

# Introduction to information theory and coding

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Set of slides No 5

## State of the art in data compression

- Stochastic processes and models for information sources
- First Shannon theorem : data compression limit
- Overview of state of the art in data compression
- Relations between data mining and data compression

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## Optimal prefix codes - Huffman Algorithm

**Note : illustrations in the binary case.**

Let  $n$  be the length of the longest codeword ( $q$ -ary code).

Complete  $q$ -ary tree of depth  $n$  : acyclic graph built recursively starting at the root (cf figure).

Oriented arcs labeled by the  $q$  code symbols.

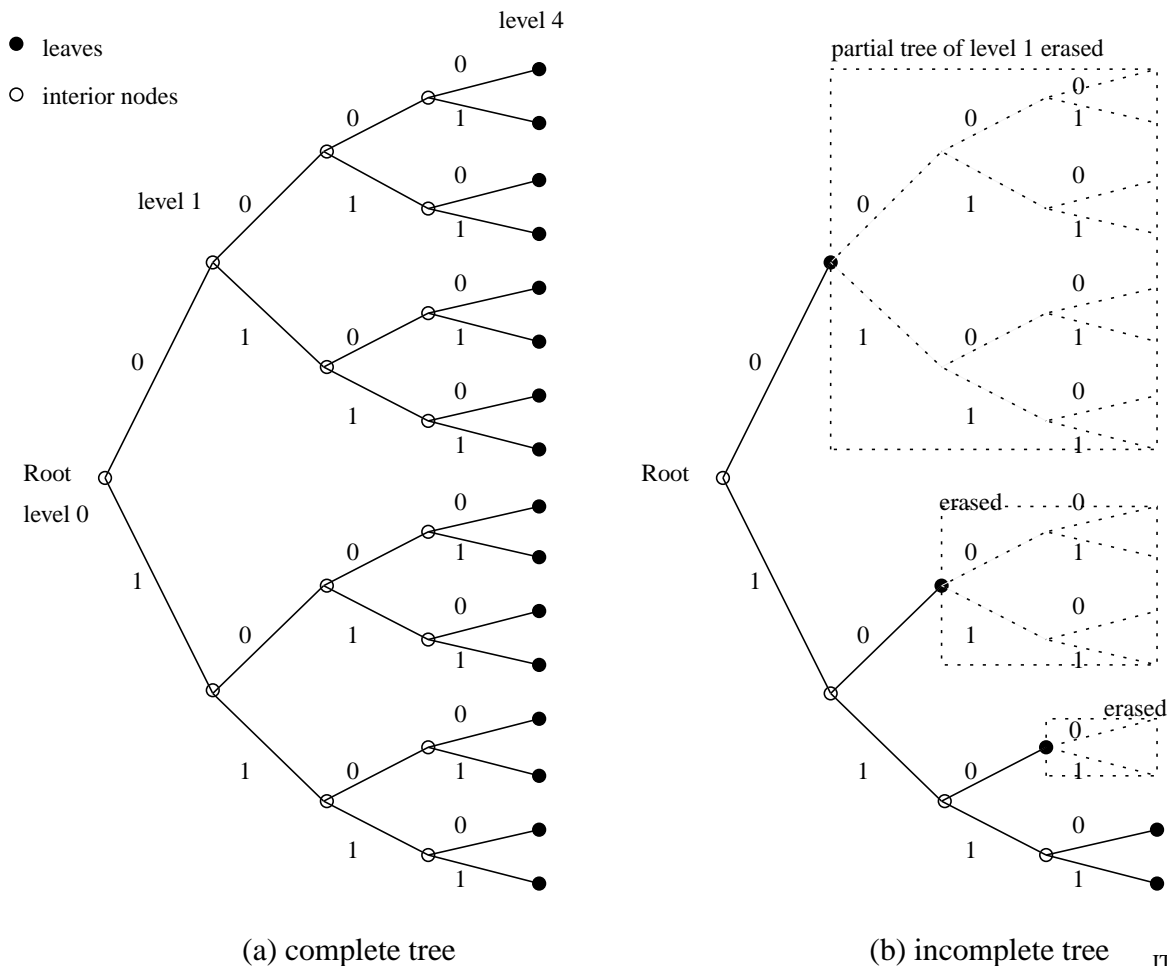
Father, son, siblings (brothers), descendants, ascendants...

Interior vs terminal nodes (leaves)

Path : sequence of arcs  $(u_{i_1}, u_{i_2})$  where  $u_{i_1} = u_{(i-1)_2}$ .

Levels : root = level 0,

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## Incomplete trees

Complete tree of which some complete subtrees have been “erased”

∃ trees which are neither complete nor incomplete

But “tree is complete or incomplete” ⇔

$$A = (L - 1) \frac{q}{q - 1}.$$

## Relations between trees and codes

The paths towards the leaves of complete  $q$ -ary tree of depth  $n$  are in one-to-one relation with sequences of length  $n$  built from a  $q$ -ary alphabet.

Each and every instantaneous (prefix-free) code may be represented by a complete or incomplete tree, and reciprocally, every such tree defines a prefix-free code.

(Nb. Some leaves may possibly not be labeled by a codeword)

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**Proof :**

Let assume we have a prefix-free code of word-lengths  $n_i$ .

Let

$$n = \max_{i=1, \dots, Q} \{n_i\}.$$

We start with a complete tree of depth  $n$  :

All codewords necessarily correspond to paths in the tree starting at the root and ending somewhere in the complete tree (at a leaf or maybe at some interior node).

All these paths are different, since the code is necessarily regular.

The complete tree initially has a total of  $q^n$  leaves.

Let us construct the uncomplete tree corresponding to the code by pruning away some of its (complete) subtrees.

How ?

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We merely insert all the codewords and mark the end-nodes of the corresponding paths.

Now, let  $m_i$  be one of these codewords : we erase all successor nodes of the last node in the corresponding path.

By doing this, we will not delete any of the marked nodes. Why ?

In addition, the number of leaves of the complete tree that we remove in this process is equal to  $q^{n-n_i}$ .

