher dimensions: $\mathbf{H} = \mathbf{N} \circ \mathbf{L}$

$$\mathbf{N}(\xi) = [h(\xi_1) \cdots h(\xi_{N_g-1})]^T$$

$$\mathbf{L} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The fixed point of $\mathbf{H}$ is **globally stable**

➔ phase-locked clustering

The fixed point of $\mathbf{H}$ is **unstable**

➔ synchronization

Dichotomic behavior of large populations: how do the stability properties extend to higher dimensions ?

In higher dimensions: $\mathbf{H} = \mathbf{N} \circ \mathbf{L}$

$$\mathbf{N}(\xi) = [h(\xi_1) \cdots h(\xi_{N_g-1})]^T \qquad \mathbf{L} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

A fundamental question arises:

Which properties of h induce the stability of $\mathbf{H}$?

- Monotone oscillators: has $\mathbf{H}$ a global contraction property ?
- QIF oscillators: has $\mathbf{H}$ global stability properties ?

Outline

Pulse-coupled firing oscillators and dichotomic behavior

Large populations of monotone oscillators:
finite and infinite cases

Beyond monotone oscillators: around the QIF conjecture

We want to find a norm which could extend the contraction property to higher dimensions

2 oscillators: $|h(\theta) - h(\psi)| < |\theta - \psi|$ if $|h'| < 1$

➔ the scalar firing map is a contraction

Large population: find a norm $\|\cdot\|$ such that

1. $\mathbf{L}$ is an isometry: $\|\mathbf{L}\boldsymbol{\Theta}\| = \|\boldsymbol{\Theta}\|$
2. $\mathbf{N}$ is a contraction:

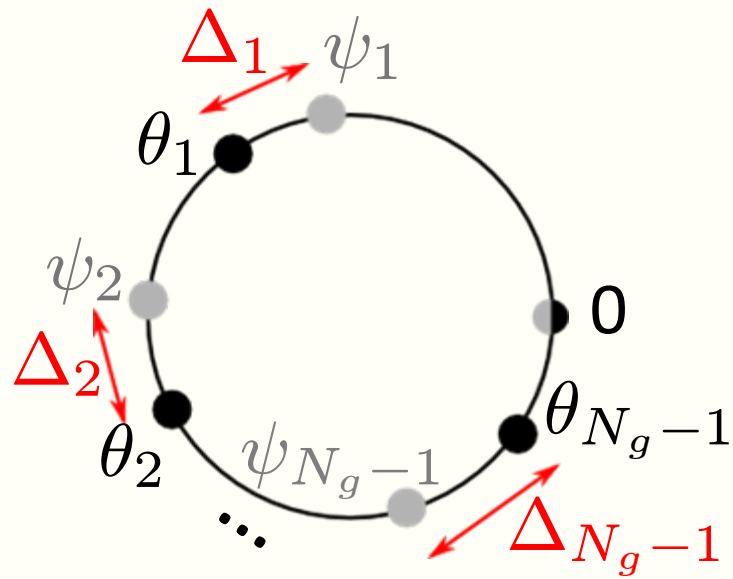
$$\|\mathbf{N}(\boldsymbol{\Theta}) - \mathbf{N}(\boldsymbol{\Psi})\| < \|\boldsymbol{\Theta} - \boldsymbol{\Psi}\| \quad \text{if} \quad |h'| < 1$$

➔ the multidimensional firing map $\mathbf{H} = \mathbf{N} \circ \mathbf{L}$ is a contraction:

$$\|\mathbf{H}(\boldsymbol{\Theta}) - \mathbf{H}(\boldsymbol{\Psi})\| < \|\boldsymbol{\Theta} - \boldsymbol{\Psi}\| \quad \text{if} \quad |h'| < 1$$

