Global analysis of pulse-coupled oscillators: 
discrete and continuous models

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1 Pulse-coupled firing oscillators

The Peskin model [1] is a simple but seminal model to study networks of interacting agents, such as pacemaker cells of the heart or neurons. Each agent corresponds to an oscillator whose state variable $x_i$ evolves according to the dynamics $\dot{x}_i = F(x_i) > 0$ between two threshold values. Upon reaching the upper threshold, the state is reset to the lower threshold — the oscillator is said to fire. The coupling is defined as follows: whenever an oscillator fires, it emits a pulse which instantaneously increases by $\varepsilon > 0$ the state of every other oscillators.

When considering standard leaky integrate-and-fire (LIF) oscillators, i.e. characterized by $F(x_i) = S + \gamma x_i$, the behavior of the network is strongly dichotomic: according to the curvature of their time evolution, identical oscillators either achieve perfect synchrony [2] or aggregate in a phase-locked clustering configuration [3]. In both situations, global convergence is established, which reinforces the evidence of dichotomy in the model. Using an appropriate 1-norm, we prove that the “distance” between two different configurations strictly increases (synchronization) or decreases (clustering) from one firing to the next one.

2 Global analysis of the continuous model

In order to investigate the case of a (very) large number of agents, the population of oscillators is approximated by a density function (mean-field approximation) which evolves according to a partial differential equation (PDE). The PDE is a standard transport equation with an additional coupling term, interpreted as a “feedback” term providing the system with stability properties. For LIF oscillators, the previous 1-norm is generalized and adapted to this infinite-dimensional case. It leads to the development of a Lyapunov functional, which is helpful and relevant for the study of the considered PDE. In particular, one shows that it is linked to the total variation distance often used to compare probability measures.

3 Extension of the LIF model

We next consider other dynamics $\dot{x}_i = F(x_i)$ characterized by even positive functions $F(x)$ verifying $F'(x) > 0$ for $x > 0$. For instance, such a model is the well-known quadratic integrate-and-fire (QIF) model, which corresponds to $F(x_i) = S + x_i^2$. In some cases, the local stability of the equilibrium configuration is proved. However, the global convergence appears to be more difficult to establish and, in particular cases, the dichotomy prevailing in the LIF model is no longer observed. We currently investigate sufficient conditions for the dichotomy to exist.

We will report on global stability results for both discrete and continuous LIF models and on preliminary results concerning the extension to more general models.

References


Acknowledgments

This work was supported by the Belgian National Fund for Scientific Research (FNRS) through a Research Fellowship at the University of Liège. This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its author(s).