Thus, we remove like this a total of  $\sum_{i=1}^Q q^{n-n_i}$  leaves of the original complete tree, by iterating the deletion process through all codewords, without removing any of the  $Q$  nodes originally marked as a codeword.

**Consequence :** Since the original tree has  $q^n$  leaves, and since (obviously) this process works, we deduce that

$$\sum_{i=1}^Q q^{n-n_i} \leq q^n$$

$\Rightarrow$  prefix-free code must satisfy the Kraft inequality.

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Now let us show that if Kraft is true there exists a uniquely decodable code with the given word length :

*If the  $n_i$  verify Kraft inequality, then  $\exists$  an instantaneous code (hence uniquely decodable) based on these lengths.*

Let us start with  $\sum_{i=1}^n r_i q^{n-i} \leq q^n : \Rightarrow \forall p \leq n : \sum_{i=1}^p r_i q^{p-i} \leq q^p$ .

Why is it true ? ( $r_i$  is the number of words of length  $i$ )

(Multiply Kraft by  $q^{p-n}$ , and truncate the sum after the  $p$ -th term.)

Now let us prove that we can build a prefix-free code with the original word lengths ( $r_i$  words of length  $i$ ,  $\forall i$ ).

1.  $r_1$  words of length 1 : is it possible ?

2.  $r_2$  words of length 2 : is it still possible ? We must have  $(q - r_1)q \leq r_2$  (?)

3.  $r_p$  words of length  $p$  : suppose we have already chosen the  $p - 1$  first groups of words, and then verify that there is still enough room to insert the  $r_p$  words of length  $p$ .

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### Questions.

If I give you the word lengths of an instantaneous code, could you build any such a code compatible ?

Under which condition can we complete the code with more words ?

If we want to add words of minimal length :

How many words of length  $\leq n$  can we certainly add ?

What is the meaning of the condition  $\sum_{i=1}^n r_i q^{-i} = 1$  ?

And in terms of code tree ?

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**Given source symbol probabilities  $P(S_i)$ , how to build an optimal code ?**

NB: optimal = average word length minimal.

**Let us explore the problem in the particular case where  $q = 2$  (binary code)**

A first idea to construct  $Q$  prefix free codewords :

Start with a complete tree of depth  $n = \lceil \log_2 Q \rceil$ .

NB : if  $Q = 2^n$  and  $P(S_i)$  uniform, it is not necessary to work further (Why ?).

Otherwise : if  $Q < 2^n$  we delete  $2^n - Q$  subtrees of depth  $n - 1$ .

Is it possible ?

⇒ **We have an algorithm to build a first, not necessarily optimal, code-tree**

**Idea : try to modify the tree, until it becomes an optimal tree.**

**Search operator : exchange subtrees in such a way that the average word length always decreases, and detect when optimality has been reached.**

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**Optimality : is  $\bar{n} = \sum_{i=1}^Q n_i P(S_i)$  minimal ?**

*How to recognize an optimal tree ?*

*How to improve the tree in order to reach optimality ?*

*Reasoning tool : node probabilities*

We decorate the leaves of the tree with the source symbol probabilities.

We propagate this information upwards towards the root : a node receives the sum of the probabilities of its sons.

The recursive structure of the code and of the tree, implies that if a codetree is optimal, than all its subtrees are also optimal (with respect to sub-source alphabets).

In particular, the partial (pending) trees must be optimal for the subset of source symbols. Why ?

And also, the reduced trees that we would get by deleting some of the subtrees, must be optimal with respect to the probabilities attached at the corresponding nodes. Why ?

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## But there is more :

An optimal tree must also respect a *non-local* condition which implies pairs of partial trees of different levels :

*If  $T_1$  and  $T_2$  are two partial trees of different levels (levels of their root) and of different probabilities, then the most probable of the two must be the least deep one.*

Indeed, otherwise we could swap the two subtrees and thereby improve (reduce) average wordlength. By how much ?

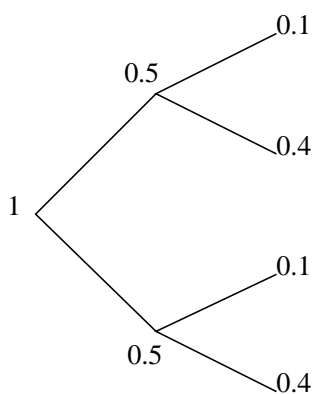
For example if  $n_1 < n_2$  and  $p_1 < p_2$  :  $\Delta\bar{n} = (n_2 - n_1)(p_2 - p_1)$ .

This criterion provides a nice criterion to improve our codetree!

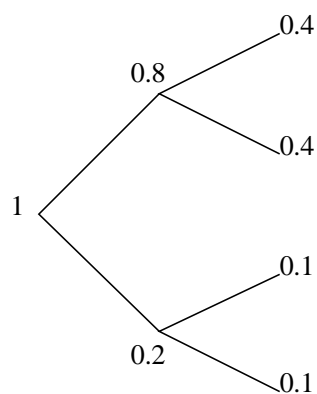
We merely need to localise subtrees of different levels and different probabilities which violate the condition, and swap them to improve our code.

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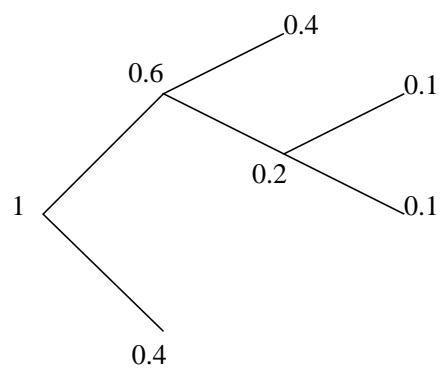
## Illustration(s)



(a) Average length : 2



(b) Average length : 2



(c) Average length : 1.8

Can we exchange the partial trees which should be exchanged while improving (strictly) the average codelength ?

Do we have an algorithm ?

Yes, but it doesn't work... (in all cases)

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## Conclusion :

We need to impose one more (at least) constraint on our codetree, in order to be optimal.