We have proposed a quadratic norm

Mauroy & Sepulchre, Chaos, 2008



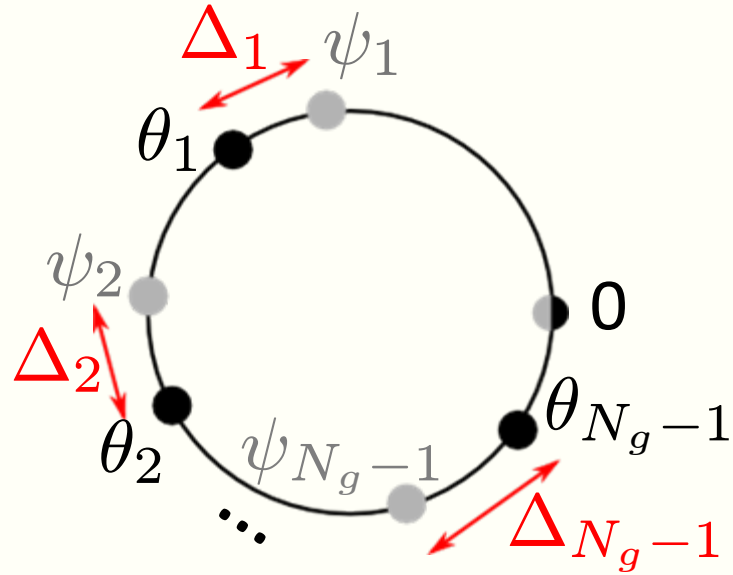
$$\|\mathbf{\Theta} - \mathbf{\Psi}\| = (\Delta_1)^2 + \sum_{i=1}^{N_g-2} (\Delta_{i+1} - \Delta_i)^2 + (\Delta_{N_g-1})^2$$

➔ The firing map is a contraction if

$$\frac{\max|h'| - \min|h'|}{1 - \min|h'|} < \frac{2\sqrt{N_g}}{N_g+1}$$ and h'' of constant sign

We finally proposed a 1-norm

Mauroy & Sepulchre, Chaos, 2008



$$\|\mathbf{\Theta} - \mathbf{\Psi}\| = |\Delta_1| + \sum_{i=1}^{N_g-2} |\Delta_{i+1} - \Delta_i| + |\Delta_{N_g-1}|$$

➔ The firing map is a contraction if

$|h'| < 1$ and h'' of constant sign

Through the 1-norm, we can prove the global stability of several monotone systems

monotone systems $\xrightarrow{\textbf{1-norm}}$ **global stability**

Pulse-coupled oscillators

1. Finite populations
2. Weak coupling
   ("Kuramoto-like" model)
3. Infinite populations
   (partial differential equation)

} dichotomic behavior

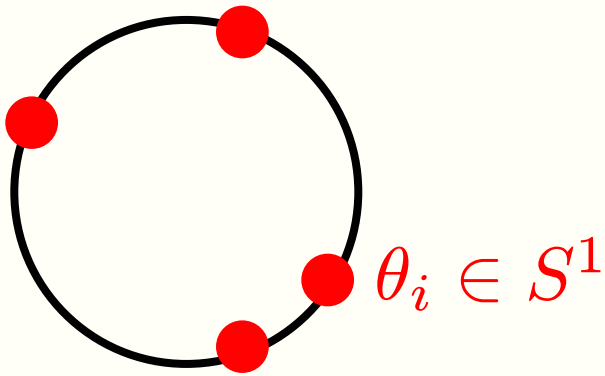
A monotone "Kuramoto-like" model is characterized by a contraction property with respect to the 1-norm

Averaged dynamics of weakly pulse-coupled monotone oscillators

$$\dot{\theta}_i = \sum_{j \neq i} \Gamma(\theta_i - \theta_j)$$

Γ monotone in $(0, 2\pi)$

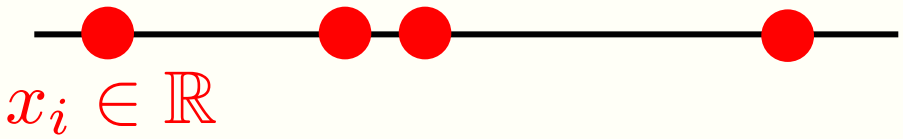
Kuramoto, Physica D, 1991



A monotone “Kuramoto-like” model is characterized by a contraction property with respect to the 1-norm

