Introduction to computability
Tutorial 7

Pushdown Automata, Context Free Languages, Turing Machines and Recursive Functions

13 November 2018

Context-free languages

The "pumping" lemma: Let L be a context-free language. Then there exists a constant K, such that for any word $w \in L$ satisfying $|w| \geqslant K$ can be written $w = uvxyz$ with v or $y \neq \varepsilon$, $|vxy| \leqslant K$ and $uv^n xy^n z \in L$ for all $n \geqslant 0$.

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1. Show that the following languages are not context-free:
   a) $L = \{a^i b^j a^k | j = max(i, k)\}$;
   b) $L = \{w\bar{w} | w \in \{0, 1\}^*\}$ where $\bar{w}$ is the complement of w, that is, the word w where every 1 is replaced by a 0 and every 0 is replaced by a 1.

2. Assume
   - that a language L is accepted by a pushdown automaton M and
   - that there exists a bound $N \in \mathbb{N}$ such that for any execution of M on a word w, the stack never contains more than N symbols.

Show that L is regular.

2. Assume
   - that a language L is accepted by a pushdown automaton M and
   - that there exists a bound $N \in \mathbb{N}$ such that for any execution of M on a word w, the stack never contains more than N symbols.

Show that L is regular.

3. Let L be a context-free language.
   a) Is L^R context-free?
   b) Is $L \cap L^R$ context-free?

Turing Machine

A Turing Machine M is a 7-tupel $M = (Q, \Gamma, \Sigma, \delta, s, B, F)$, where:

- Q is a finite set of states,
- Γ is the tape alphabet,
- $\Sigma \subseteq \Gamma$ is the input alphabet,
- δ: $Q \times \Sigma \to Q$, is the transition function,
- $s \in Q$ is the initial state,
- $F \subseteq Q$ is the set of accepting states,
- $B \in \Gamma \backslash \Sigma$ is the "blank symbol" ($\#$),
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the transition function.

4. Let $M = (Q, \Gamma, \Sigma, \delta, q_0, \sharp, \emptyset)$ be a Turing machine where $Q = \{q_0, q_1, q_2, q_3\}$, $\Gamma = \{a, b, A, B, A', B'\}$, $\Sigma = \{a, b\}$ and δ contains the following transitions:

$(q_0, a) \rightarrow (q_1, A', R)$	$(q_1, \#) \rightarrow (q_2, A, L)$	$(q_2, B') \rightarrow (q_0, B, R)$
$(q_0, b) \rightarrow (q_3, B', R)$	$(q_2, a) \rightarrow (q_2, a, L)$	$(q_3, a) \rightarrow (q_3, a, R)$
$(q_1, a) \rightarrow (q_1, a, R)$	$(q_2, b) \rightarrow (q_2, b, L)$	$(q_3, b) \rightarrow (q_3, b, R)$
$(q_1, b) \rightarrow (q_1, b, R)$	$(q_2, A) \rightarrow (q_2, A, L)$	$(q_3, A) \rightarrow (q_3, A, R)$
$(q_1, A) \rightarrow (q_1, A, R)$	$(q_2, B) \rightarrow (q_2, B, L)$	$(q_3, B) \rightarrow (q_3, B, R)$
$(q_1, B) \rightarrow (q_1, B, R)$	$(q_2, A') \rightarrow (q_0, A, R)$	$(q_3, \#) \rightarrow (q_2, B, L)$

a) What is on the tape after an execution of M on the word *abab*?

b) Describe what the Turing machine M does on a word $w \in \{a, b\}^*$.

