Introduction to the Theory of Computation

Final exam

August 21$^{\text{th}}$, 2018

Closed-book. Duration: 3h30.
Please answer each question on a separate sheet of paper with your name and section. ***Motivate all your answers and give sufficient details.***

1. a) Is the set of well formed arithmetic expressions denumerable? For example $3 * (2 + 4)$ is well formed and $3 + *5$ is not.

   b) Is the set of all closed intervals of $\mathbb{R}$ with real bounds denumerable?

2. Let $\Sigma = \{a, b, c\}$ and L be the language over Σ generated by the following regular expression $[c^+(a^*b \cup (ba)^*)]^*$

   a) Give an NFA accepting L.

   b) Give a DFA accepting L.

   c) Give a regular grammar that generates $\overline{L}$, the complement of L.

3. a) Is the language $L = \{a^n b^m \mid n \text{ divides } m\}$ regular? Is this also the case if we invert the direction of the divisibility, i.e. consider the language $L' = \{a^n b^m \mid m \text{ divides } n\}$

   b) Is the union of two non-regular languages also a non-regular language?

4. a) Is the language $L = \{a^i b^j c^k \mid i = j \text{ or } j = k \text{ where } i, j, k \geqslant 0\}$ context-free?

   b) Show that the intersection of two context-free languages is not necessarily context-free. Use this to deduce that the complement of a context-free language is not necessarily context-free. Give a sufficient criterion for the intersection of two context-free languages to be context-free.

5. a) Give a Turing machine that computes the function $f(x, y) = x + 2y$. Consider that numbers are encoded using a single symbol, so that x is represented by x repetitions of this single symbol. To simplify manipulating numbers encoded in this way, we will also use another symbol to mark the beginning and the end of the number's encoding. If q_0 is an initial state and q_f a final state of the Turing machine, you must for instance have

$$q_0 . \underline{0}11010\# \vdash^* q_f . 011110\underline{\#}$$

which represents $f(2, 1) = 2 + 2 * 1 = 4$.

b) Give an example of a language that is accepted but not decided by a Turing Machine and an example of a language that is decided but not accepted by a Turing Machine, or explain why such an example does not exist.

6. a) Define the notion of *primitive recursive* functions as well as the concepts used in this definition.

b) Show that $\text{IntegerSqrt}(n) = \lceil \sqrt{n} \rceil$ is primitive recursive. Is IntegerSqrt μ-recursive?

7. a) Let M_1 and M_2 be two Turing Machines that accept the languages L_1 and L_2 respectively. Show that determining whether there exists $w \in L_1$ such that M_2 stops on w is undecidable.

b) i) Define the decidability classes R and RE.

ii) Prove that if a language L and its complement $\overline{L}$ both belong to RE, then both, L as well as $\overline{L}$, belong to R.

8. a) Define the complexity classes P, NP and NPC.

b) Consider a language L. If $L \in NP$, is L decidable? And if $L \in co - NP$, is L decidable? Prove your answers.