Consensus problem on the real line

$$\dot{x}_i = \sum_{j \neq i} \Gamma(x_i - x_j)$$



➔ trivial behavior if Γ is monotone

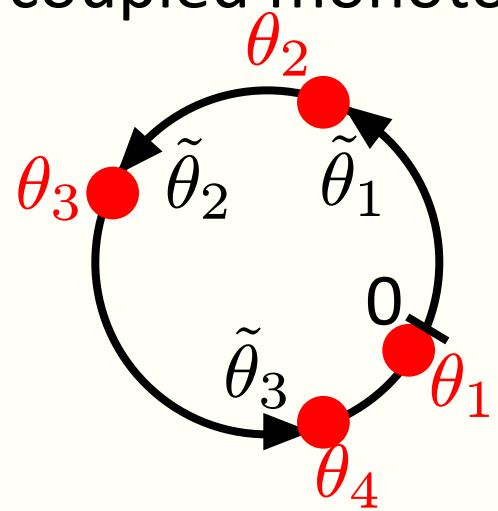
A monotone "Kuramoto-like" model is characterized by a contraction property with respect to the 1-norm

Averaged dynamics of weakly pulse-coupled monotone oscillators

$$\dot{\theta}_i = \sum_{j \neq i} \Gamma(\theta_i - \theta_j)$$

Γ monotone in $(0, 2\pi)$

Kuramoto, Physica D, 1991



$$\tilde{\theta}_i = \theta_{i+1} - \theta_1$$

The dynamics of the phase differences $\tilde{\theta}_i$ is a contraction provided that Γ'' is of constant sign:

$$\frac{d}{dt} \left\| \tilde{\mathbf{\Theta}} - \tilde{\mathbf{\Psi}} \right\| \lessgtr 0 \quad \text{if} \quad \Gamma' \lessgtr 0$$

➔ dichotomic behavior (splay state or synchronization)

We consider infinite populations of monotone oscillators

The continuum of oscillators is represented by a density $\rho(\theta, t)$

The flux of oscillators is given by $J(\theta, t) = \rho(\theta, t)\,\dot{\theta}$

The continuum evolves according to the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial \theta}$$

with the boundary condition $J(0, t) = J(2\pi, t) \triangleq J_0(t)$

The impulsive coupling is proportional to the firing rate $J_0(t)$

For a finite population: $$\dot{x} = F(x) + \epsilon \sum_{k=1}^{N} \delta(t - t_k)$$

firing times

But $$J_0(t) = \frac{1}{N} \sum_{k=1}^{N} \delta(t - t_k) \quad \Rightarrow \quad \dot{x} = F(x) + \epsilon N \, J_0(t)$$

For infinite populations: $$\begin{array}{c} N \to \infty \\ \epsilon \to 0 \end{array} \quad \Rightarrow \quad \begin{array}{c} \epsilon N \to K \\ (0 < K < \infty) \end{array}$$

$$\dot{x} = F(x) + K \, J_0(t)$$

We obtain a nonlinear partial differential equation with monotonicity properties

$$\dot{x} = F(x) + K\, J_0(t) \quad \Rightarrow \quad \dot{\theta} = \omega + K \frac{\omega}{F(x)} J_0(t)$$

We obtain a nonlinear partial differential equation with monotonicity properties

$$\dot{x} = F(x) + K\, J_0(t) \quad \rightarrow \quad \dot{\theta} = \omega + K \underbrace{Z(\theta)}_{\text{phase response curve}} J_0(t)$$

F monotone $\rightarrow$ Z monotone

The continuity equation yields a nonlinear PDE

$$\frac{\partial \rho(\theta,t)}{\partial t} = -\omega \frac{\partial \rho(\theta,t)}{\partial \theta} - J_0(t)\, \frac{\partial}{\partial \theta}[K Z(\theta) \rho(\theta,t)]$$

