Logic

Tutorial 4
29 November 2018

1. Write the following reasoning formally:
   - All men are mortal
   - Socrates is a man
   - Therefore, Socrates is mortal.
2. Is the following reasoning correct?

$$\frac{\begin{array}{l}\text{Some students do not work}\\ \text{All students want to pass}\end{array}}{\text{Some people want to pass without working}}$$

3. What is the link (in terms of logical consequences) between the following formulas?
   (a) $\forall x\, p(x) \vee \forall x\, q(x)$ et $\forall x\, (p(x) \vee q(x))$
   (b) $p(x)$ et $\exists x\, p(x)$
4. What is the link between the following couples of formulas?
   (a) $p(x)$ and $\forall x\, p(x)$
   (b) $\forall x\, p(x) \wedge \forall x\, q(x)$ and $\forall x\, [p(x) \wedge q(x)]$
   (c) $\forall x\, \forall y\, p(x, y)$ and $\forall x\, \forall y\, p(y, x)$
   (d) $\forall x\, (p(x) \Rightarrow q(x))$ and $\forall x\, p(x) \Rightarrow \forall x\, q(x)$
5. Using the semantic tableaux method, determine whether the following formulas are valid, consistent or inconsistent.
   (a) $\forall y\, [p(y) \Rightarrow \forall x\, p(x)]$
   (b) $\forall x\, [p(x) \Rightarrow q(x)] \Rightarrow [\forall x\, p(x) \Rightarrow \forall x\, q(x)]$
6. Tony, Mike and John are members of an alpine club. Each member is a skier or an alpinist or both. No alpinist likes rain but all skiers like snow. Mike likes nothing that Tony likes and likes everything that Tony doesn't like. Tony likes rain and snow. Is there a member of the alpine club that is an alpinist but not a skier?

7. Determine whether the following formulas are valid, consistent or inconsistent.

   (a) $\forall x\,[p(x) \Rightarrow p(a)]$
   (b) $\forall x\,[p(x) \Rightarrow p(x)]$
   (c) $\forall x\,[p(y) \Rightarrow q(x)] \Rightarrow [p(y) \Rightarrow \forall x\, q(x)]$
   (d) $\forall x\,[p(x) \Rightarrow q(x)] \Rightarrow [p(x) \Rightarrow \forall x\, q(x)]$

8. What can you say about the following inference rule?

$$\frac{P(a), \forall x\,[P(x) \Rightarrow P(f(x))]}{\forall x P(x)}$$

9. Using the semantic tableaux method, determine whether the following formulas are valid, consistent or inconsistent.

   (a) $[\forall x\, p(x) \wedge \neg\forall y\, q(y)] \vee \forall z\,[p(z) \Rightarrow q(z)]$
   (b) $\forall x \exists y\, p\,(x, y) \wedge \forall x\, \neg p(x, x) \wedge \forall x \forall y \forall z\,[(p(x, y) \wedge p(y, z)) \Rightarrow p(x, z)]$

10. What is the link between the following formulas?

   (a) $A \triangleq \forall x\, P(x) \Rightarrow \forall x\, Q(x)$
   (b) $B \triangleq \exists x\, P(x) \Rightarrow \forall x\, Q(x)$
   (c) $C \triangleq \forall x\, P(x) \Rightarrow \exists x\, Q(x)$
   (d) $D \triangleq \forall x\,[P(x) \Rightarrow Q(x)]$

11. What can you say about the following inference rule?

$$\frac{\forall x P(x, x), \forall x \forall y\,[P(x, y) \Rightarrow P(x, f(x))]}{\forall x \forall y P(x, y)}$$

12. What is the link between the following formulas?

   (a) $A \triangleq \forall x \exists y\,[P(x) \Rightarrow Q(x, y)]$
   (b) $B \triangleq \forall x\,[P(x) \Rightarrow \exists y\, Q(x, y)]$
   (c) $C \triangleq \forall x P(x) \Rightarrow \exists y\, Q(x, y)$
   (d) $D \triangleq \forall x\,[P(x) \Rightarrow \forall x \exists y\, Q(x, y)]$

13. What is the link between the following formulas?

    (a) $\alpha \triangleq \exists x\, \exists y\, \exists z\, [P(x, y) \Rightarrow [Q(x, z) \Rightarrow R(y, z)]]$
    (b) $\beta \triangleq \exists x\, \exists y\, [P(x, y) \Rightarrow [\forall z\, Q(x, z) \Rightarrow \exists z\, R(y, z)]]$
    (c) $\gamma \triangleq \forall x\, \forall y\, P(x, y) \Rightarrow [\forall x\, \forall z\, Q(x, z) \Rightarrow \exists y\, \exists z\, R(y, z)]$

14. What can you say about the following inference rule?

$$\frac{H \Rightarrow \forall x A(x), H \Rightarrow \exists x\, [A(x) \Rightarrow \forall y B(x, y)]}{H \Rightarrow \exists x \forall y B(x, y)}$$

15. If $A \models \forall x P(x)$ and $\exists x P(x) \models B$, then $A \Rightarrow \exists x P(x) \models \forall x P(x) \Rightarrow B$.
    Is this statement correct for all formulas A and B?