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4. Let $M = (Q, \Gamma, \Sigma, \delta, q_0, \sharp, \emptyset)$ be a Turing machine where $Q = \{q_0, q_1, q_2, q_3\}$, $\Gamma = \{a, b, A, B, A', B'\}$, $\Sigma = \{a, b\}$ and δ contains the following transitions:

$(q_0, a) \rightarrow (q_1, A', R)$	$(q_1, \#) \rightarrow (q_2, A, L)$	$(q_2, B') \rightarrow (q_0, B, R)$
$(q_0, b) \rightarrow (q_3, B', R)$	$(q_2, a) \rightarrow (q_2, a, L)$	$(q_3, a) \rightarrow (q_3, a, R)$
$(q_1, a) \rightarrow (q_1, a, R)$	$(q_2, b) \rightarrow (q_2, b, L)$	$(q_3, b) \rightarrow (q_3, b, R)$
$(q_1, b) \rightarrow (q_1, b, R)$	$(q_2, A) \rightarrow (q_2, A, L)$	$(q_3, A) \rightarrow (q_3, A, R)$
$(q_1, A) \rightarrow (q_1, A, R)$	$(q_2, B) \rightarrow (q_2, B, L)$	$(q_3, B) \rightarrow (q_3, B, R)$
$(q_1, B) \rightarrow (q_1, B, R)$	$(q_2, A') \rightarrow (q_0, A, R)$	$(q_3, \#) \rightarrow (q_2, B, L)$

a) What is on the tape after an execution of M on the word $abab$?

b) Describe what the Turing machine M does on a word $w \in \{a, b\}^*$.

5. For each of the following languages, give a Turing machine that decides the language:

a) $\{a^n b^n c^n | n \geqslant 0\}$;

b) $\{a^n b^n c^m | n \leqslant m < 2n\}$.

- Configuration: 3-tuple $(q, \alpha_1, \alpha_2) \in Q \times \Gamma^* \times (\varepsilon \cup \Gamma^*(\Gamma \backslash \{B\}))$.
- Derivation:
  A configuration (q, α_1, α_2) written as $(q, \alpha_1, b\alpha_2')$ with $(b = \# \text{ if } \alpha_2 = \varepsilon)$.

  If $\delta(q, b) = (q', b', \boxed{\text{R}})$ we have

  $$(q, \alpha_1, b\alpha_2') \vdash_M (q', \alpha_1\boxed{\text{b'}}, \alpha_2').$$

  If $\delta(q, b) = (q', b', \boxed{\text{L}})$ and if $\alpha_1 \neq \varepsilon$and thus of the form $\alpha_1' a$ we have

  $$(q, \alpha_1' a, b\alpha_2') \vdash_M (q', \alpha_1', \boxed{\text{ab'}}\alpha_2').$$

- The language accepted by the Turing Machine M is the set

  $$L(M) = \{w \mid (s, \varepsilon, w) \vdash_M^* (p, \alpha_1, \alpha_2), p \in F\}.$$

- A language L is decided by a Turing Machine M if M accepts L and M has no infinite executions.

6. One can define Turing machines that have the possibility to write a symbol without moving the read head. Such a Turing machine is thus a septuple

$$M = (Q, \Gamma, \Sigma, \delta, s, B, F)$$

where Q, Γ, Σ, s, B and F have the usual meaning and where $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$ (S meaning "stay in the same place").

a) Define the derivation relation "$\vdash_M$" (in one step) for these Turing machines.
b) Show that this extension of the Turing machines does not change the class of recognized languages.

7. Show that the following functions are primitive recursive:

a) $square(x)$ that is equal to 1 if x is a square and 0 otherwise;

b) $sumSquares(x) = \sum_{i=0}^{x} i^2$;

c) $mod(x, y)$ that computes the remainder of the division of x by y;

d) $divides(x, y)$ that is equal to 1 if x divides y and 0 otherwise;

e) bounded maximization:

$$\nu i \leqslant m\, q(\bar{n}, i) = \begin{cases} \text{the greatest } i \leqslant m \text{ such that } q(\bar{n}, i) = 1 \\ 0 \text{ if there is no such } i \end{cases} \quad ;$$

f) $gcd(x, y)$ that computes the greatest common divisor of x and y;

g) $lcm(x, y)$ that computes the least common multiple of x and y.

Bonus Exercise 7

Give a Turing machine that for the following initial configuration $(s_0, \varepsilon, \$1^n01^m\$)$ terminates for any natural numbers n and m with the configuration $(q_f, \varepsilon, \$1^{n+m}\$)$ where s_0 is an initial and q_f an accepting state of the machine.

Bonus Exercise 8

Show that $lcm(x, y)$ that computes the least common multiple of x and y is primitive recursive.