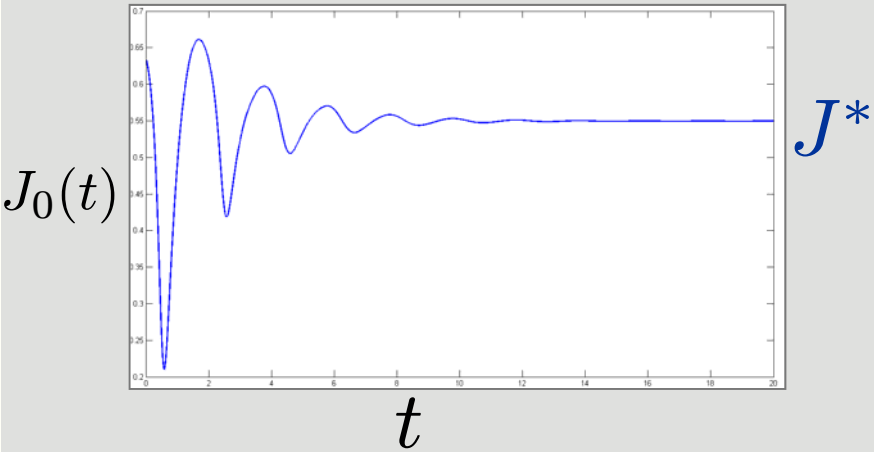
with the stationary solution $J^*(\theta) = J^*$ $\quad \rho^*(\theta) = J^*/\dot{\theta}$

According to the stability of the stationary solution, we observe a dichotomic behavior

Stable stationary solution

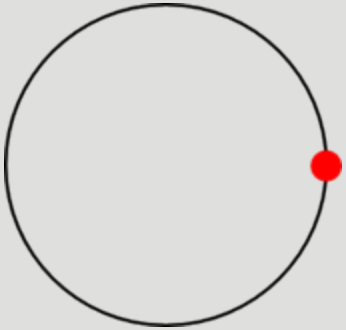
$K\,dF/dx > 0 \qquad KZ' < 0$

asynchronous state

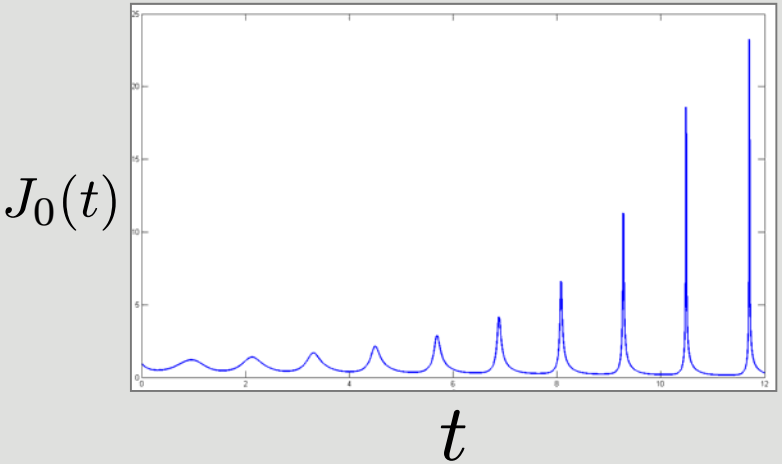


Unstable stationary solution

$K\,dF/dx < 0 \qquad KZ' > 0$



synchronization



For finite populations, a Lyapunov function induced by our 1-norm is a **total variation distance**

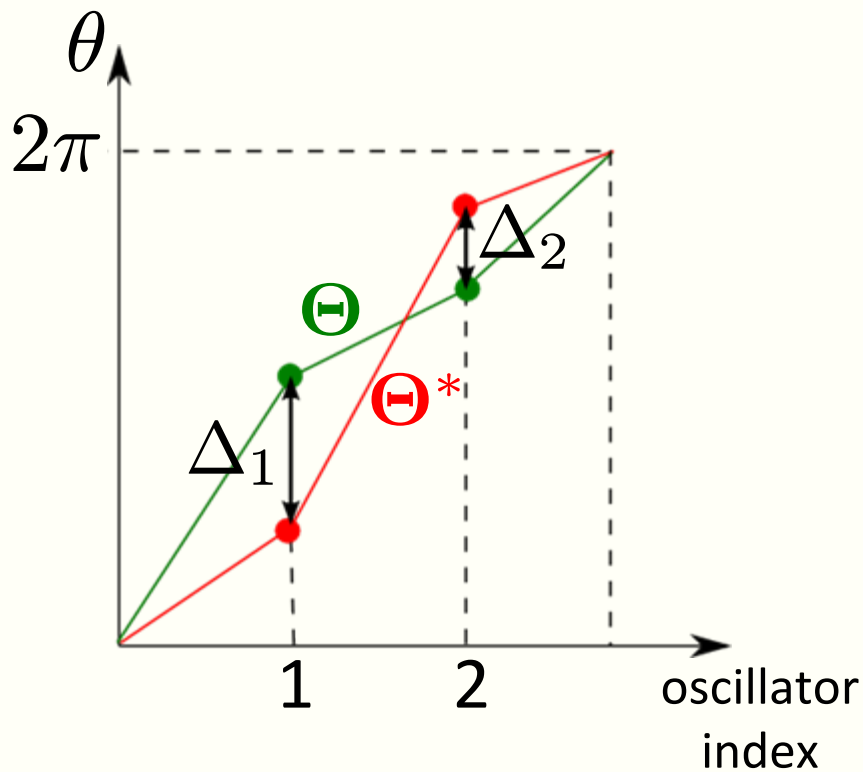
Finite populations:

The 1-norm yields the Lyapunov function

$$\mathcal{V} = \|\boldsymbol{\Theta} - \boldsymbol{\Theta}^*\|$$

which corresponds to the total variation distance

$$\mathcal{V} = 2(\Delta_1 + \Delta_2)$$



For infinite populations, a Lyapunov function is the total variation distance **between quantile densities**

Infinite populations:

The 1-norm yields the Lyapunov function

$$\mathcal{V} = \|q - q^*\|_{L^1}$$

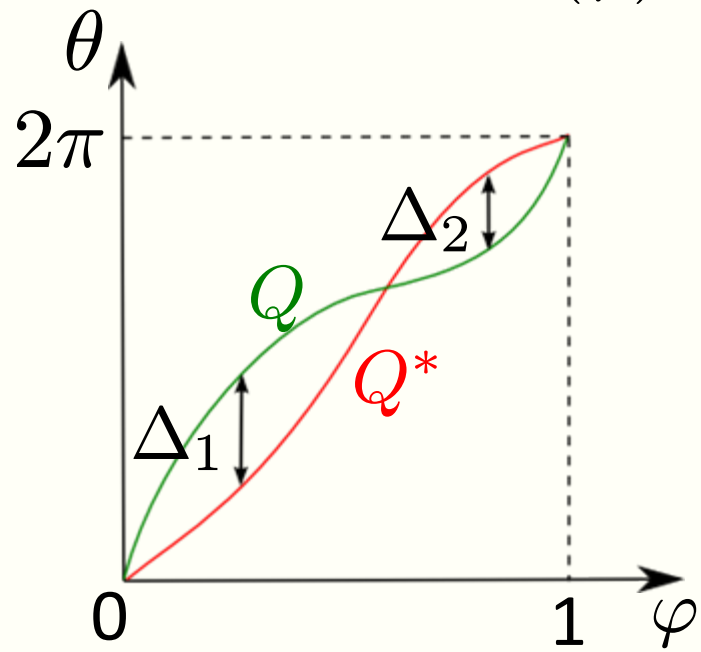
which corresponds to the total variation distance

$$\mathcal{V} = 2(\Delta_1 + \Delta_2)$$

cumulative density function:

$$P(\theta) = \int_0^\theta \rho(s)ds$$

quantile function: $P^{-1}(\varphi) \triangleq Q(\varphi)$



quantile density $q = \frac{dQ}{d\varphi}$

According to our global analysis, infinite populations of monotone oscillators have a dichotomic behavior

Provided that Z'' is of constant sign, $\mathcal{V}$ evolves according to

$$J_0 \min\left(K\,Z'\right)\mathcal{V} \leq \dot{\mathcal{V}} \leq J_0 \max\left(K\,Z'\right)\mathcal{V}$$

between two values:

$$\mathcal{V} = 0$$

(asynchronous state)

if $K\,Z' < 0$ ➔ $\dot{\mathcal{V}} < 0$

➔ exponential convergence to the asynchronous state

$$\mathcal{V} = 4\pi$$

(synchronization)

if $K\,Z' > 0$ ➔ $\dot{\mathcal{V}} > 0$

➔ synchronization in finite time

Mauroy & Sepulchre, submitted to IEEE TAC

We have proposed a **1-norm** to study the global stability of several **monotone** systems