Prenex, Skolem and Clausal forms

A formula is in *prenex form* if it has the form

$\underbrace{Q_1 x_1 \times \cdots \times Q_n x_n}_{\text{prefix}} \underbrace{M}_{\text{matrix}}$ $\quad Q_i \in \{\forall, \exists\} \forall i$ and M is a quantification-free formula

The scope of the prefix must be the whole matrix.

Theorem: For every predicate formula, some logically equivalent prenex form exists.

Reduction to the prenex form

1. Eliminate all boolean connectives except $\neg$, $\vee$, $\wedge$
2. Rename bound variables (if necessary) so that no variable has both free and bound occurences in any subformula
3. Eliminate spurious quantifications
4. Propagate $\neg$ downwards and eliminate double negations
5. Propagate quantidications upwards

A *Skolem form* is a prenex form with only unversal quantifications

From prenex to Skolem form
For each existential quantification $\exists x$ in the scope of $k \geqslant 0$ universal quantifications $(\forall x_a \ldots \forall x_k))$

1. replace each occurence of x in the matrix by $f(x_1, \ldots, x_k)$ where f is a fresh k-ary function symbol ($k = 0$: replace x by a fresh constant)
2. delete the quantification $\exists x$.

Theorem: The Skolem form S_A associated with the prenex form A is consistent if and only if A is consistent.

A formula is in *clausal form* if it is in Skolem form and if its matrix is in conjunctive normal form.

1. Give the prenex, Skolem and clausal form of the following formulas:
   - $p(a) \wedge \exists x \neg p(x)$
   - $\forall x\ [p(x) \Rightarrow \forall y\,[\forall z\, q(x, y) \Rightarrow \neg \forall z\, r(y, x)]]$
   - $\forall x\, p(x) \Rightarrow \exists x\,[\forall z\, q(x, z) \vee \forall z\, r(x, y, z)]$
   - $\exists x\, p(x, z) \Rightarrow \forall z\,[\exists y\, p(x, z) \Rightarrow \neg \forall x \exists y\, p(x, y)]$
   - $[\exists x\, p(x) \vee \exists x\, q(x)] \Rightarrow \exists x\,[p(x) \vee q(x)]$

2. Consider the following inference rules:

$$\frac{H \Rightarrow \forall x A \quad H \Rightarrow \forall x\,(A \Rightarrow B)}{H \Rightarrow \forall x B} \qquad \frac{H \Rightarrow \exists x A \quad H \Rightarrow \forall x\,(A \Rightarrow B)}{H \Rightarrow \exists x B}$$

$$\frac{H \Rightarrow \exists x A \quad H \Rightarrow \exists x\,(A \Rightarrow B)}{H \Rightarrow \exists x B} \qquad \frac{H \Rightarrow \forall x A \quad H \Rightarrow \exists x\,(A \Rightarrow B)}{H \Rightarrow \exists x B}$$

   Are they correct?
   If not, do they become correct if one adds restrictions on the occurences of the variable x within A and/or B? Motivate your answers.

3. In the definition of the interpretation of a predicate formula, what hypothesis is absolutely necessary in order for $\exists x\,(p(x) \Rightarrow p(x))$ to be valid.

Syllogism Theory

- 4 basic formulas

$A : \forall x(P(x) \Rightarrow Q(x))$	universal afffirmative
$E : \forall x(P(x) \Rightarrow \neg Q(x))$	universal negative
$I : \exists x(P(x) \wedge Q(x))$	particluar affirmative
$O : \exists x(P(x) \wedge \neg Q(x))$	particular negative

- 3 predicates: $P(x)$ is the *minor*, $R(x)$ the *major* and $Q(x)$ the *midterm*
- The *(categorical) syllogism* is the inference rule

$$\frac{\text{Major } \{Q, R\} \qquad \text{Minor } \{P, Q\}}{\text{Conclusion } \{P, R\}}$$

- 64 modes: records the nature of the premises and the conclusion: of the form XYZ where X is the major, Y is the minor and Z is the conclusion
- 4 figures (order of the terms / predicates within the premises)

Figure	1	2	3	4
Major	QR	RQ	QR	RQ
Minor	PQ	PQ	QP	QP
Conclusion	PR	PR	PR	PR

- A syllogism is *quasi-valid* if it is a syllogism that is not valid, but that becomes valid by adding $\exists x P(x)$ or $\exists x Q(x)$ or $\exists x R(x)$

1. Determine the predicates and the formulas of the following syllogisms, state their mode and figure. Using a Venn diagram, determine whether these syllogisms are valid, quasi-valid, ...

$$\frac{\forall x\,(Q(x) \Rightarrow R(x)) \quad \forall x\,(P(x) \Rightarrow Q(x))}{\forall x\,(P(x) \Rightarrow R(x))}$$

$$\frac{\forall x\,(A(x) \Rightarrow B(x)) \quad \exists x\,(B(x) \wedge C(x))}{\exists x\,(C(x) \wedge \neg A(x))}$$

2. Is the following rule a syllogism? Can it be transformed into a syllogism? Is it correct?

$$\frac{\forall x \exists y\,[\neg Q(x,y) \vee R(x)] \quad \exists x \forall y\,[P(x) \wedge Q(x,y)]}{\exists y\,[R(y) \wedge P(y)]}$$