Actually, we can (easily) prove the following :

*For any source probability distribution, there exists an optimal prefix-free code that satisfies the following properties :*

1. *If  $p_j > p_i$ , then  $n_j \leq n_i$ .*
2. *The two longest codewords have the same length*
3. *The two longest codewords differ only in the last bit and correspond to the least likely symbols.*

Summary : if  $p_1 \geq p_2 \geq \dots \geq p_Q$ , then there exists an optimal (binary code) with length satisfying  $n_1 \leq n_2 \leq \dots \leq n_{Q-1} = n_Q$ , and codewords  $m_{Q-1}$  and  $m_Q$  differing only in the last bit.

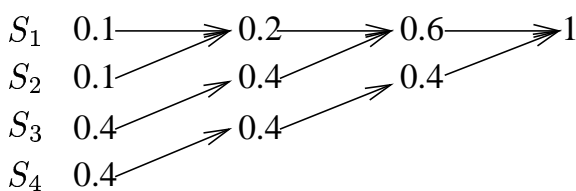
⇒ **We can restrict our search in the class of codes which satisfy these properties.**

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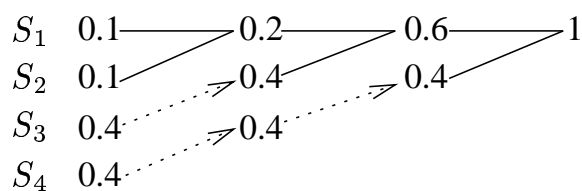
## Huffman Algorithm

Who has guessed ?

Who does remember ?



(a) Code construction



(b) Building of code-tree

This produces an optimal prefix-less code (not unique in general).

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## Synthesis.

Huffman produces an optimal code tree and prefix-free code, such that

$$\frac{H(S)}{\log q} \leq \bar{n} < \frac{H(S)}{\log q} + 1$$

**Absolutely optimal code** : If  $\bar{n} = \frac{H(S)}{\log q}$

Iff  $n_i = -\log_q P(S_i), \forall i \Leftrightarrow P(S_i) = q^{-n_i}, \forall i.$

Reciprocally :

For every set of word lengths  $n_i$  which respect the Kraft equality, there exists a source probability distribution  $q_i$  such that the optimal code has these word lengths and is absolutely optimal.

What if  $p_i \neq q_i$ ? One can show that in this case (binary code)

$$\bar{n} = H_Q(p_1, \dots, p_Q) + D((p_1, \dots, p_Q) \parallel (q_1, \dots, q_Q)).$$

What if  $q$ -ary code ?

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## Data compression algorithms

### 1. Reversible text compression

- Zero order methods  $\rightarrow$  arithmetic coding
- Higher order methods
- Adaptative methods
- Dictionary methods

### 2. Image compression

- Multi-dimensional information structures
- Sources of redundancy
- Image transform based methods

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## 1. Reversible text compression

Let us assume binary input and output alphabets.

$T$  : input text (sequence of bits)

$U = C(T)$  : coded text (sequence of bits)

$\ell(\cdot)$  : length

Compression rate of  $C$  on text  $T$  :  $\frac{\ell(T)}{\ell(C(T))}$

Realized rate : average of texts  $T$ .

### Preliminary stage :

Choice of a source alphabet  $\rightarrow$  segmentation of text into a sequence of words

$S = \{s_1, \dots, s_m\}$

Parsing of  $T$  :  $T \approx s_{i_1} \cdots s_{i_t}$

## Zero order methods

Intuitively : we take into account only the frequencies of the  $s_i$ .

(Higher order : we take also into account correlations among successive symbols)

Non-adaptative : code independent of the position in the text.

(Adaptative : code evolves as the text is screened)

NB. Text screened by increasing order of indexes

$\Rightarrow$  One-dimensional (oriented) structure (time : from left to right)

### 1. Replacement schemes

Idea : replace each  $s_i \rightarrow w_i$  (Shannon, Fano, Huffman)

### 2. Arithmetic codes

Remplace the whole text :  $T \rightarrow r \in [0, 1[$ .

**Example :**  $T = 111110111111101110111101110110$

Let's suppose that the  $s_i$  are chosen as follows

$$\begin{aligned} s_1 &= 0 \\ s_2 &= 10 \\ s_3 &= 110 \\ s_4 &= 1110 \\ s_5 &= 1111, \end{aligned} \tag{1}$$

$\Rightarrow$  parsing of  $T$  gives :  $T = s_5 s_2 s_5 s_4 s_4 s_5 s_1 s_4 s_3$ .

Let the code be  $C : s_i \rightarrow w_i$

$$\begin{aligned} s_1 &\rightarrow 1111 \\ s_2 &\rightarrow 1110 \\ s_3 &\rightarrow 110 \\ s_4 &\rightarrow 10 \\ s_5 &\rightarrow 0. \end{aligned} \tag{2}$$

$\Rightarrow C(T) = 01110010100111110110$ .

Thus  $\ell(C(T)) = 20$  (et  $\ell(T) = 30$ )  $\Rightarrow$  compression rate of  $3/2$ .

## Questions

How to chose the  $s_i$  ? (source alphabet)

How to chose the  $w_i$  ? (code)

NB: both influence the compression rate.

E.g. : if  $S = \{0, 1\}$ , replacement scheme always gives a compression rate  $\leq 1$ .

### 1. Choice of $S$

Which types of  $S$  make sense ?

We restrict our choice to the sets which are sufficiently rich to parse any sequence of bits, and at the same time sufficiently small to do the parsing in a single way (no ambiguity).

This will mean that for such an  $S$ , every binary text may be parsed in a single way into a sequence of symbols in  $S$ .

**Definition : SPP.**

$S = \{s_1, s_2, \dots, s_m\}$  (binary words) verify the *strong parsing property* SPP)  
 $\Leftrightarrow$  every binary text is representable in the form of a *unique* concatenation,

$$T = s_{i_1} \cdots s_{i_t} \nu, \quad (3)$$

of some of the  $s_i$  and a suffix  $\nu$  (possibly empty), such that none of the  $s_i$  is a prefix of  $\nu$ , and  $\ell(\nu) < \max_{1 \leq i \leq m} \ell(s_i)$ .