Large populations of pulse-coupled monotone oscillators:

the firing map is a global contraction

Weakly pulse-coupled monotone oscillators:

a “Kuramoto-like” model is a global contraction

Infinite populations of monotone pulse-coupled oscillators:

a nonlinear PDE has global stability properties

➔ All these monotone systems have a dichotomic behavior

Outline

Pulse-coupled firing oscillators and dichotomic behavior

Large populations of monotone oscillators:
finite and infinite cases

Beyond monotone oscillators: around the QIF conjecture

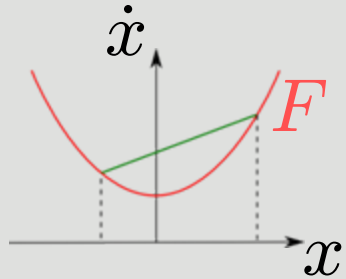
Numerical simulations suggest a dichotomic behavior for large populations of QIF oscillators

Numerical simulations: the fixed point of the firing map $\mathbf{H}$ is

globally stable when

$$F(\underline{x}) < F(\overline{x})$$

$$\underline{x} + \overline{x} > 0$$

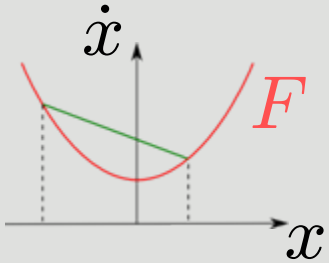


➔ phase-locked clustering

unstable when

$$F(\underline{x}) > F(\overline{x})$$

$$\underline{x} + \overline{x} < 0$$



➔ synchronization

When the thresholds are reversed, the stability is reversed:

$$\mathbf{H} \xrightarrow{[\underline{x}, \overline{x}] \to [-\overline{x}, -\underline{x}]} \mathbf{T}^{-1}\mathbf{H}^{-1}\mathbf{T}$$

The global stability problem in large populations of QIF oscillators leads to a standing conjecture

dichotomic behavior
of 2 oscillators

dichotomic behavior
of large populations

global stability of h

OK ☺

global stability of $\mathbf{H} = \mathbf{N} \circ \mathbf{L}$

open problem

QIF oscillators are not monotone ➔ the problem is more involved

$$\mathbf{H}^{\mathbf{n}} = \mathbf{N} \circ \mathbf{L} \circ \cdots \circ \mathbf{N} \circ \mathbf{L}$$

$\mathbf{N}$ contracting "on average"

$\mathbf{L}$ mixing: $(-\mathbf{L})^{N_g} = \mathbf{I}$

It is difficult to find a global contraction property "on average"

A local analysis suggests great evidence of the dichotomic behavior

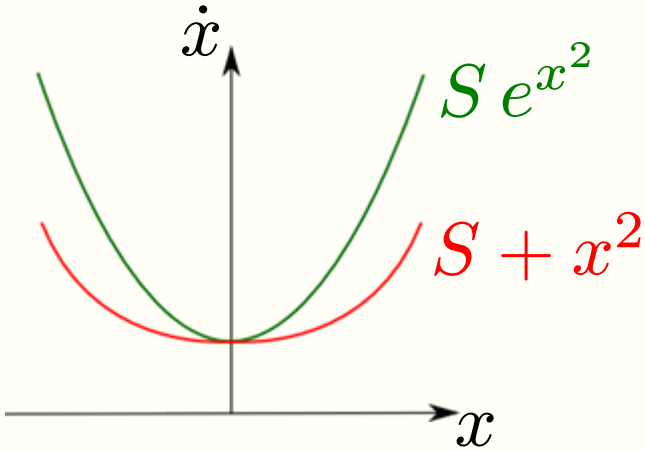
Far from the fixed point: the behavior is always compatible with a dichotomic behavior

In the vicinity of the fixed point:

- local stability is proved for three QIF oscillators
- local stability is proved for oscillators satisfying $h''(\theta) < 0 \quad \forall \theta$

