Introduction to computability

Final exam

27 August 2014

Closed-book. Duration: 3h30

Please answer each question on a separate sheet with your name and section. ***Motivate all your answers and give sufficient details.***

1. a) Is the set $\mathbb{N} \times \{0,1\}$ denumerable?

   b) Is the set of Java programs denumerable?

2. a) Give a DFA that accepts the language L_1 containing all the words on the alphabet $\{a,b\}$ that begin with a and end with b.

   b) Give a DFA that accepts the language L_2 containing all the words on the alphabet $\{a,b\}$ that contain at least three a's. In other words,
   $$L_2 = \{w \mid N_a(w) \geqslant 3\}$$
   where $N_\sigma(w)$ is the number of letters σ contained in the word w.

   c) Give a DFA that accepts $L_1 \cap L_2$.

3. a) Show that the language $L = \{www \mid w \in \{a,b\}^*\}$ is not regular.

   b) Does there exist a regular language L_1 such that for every context-free language L_2 one has $L_1 \subseteq L_2$? And does there exist a regular language L_1 such that for every context-free language L_2 one has $L_2 \subseteq L_1$?
   Let L_1 be a regular language and L_2 a language such that $L_2 \subseteq L_1$. Does this imply that L_2 is a regular language?

4. a) Let $\Sigma = \{a,b,c\}$, show that
   $$L = \{w \mid \text{the length of } w \text{ is odd and its middle symbol is a } c\}$$
   is context-free by giving a pushdown automaton that accepts L as well as a context-free grammar that generates L.

   b) State the pumping lemma for context-free languages.

5. a) For a Turing machine M, define the notions of *configuration*, *derivation*, *execution*, *accepted language* and *decided language*.

   b) Give a Turing machine that computes the function $n \mapsto 2n$. Consider that numbers are encoded using a unary alphabet, so that n is represented by n repetitions of the unique letter of the alphabet. If q_0 is an initial state and q_f a final state of the Turing machine, you must for instance have

$$q_0 \,.\, \underline{1}11\# \vdash^* q_f \,.\, 111111\underline{\#}$$

6. a) Show that the function nbfact(n, f), where n is an integer and f a prime number, that computes the multiplicity of f in the prime factorization of n, is primitive recursive.
   For example, nbfact$(18, 3) = 2$ as $18 = 2 * 3 * 3$ and nbfact$(18, 5) = 0$.
   Remark: You can use any primitive recursive functions and predicates seen in the lectures.

   b) Define *μ-recursive predicates*. Why are they of interest in computability theory?

7. a) Given two Turing machines M_1 and M_2, accepting respectively the languages L_1 and L_2. Show that the problem of determining if $L_1 \cap L_2 = \varnothing$ is undecidable.

   b) Define the classes R and RE. What can you say about a language L and its complement $\overline{L}$ with respect to membership in R and RE?

8. a) Give an explicit statement of *Cook's theorem* and explain its importance.

   b) Prove that 3SAT is NP-complete.

   c) What is the difference between NP-complete and NP-hard problems?