$\nu = \text{leaf}$  of the parsing of  $T$  by the  $s_i$ .

**Uniqueness : If**

$$T = s_{i_1} \cdots s_{i_t} \nu = s_{j_1} \cdots s_{j_r} \mu$$

with  $\nu$  and  $\mu$  having none of the  $s_i$  as prefix and  $\ell(\nu), \ell(\mu) < \max_{1 \leq i \leq m} \ell(s_i)$ , then  $t = r, i_1 = j_1, \dots, i_t = j_r$ , and  $\nu = \mu$ .

**Necessary and sufficient condition for SPP : no prefix + Kraft equality**

$$\sum_{s_i \in S} 2^{-\ell(s_i)} = 1. \quad (4)$$

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Avantage prefix-less : efficient and on-line parsing

Avantage completeness : works for any text

**Examples :**

1.  $S = \{0, 1\}^L$  : fixed block lengths

E.g.  $L = 8$  computer files (cf bytes).

Interest : natural redundancy (8 bits for 60 ASCII characters)

$\Rightarrow$  free compression :  $m = 60$ .

2. Complete prefix-free codes (cf. codetrees).

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NB: we are going to neglect the parsing leaf in what follows...

How to choose the  $s_i$  ? Standard solution :  $S = \{0, 1\}^8$ , but...

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## 2. Choice of the (data compression) code

$T$  given in binary, then parsed into  $Z$  using a source alphabet  $S$  (SPP) :

$$Z = s_{i_1} \cdots s_{i_n} \quad (+ \text{possibly } \nu)$$

Choice of a  $w_i$  for each  $s_i$ , such that  $\ell(U) = \ell(w_{i_1} \cdots w_{i_n})$  is minimal

$$\ell(U) = \sum_{j=1}^n \ell(w_{i_j}) = n \sum_{i=1}^m f_i \ell(w_i). \quad (5)$$

NB: mathematically same problem than source coding ( $p_i \rightarrow f_i$ ).

Conclusion : same solutions applicable, and same limitations (Shannon).

Optimal solution : Huffman using the  $f_i$ .

NB:

If  $f_i$  change from text to text

$\rightarrow$  code changes  $\rightarrow$  must transmit the  $f_i$  or the code  $\rightarrow$  overhead.

Or we take a fixed source model.

## Binary expansion of a number $r \in [0, 1[$

$$r = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j 2^{-j}, \quad a_j \in \{0, 1\} \quad (6)$$

The  $n$  first bits :  $\rightarrow$  a word  $a_1 a_2 \cdots a_n$ .

Notation :  $0.a_1 a_2 \cdots a_n = \sum_{j=1}^n a_j 2^{-j}$ .

Dyadic fraction, if  $\exists$  'exact' finite expansion.

Dyadic fraction  $\Rightarrow$  two binary representations :

$$0.a_1 a_2 \cdots a_{n-1} 1 = 0.a_1 a_2 \cdots a_{n-1} 011111 \dots$$

### Convention

If dyadic : we use finite expansion

Otherwise :  $\exists$  1 single expansion (infinite).

## Shannon code

We have already seen the word lengths (proof of first Shannon theorem), but not the method invented by Shannon to build the prefix-free code using these word-lengths.

Let the  $s_1, \dots, s_m$  be sorted such that  $f_1 \geq f_2 \geq \dots \geq f_m > 0$

(we can remove those  $s_j$  which do not appear at all after parsing text  $T$ .)

Let  $F_1 = 0$  and  $F_k = \sum_{i=1}^{k-1} f_i$ ,  $2 \leq k \leq m$  and denote by  $\ell_k = \lceil \log_2 f_k^{-1} \rceil$ .

The Shannon code  $s_i \rightarrow w_i$  consists in using for  $w_i$  the  $\ell_i$  first bits of the binary expansion of  $F_i$ .

Question : is this code prefix-free ?

Convince yourself...

Average length:

$$\bar{\ell}_{\text{Shannon}} \leq H_m(f_1, \dots, f_m) + 1. \quad (7)$$

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**Example :**  $s_5 s_2 s_5 s_4 s_4 s_5 s_1 s_4 s_3 \rightarrow f_1 = f_2 = f_3 = 1/9, f_4 = f_5 = 3/9$

Sort :  $s'_i = s_{5-i+1} \rightarrow \ell'_1 = \ell'_2 = 2$  and  $\ell'_3 = \ell'_4 = \ell'_5 = 4$

$$\begin{aligned} F'_1 &= 0 &= (.00\dots) \\ F'_2 &= 3/9 &= (.01\dots) \\ F'_3 &= 6/9 &= (.1010\dots) \\ F'_4 &= 7/9 &= (.1100\dots) \\ F'_5 &= 8/9 &= (.1110\dots) \end{aligned} \quad (8)$$

Shannon code

$$\begin{aligned} s_5 &= s'_1 \rightarrow 00 \\ s_4 &= s'_2 \rightarrow 01 \\ s_3 &= s'_3 \rightarrow 1010 \\ s_2 &= s'_4 \rightarrow 1100 \\ s_1 &= s'_5 \rightarrow 1110 \end{aligned} \quad (9)$$

Average length is  $8/3 = 2.666$ , to compare with  $H = 2.113$ .

Compression rate  $5/4$  (24 bits to represent the text  $U$ ).

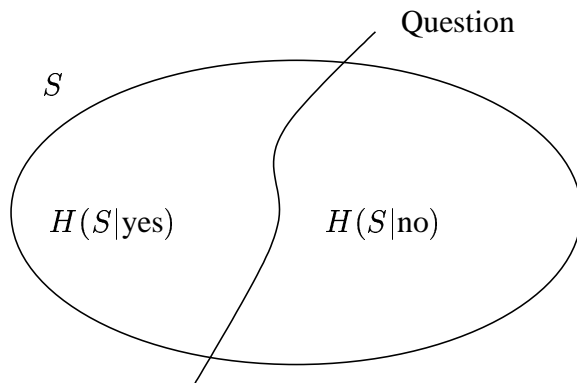
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## Fano code (philosophy)

Construction of a codetree in a “top-down” fashion.