  ➔ QIF oscillators with low thresholds

  ➔ exponential oscillators $\dot{x} = S\, e^{x^2}$



Mauroy et al., MTNS, 2010

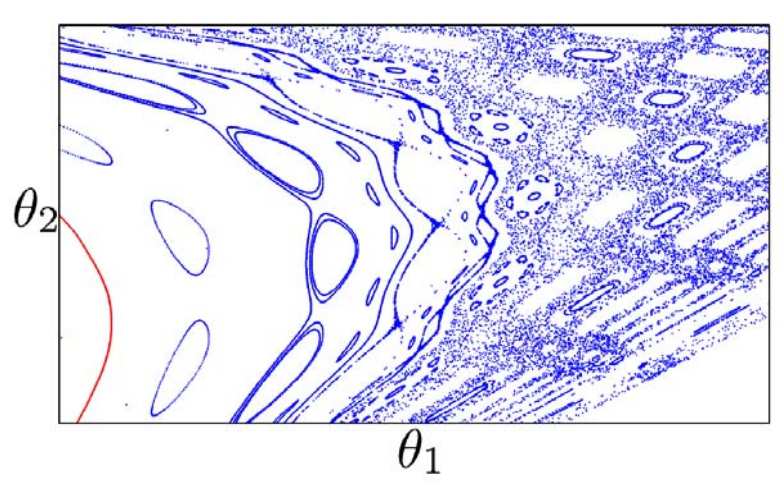
The local analysis proves that the dichotomic behavior is not a general property

Three (or more) piecewise linear oscillators $\dot{x} = S + \gamma|x|$ have not a dichotomic behavior:

Hopf bifurcation ➔ dense orbit

no global behavior

complex patterns

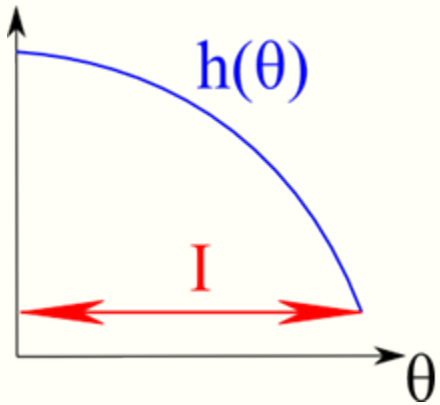


The dichotomy property of 2 oscillators does not always extend to larger networks !

QIF oscillators are a limit case between dichotomic and non-dichotomic oscillators

The length of $I = \{\theta | h''(\theta) < 0\}$ plays a key role to extend the stability of h to higher dimensions

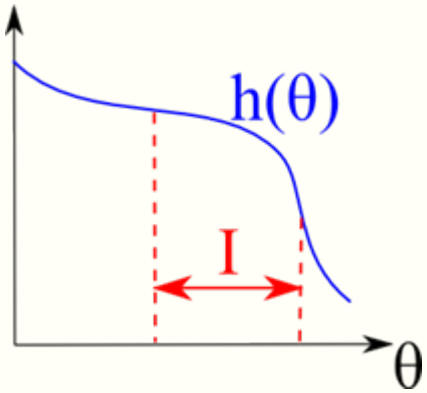
Exponential



$|I| = 2\pi$

dichotomic behavior

QIF

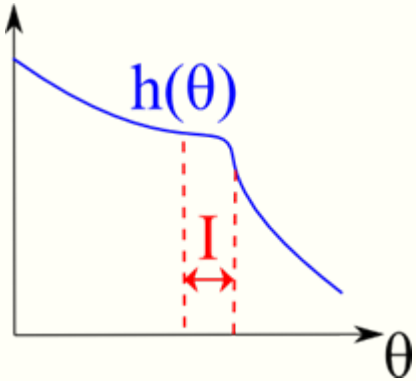


$|I| < 2\pi$

dichotomic behavior

➔ conjectures (global/local)

Piecewise linear



$|I| \ll 2\pi$

non-dichotomic behavior

In the weak coupling limit, the local stability problem is solved for QIF oscillators

Global stability: the problem is still elusive
but can be reformulated ("Kuramoto-like" model)

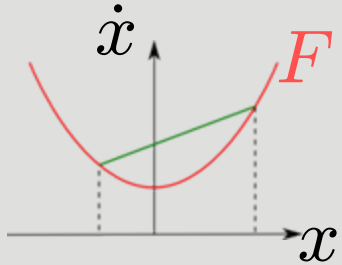
Local stability: the problem is solved

For finite populations, the fixed point of the firing map is

locally stable when

$$F(\underline{x}) < F(\overline{x})$$

$$\underline{x} + \overline{x} > 0$$

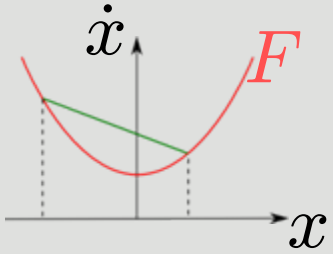


➔ phase-locked configuration

locally unstable when

$$F(\underline{x}) > F(\overline{x})$$

$$\underline{x} + \overline{x} < 0$$



➔ synchronization

Mauroy & Sepulchre, CDC, 2011

In the weak coupling limit, the local stability problem is solved for QIF oscillators

Global stability: the problem is still elusive
but can be reformulated ("Kuramoto-like" model)

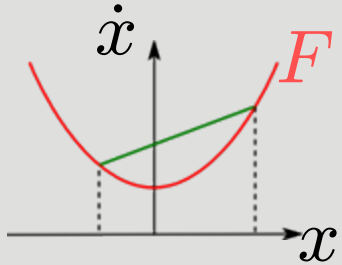
Local stability: the problem is solved

For infinite populations, the stationary density is

locally stable when

$$F(\underline{x}) < F(\overline{x})$$

$$\underline{x} + \overline{x} > 0$$

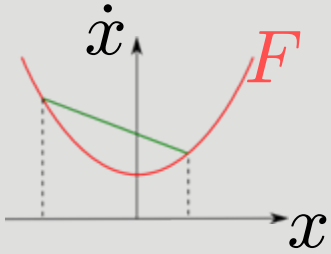


➔ asynchronous state

locally unstable when

$$F(\underline{x}) > F(\overline{x})$$

$$\underline{x} + \overline{x} < 0$$



➔ synchronization

Mauroy & Sepulchre, CDC, 2011

In the weak coupling limit, we obtain general criteria for the stability of finite and infinite populations

The stability criteria only depend on the phase response curve:

Finite populations: local stability if and only if

$$-\sum_k Z'\left(k\frac{2\pi}{N}\right) + \sum_k Z'\left(k\frac{2\pi}{N}\right)\cos\left(\frac{kn2\pi}{N}\right) \geq 0 \qquad \forall n \in \mathbb{N}$$

Infinite populations: local stability if and only if

$$K\int_0^{2\pi} Z(\theta)\sin(n\theta)d\theta \geq 0 \qquad \forall n \in \mathbb{N}$$

The results apply to general (multidimensional) oscillators (e.g. Hodgkin-Huxley model) and to delayed models

Non-monotone oscillators lead to standing conjectures and non-dichotomic behaviors

The global stability analysis of non-monotone oscillators is an involved problem (QIF conjecture(s))

Local results clearly suggest that the QIF oscillators have a dichotomic behavior

Populations of non-monotone oscillators may exhibit a non-dichotomic behavior

We have linked the (microscopic) dynamics of pulse-coupled oscillators to their (macroscopic) collective behavior

microscopic level → **macroscopic level**

1 oscillator	**2 oscillators**	**large population**	**infinite population**
monotone dynamics (e.g. leaky LIF)	dichotomy	dichotomy	equivalent behaviors
non-monotone dynamics (e.g. QIF)		QIF conjectures	

1-norm

local analysis

non-dichotomic

The present study leads to several perspectives for future research on pulse-coupled oscillators

QIF conjecture: a promising lead is to consider statistical time averages of the dynamics (operator theoretic methods)

Other dynamics: further research could focus on other application-oriented dynamics (neuroscience, earthquakes, etc.)

Excitable units: an interesting extension is to consider excitable units instead of oscillators (or mix both)

Non-identical oscillators: considering heterogeneous populations is a challenging problem that is still open

On the dichotomic collective behaviors of large populations of pulse-coupled firing oscillators

PhD defense

A. Mauroy

Université de Liège

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