Strategy is similar to decision tree building techniques.

We start with source entropy  $H(S)$  (given) : how to divide the set of symbols so as to minimize average conditional entropy ?



$$H(S) = H(S, Q) = H(S|Q) + H(Q)$$

$\Rightarrow$  maximize  $H(Q)$

$\Rightarrow$  equilibrate probabilities

**Code : decision tree**

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## Fano code (algorithm)

- $f_i$  and  $s_i$  sorted by decreasing order of the  $f_i$ .
- we split according to this order, so as to maximize  $H(Q)$  :  
 $\Rightarrow \sum_{i=1}^k f_i$  and  $\sum_{i=k+1}^m f_i$  as close as possible
- one proceeds recursively with each subset  $\rightarrow$  singletons

**Example :**

$$\begin{array}{rcl}
 s_1 = & 3/9 & 0 \ 0 \quad \rightarrow 00 \\
 s_2 = & 3/9 & 0 \ 1 \quad \rightarrow 01 \\
 s_3 = & 1/9 & 1 \ 0 \quad \rightarrow 10 \\
 s_4 = & 1/9 & 1 \ 1 \ 0 \quad \rightarrow 110 \\
 s_5 = & 1/9 & 1 \ 1 \ 1 \quad \rightarrow 111.
 \end{array} \tag{10}$$

Average length  $20/9 = 2.222$ .

**In general :**

One can show that :  $\bar{\ell}_{\text{Fano}} \leq H_m(f_1, \dots, f_m) + 2$ .

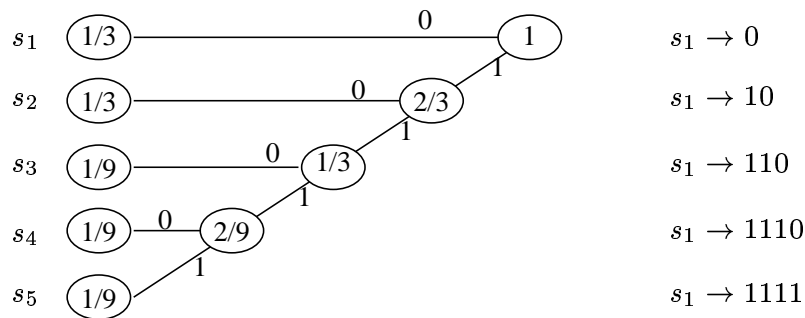
NB: if for each question  $H(Q) = 1$ , then  $\bar{\ell}_{\text{Fano}} = H_m(f_1, \dots, f_m)$

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## Huffman code

Bottom-up construction of the codetree  $\Rightarrow$  optimal.

### Example



Average length  $20/9 = 2.222$ .

We have :  $\bar{\ell}_{\text{Huffman}} \leq H_m(f_1, \dots, f_m) + 1$ .

## Summary (symbol codes) :

What about the choice of the source alphabet  $s_i$  ?

What about the  $f_i$  ?

Given by a source model

Or estimated from each given text :

- necessity to transmit code (what is the overhead ?)
- necessity to adopt some conventions in code construction algorithms
- is not an on-line method.

Optimality :

$$\bar{\ell}_{\text{Huffman}} \leq \min\{\bar{\ell}_{\text{Shannon}}, \bar{\ell}_{\text{Fano}}\} \quad \text{but} \quad \bar{\ell}_{\text{Shannon}} \geq \bar{\ell}_{\text{Fano}}$$

What if the  $p_i \neq f_i$  ?

What if we consider the extended source ?

## Arithmetic coding (the Rolls)

**Idea** (stream code)

For a given source text length  $N$ .

We associate to each possible text a sub-interval of  $[0, 1[$  (they don't overlap)

Sub-interval defined by a probabilistic model of the source.

$C(T) = r \in$  sub-interval, represented in binary with just enough bits to avoid confusion among different numbers corresponding to different texts.

Small sub-intervals = unlikely texts :

$\Rightarrow$  unlikely texts : need many bits to specify  $r$ .

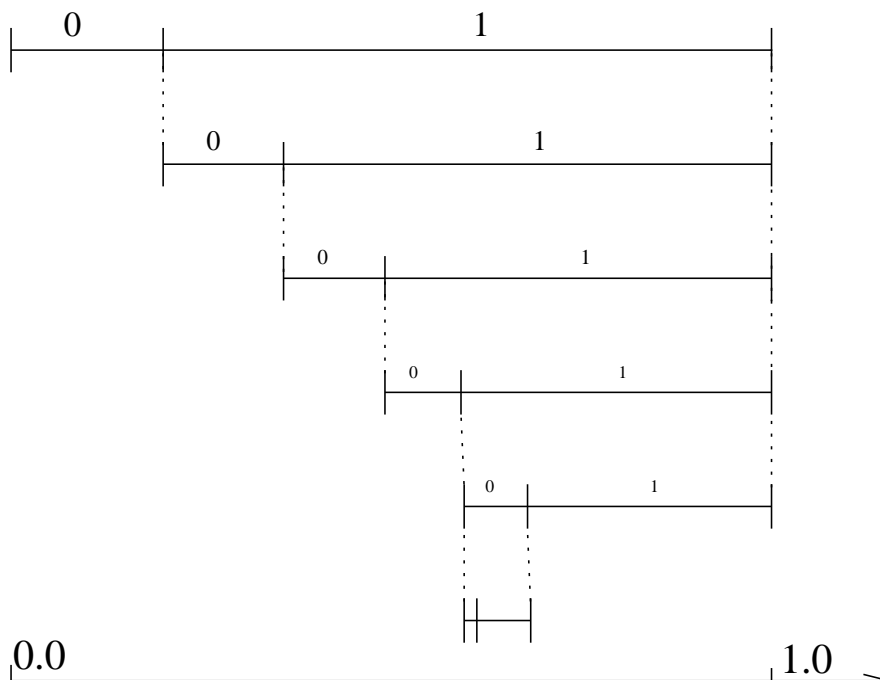
Sub-interval of a text is included in the sub-interval of any prefix of this text :

$\Rightarrow$  recursive (and on-line) construction

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## Illustration

$T = 111110111111101110111101110110$  thus  $f(0) = 6/30 = 0.2$  ,  $f(1) = 0.8$

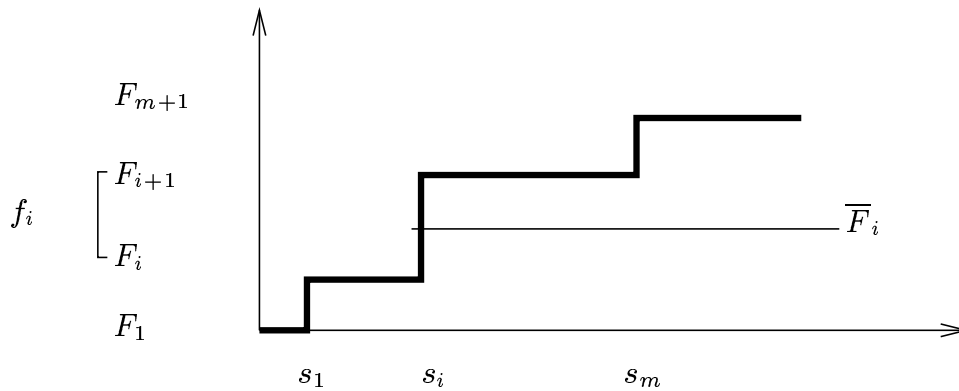


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## How does it work ?

**Shannon-Fano-Elias** → starting point

Symbol code : cumulative symbol frequency diagram (NB: symbols are in arbitrary order.)



$$\bar{F}_i = F_i + \frac{1}{2}f_i.$$

$\lfloor r \rfloor_\ell$  : keep the  $\ell$  first bits of the binary expansion of  $r$ .

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Let us take for  $r = \bar{F}_i$  and  $\ell_i = \lceil -\log f_i \rceil + 1$ .

One can check that :  $0.w_i = \lfloor \bar{F}_i \rfloor_{\ell_i}$  in  $]F_i, F_{i+1}[$ .

One can also check that code  $s_i \rightarrow w_i$  is prefix-free.

Average length :  $\bar{\ell} < H_m(f_1, \dots, f_m) + 2$ .

NB: we pay for 1 bit because of the prefix condition, which is imposed by the nature of a symbol code (can be dropped for a stream code).

By itself not very efficient, but the idea is at the basis of arithmetic coding.

If, instead of coding source symbols we code blocs of source symbols : same idea still works but the overhead of the rounding and prefix bits become less dramatic.

If we code the whole text (Mega-Block) : no need for the prefix condition : we can assume  $\ell(T) = \lceil -\log f(T) \rceil$ .

⇒ **Arithmetic code**

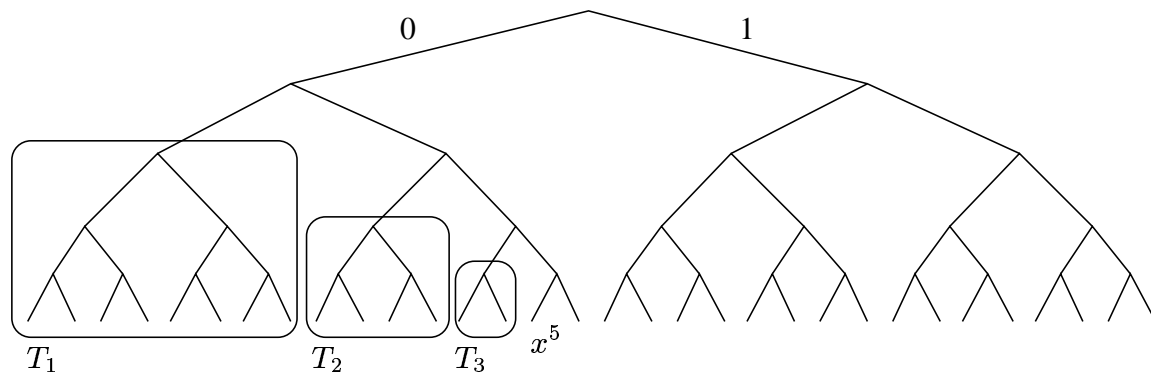
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## How to encode and decode efficiently

NB. For a long text, explicit method doesn't work.

Let  $n$  be the length of the text  $T$ .

We use a tree of depth  $n$  to represent (implicitly) all possible texts of length  $n$ .



The leaves sorted from left to right correspond to the lexicographic order of all possible texts.

Let  $n = 5$  and let's suppose that  $x^5$  represents our text.

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We must determine  $F(x^5)$  and  $f(x^5)$  to encode the text.

$f(x^n)$  : given by the source model (see also subsequent discussion).

$F(x^n)$  : is in principle laborious, since it is defined as a sum of  $f(\cdot)$  over all texts which are on the left of  $x^n$  : about  $2^{n-1}$  terms (in the average).

But, this sum can be decomposed in a different way : sum of the probs of the subtrees which are on the left of  $x^n$  : only about  $\frac{n}{2}$  terms.

Let  $T_{x_1x_2\cdots x_{k-1}0}$  denote the sub-tree pending below the prefix  $x_1x_2\cdots x_{k-1}0$ . The frequency of this sub-tree is

$$f(T_{x_1x_2\cdots x_{k-1}0}) = \sum_{y_{k+1}\cdots y_n} f(x_1x_2\cdots x_{k-1}0y_{k+1}\cdots y_n) \quad (11)$$

$$= f(x_1x_2\cdots x_{k-1}0), \quad (12)$$

$\Rightarrow$  coding reduces to the computation of order  $n$  values of  $f(\cdot)$ .

For example, if we use a zero order model, we compute  $f(x^n) = \prod_{i=1}^n f(x_i)$ .

Thus, this improved version will require order  $n^2$  operations.

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## Example

Binary text of the figure, with  $f(1) = \theta$  et  $f(0) = 1 - \theta$ .

Order zero hypothesis (successive symbols independent) :  $f(s_1, \dots, s_n) = f(s_1) \cdots f(s_n)$

Let us compute the value of  $F(01110)$  ( $x^5$  on the figure).

We find that

$$\begin{aligned} F(01110) &= f(T_1) + f(T_2) + f(T_3) \\ &= f(00) + f(010) + f(0110) \\ &= f(0)f(0) + f(0)f(1)f(0) + f(0)f(1)f(1)f(0) \\ &= f(0)(1 + f(1)(1 + f(1)))f(0) \\ &= (1 - \theta)(1 + \theta(1 + \theta))(1 - \theta), \end{aligned}$$

Observation : many identical terms in the  $f(\cdot)$  which have to be recomputed.

⇒ Recursive computation of the  $f(\cdot)$  : linear time complexity

⇒ reduction of the number multiplications/additions by avoiding to recompute common factors of the  $f_i$ .

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## Encoding algorithm

The source symbols are treated sequentially :

1. Let  $x^k$  be the prefix already treated at stage  $k$ ,  $f(x^k)$  the corresponding relative frequency,  $F(x^k)$  the cumulative frequency (left trees), and  $u_k$  the current node.
2. **Initialization** :  $k = 0$  ;  $x^0$  empty string;  $f(x^0) = 1$ ;  $F(x^0) = 0$
3. **Updating** : Let  $b$  denote  $(k + 1)$ -the bit read of the source text.
  - if  $b = 1$ ,  $F(x^{k+1}) = F(x^k) + f(x^k0)$ .
  - if  $b = 0$ ,  $F(x^{k+1}) = F(x^k)$ .
  - $x^{k+1} = x^k b$ ;  $f(x^{k+1})$  see comments below; current node  $u_{k+1}$  is implicitly updated (following branch  $b$ ) from node  $u_k$ .
4. **Iteration** : if  $k < n$ , we iterate, otherwise the values  $F(x^n)$  and  $f(x^n)$  are returned.
5. **Termination** : the codeword  $\lfloor F(x^n) + f(x^n) \rfloor_{\lceil \log f(x^n) \rceil}$  is constructed.

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## Computing the $f(x^k)$ recursively

Independent symbols :  $f(x^{k+1}) = f(x^k)f(x_{k+1})$

In general :  $f(x^{k+1}) = f(x_{k+1}|x^k)f(x^k)$

Markov :  $f(x^{k+1}) = f(x_{k+1}|x_k)f(x^k)$

**$m$ -ary source alphabet** : cumulate frequencies of all left subtrees at each stage.

**Decoding** : works symmetrically.

The decoder uses the binary expansion of the number  $r = 0.w_i$  in order to select branches in the tree.

Same computations, leaving on the left all subtrees such that  $F(x^k) < 0.w_i$ .

At each transition the encoder will produce one source symbol.

The decoding process stops after  $n$  symbols

⇒ the decoder needs to be informed of the source message length.

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## On-line Algorithm

Stopping criterion is problematic.

Transmission of length  $n$  vs special end of text symbol “.”

Why on-line ?

### Average length

Zero order mode :  $\frac{1}{n}$  bits more than the zero-order entropy limit  $H_m(f_1, \dots, f_m)$ .

### Remarkably flexible

Can easily adapt to any left-to-right oriented probabilistic source model.

### Technicalities (...)

Mainly : computing with the very-long dyadic fractions (high-precision real-number computations)

E.g: text of 1MB → a real number with about  $10^6$  bits precision.

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## Data compression with higher order models

Instead of using the model of “monogram”  $f(s_i)$ , one uses the frequencies of multi-grams  $f(s^{k+1})$  (for an order  $k$  model).

Models : either provided a priori or determined from the given text, or from a sample of representative texts.

If model depends on encoded text :  $\Rightarrow$  overhead (transmit multigram frequencies).

## Higher order Huffman encoding

How would you do ? In practice : two possible solutions

1. Code blocs of length  $k + 1 \Rightarrow$  big Huffman tree.
2. Construct  $m^k$  small Huffman trees for the conditional distributions  $f(s_{k+1}|s^k)$ , and take into account the previous  $k$  symbols to encode/decode  $\Rightarrow$  border effects (initialization)

Which one is better : no general rule.

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## Adaptative data compression

These techniques allow us to treat two problems :

1. Non stationary sources : a single code is not good for the whole text.
2. Don't need to transmit probabilistic model to decoder (on-line...)

Very simple generic idea :

Let  $T = s^N$  be the text to encode/decode.

When coding (and hence also when decoding) the  $k$ -th symbol we use a probabilistic model determined from the already seen symbols (prefix  $s^{k-1}$ ).

Model initialized e.g. with a uniform distribution, and then updated sequentially after each source symbol.

NB: idea is also compatible with higher order source models.

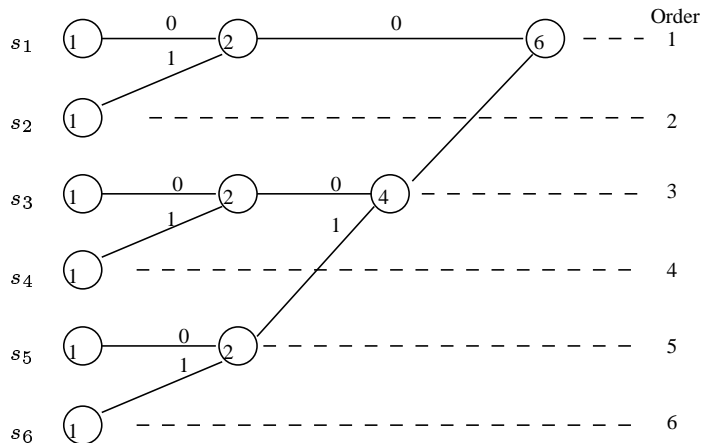
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## Implementation details (...)

### Huffman

Since the source model changes after each symbol, the codetree must be recomputed  
 $\Rightarrow$  not very practical.

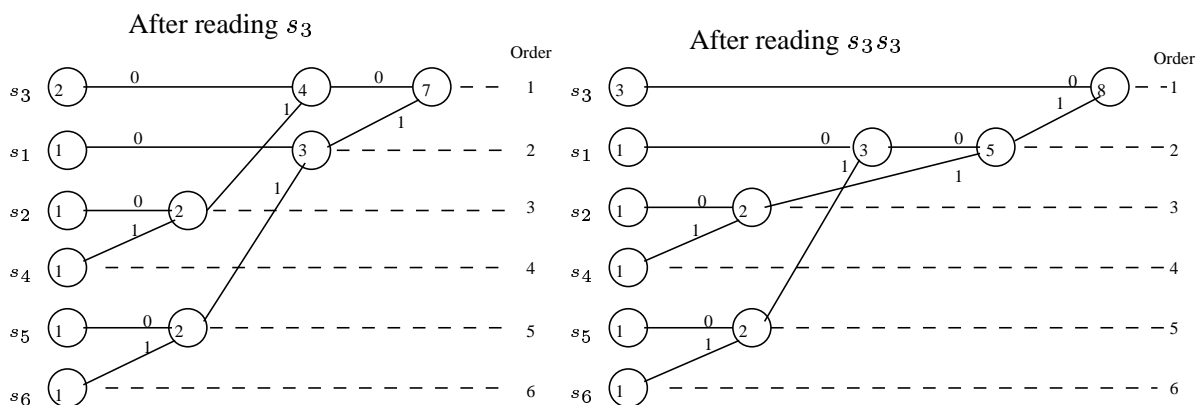
**Example :** a source with six symbols : initialization of the tree : used to code the first symbol.



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**NB:** we need some conventions to treat multiple possibilities.

After the next symbols have been read :



**What you should remember :**

$\exists$  an efficient algorithm to update Huffman trees incrementally (Knuth-Gallager)

**Adaptive arithmetic coding (the adaptive Rolls)**

Think about it yourself...

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## **Dictionary methods for data compression**

Basic idea :

Use a dictionary (set of frequently used words)

Parse text using the dictionary :

→ encode text as a sequence of addresses in the dictionary

NB: similar (but not identical) to source alphabet parsing idea.

NB: but no (e.g. SPP) hypothesis on the contents of the dictionary.

Solutions: use a “library” of specialized dictionaries

E.g. : one dictionary for English texts, one for L<sup>A</sup>T<sub>E</sub>X source code...

Problem: maintenance of dictionaries; does not work for a “random” text

NB: dictionary methods → a generic approach in AI...

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## **“Universal” dictionary methods**

Rebuild the dictionary on the fly for each text, incrementally as the text is read.

→ “universal” adaptive methods

⇒ algorithms invented by Lempel and Ziv (1977-78)

Two basic methods : LZ77 et LZ78

⇒ numerous implementations (e.g. GNUzip, PKZIP, COMPRESS, GIF...)

Basic principle and a few discussions follow.

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## Basic Lempel-Ziv Algorithm

- one starts with an empty dictionary;
- then, at each step one reads symbols as long as current prefix belongs to the dictionary;
- the prefix together with the next source symbol form a word which is not yet in the dictionary  $\Rightarrow$  this new word is inserted in the dictionary  
E.g. if  $T = 1011010100010 \dots$ , this yields 1, 0, 11, 01, 010, 00, 10,  $\dots$
- The present word is encoded : address of prefix in the dictionary + last bit

Let  $c(n)$  denote the address (integer) in the dictionary. We have the following for our example text :

source words	$\lambda$	1	0	11	01	010	00	10
$c(n)$	0	1	2	3	4	5	6	7
$c(n)$ binary address	000	001	010	011	100	101	110	111
(address, bit)	–	(000,1)	(000,0)	(001,1)	(010,1)	(100,0)	(010,0)	(001,0)

$\Rightarrow$  encoded text :  $U = 0001, 0000, 0011, 0101, 1000, 0100, 0010.$

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## Why does this idea allow to compress ?

Because the size of the dictionary grows “slowly” with the size of the source text.

Let  $c(N)$  be the number of encoded entries for a text of length  $N$ .

$\Rightarrow \lceil \log c(N) \rceil + 1$  bits for every word

$\Rightarrow$  in average :  $\frac{c(N)(\log c(N)+1)}{N}$  bits/symbol

One can show that :  $\Rightarrow$  asymptotically  $\lim_{n \rightarrow \infty} \frac{c(n)(\log c(n)+1)}{n} = H(\mathcal{S})$

almost surely for messages of any stationary ergodic source

$\Rightarrow$  “universale” algorithm

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**On-line character :** problem = address coding

Solution  $\Rightarrow$  use current dictionary size to determine number of bits.

source words	$\lambda$	1	0	11	01	010	00	10
$c(n)$	0	1	2	3	4	5	6	7
$c(n)$ binary address	000	001	010	011	100	101	110	111
$\lceil \log_2 c(n) \rceil$	-	0	1	2	2	3	3	3
(address, bit)	-	(,1)	(0,0)	(01,1)	(10,1)	(100,0)	(010,0)	(001,0)

$\Rightarrow U = 1, 00, 011, 101, 1000, 0100, 0010$

**Adaptativity :**  $\Rightarrow$  local dictionary

**Variants :** dictionary management, address coding (e.g. Huffman)

**Relative optimality :**  $\Rightarrow$  not very competitive in general but very robust (no assumption about source behavior).

The asymptotic performances are reached only when the dictionary starts to become representative : contains a significant fraction of sufficiently long typical messages.  
 $\Rightarrow$  for very long texts

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**Summary of text compression :** we have seen the state of the art.

Complementarity of good source models and good coding algorithms  $\Rightarrow$  need both

**Codes of fixed (given) word lengths :**  $\Rightarrow$  conceptual tool for AEP

**Symbol codes :** Huffman

**Stream codes :**

1. Are not constrained to use at least one bit per source symbol  
 $\Rightarrow$  work also for a binary source alphabet
2. Arithmetic coding : a nice probabilistic approach (source modeling)  
 $\Rightarrow$  allow one to exploit a priori knowledge about the real world.
3. Lempel-Ziv : universal method, able to learn “everything” about a stationary ergodic source, at the expense of more data (longer messages).

**Data compression  $\simeq$  Automatic learning**

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