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Chapter 6
Conic Programming

Initial Question : Which class of problems does keep the good properties of linear programming, namely strong duality ?

Conic Programming

Replace the $x \geq 0$ constraint by another type of nonnegativity $x \succeq 0$.

$$\begin{array}{ll} \min c^T x & \min c^T x \\ \text{s.t. } Ax = b & \text{s.t. } Ax = b \\ x \geq 0 & x \succeq 0 \end{array}$$

Question : which important properties on the ordering ≥ 0 do we need to keep ?

Properties of the ordering that are carried over

(1) Reflexivity
$a \succeq a$

(2) Antisymmetry
If both $a \succeq b$ and $b \succeq a$ then $a = b$

(3) Transitivity
If $a \succeq b$ and $b \succeq c$ then $a \succeq c$

(4) Compatibility with linear operations

- If $a \succeq b$ then $\lambda a \succeq \lambda b$ for all $\lambda > 0$
- If $a \succeq b$ and $c \succeq d$ then $a + b \succeq c + d$

An ordering that satisfies (1)-(4) keeps the good properties of linear programming.

Properties of the set $K = \{x \mid x \succeq 0\}$

We now write equivalently

$$x \succeq 0 \qquad \Leftrightarrow \qquad x \in K$$

What are the properties of K ?

(1) K is a cone
We must have $a \in K, \lambda \geq 0 \Rightarrow \lambda a \in K$

(2) K is a convex cone
We must have $a, b \in K \Rightarrow a + b \in K$

(3) K is a pointed cone
$a \in K$ and $-a \in K \quad \Rightarrow \quad a = 0$

Two additional useful properties are

(4) K is closed
This way, if we have a sequence (x_i) with $x_i \in K$ for all i, and if $\lim_{i \to \infty} x_i = \bar{x}$ then $\bar{x} \in K$ as well.

(5) K has a nonempty interior

Three useful cones

The following three cones are the ones that are mostly used.

- The nonnegative orthant
  $K_1 = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}$.
  Optimizing over this cone corresponds to linear programming.
- The second-order cone
  ... also called the Lorentz cone or the ice-cream cone.
  $K_2 = \mathbb{L}^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 \leq x_0^2, x_0 \geq 0\}$
- The semidefinite cone
  This cone is defined over the square matrices.

$$K_3 = \mathbb{S}^n_+ = \{X \in \mathbb{R}^{n \times n} \, | X \text{ is symmetric and positive semidefinite} \\ \text{i.e. all eigenvalues of } X, \lambda_i(X) \geq 0. \}$$

Conic programming

$$\begin{aligned} \min \ & c^T x \\ \text{s.t.} \ & Ax = b \\ & x \in K \end{aligned}$$

where K is a closed, convex, pointed cone for example $\mathbb{R}^n_+, \mathbb{L}^n, \mathbb{S}^n_+$.

Conic duality

Consider the conic program

$$\min c^T x$$
$$\text{s.t. } (Ax - b) \in K$$

Question : what combinations of the rows allow us to derive a valid consequence concerning the objective function ?

We look for $p \in \mathbb{R}^m$ such that

$$\text{for all } x \in K \subseteq \mathbb{R}^m (x \succeq 0) \Rightarrow p^T x \geq 0 \text{ (in the traditional sense)}$$

If p is of that type then we can obtain the implications $Ax \succeq b \Rightarrow p^T Ax \geq p^T b$.

The dual cone

Let $K \subseteq \mathbb{R}^n$ be a closed convex pointed cone, the dual cone is defined by

$$K_* = \{p \in \mathbb{R}^n \mid p^T a \geq 0 \text{ for all } a \in K\}.$$

Properties of the dual cone

- $\mathrm{int}(K) \neq \emptyset \Rightarrow K_*$ is pointed
- If K is a closed convex pointed cone then $\mathrm{int}(K_*) \neq \emptyset$
- If K is closed then K_* is closed
- If K is closed then $(K_*)_* = K$

Deriving conic duality

The primal problem

$$\min c^T x$$
$$\text{s.t. } (Ax - b) \in K$$

If $p \in K_*$ and $p^T A = c^T$, we have

$$p^T(Ax - b) \geq 0$$
$$p^T Ax \geq p^T b$$
$$c^T x \geq p^T b$$

If we want to obtain the best bound possible, we solve the problem

$$\max p^T b$$
$$\text{s.t. } p^T A = c^T$$
$$p \in K_*$$

This is the dual conic problem

The conic duality theorem

Consider the primal-dual pair

$$c^* = \min c^T x \quad \text{s.t. } (Ax - b) \in K$$

PRIMAL

$$b^* = \max p^T b \quad \text{s.t. } p^T A = c^T, \quad p \in K_*$$

DUAL

Theorem

(i) The value $p^T b$ for a feasible solution for the dual is less or equal to the value $c^T x$ for a feasible solution of the primal.
The duality gap $c^T x - p^T b \geq 0$ for every primal-dual pair (x, p).

(ii) If the primal is strictly feasible ($Ax - b \succ_K 0$) and bounded below then the dual is feasible and $b^* = c^*$.

(iii) If the dual is strictly feasible and bounded above then the primal is feasible and $b^* = c^*$.

(iv) Assume one of the problems is bounded and strictly feasible then (x^*, p^*) is an optimal primal-dual pair
- if and only if $c^T x^* = (p^*)^T b$
- if and only if $p^T(Ax - b) = 0$ (complementarity slackness)

Dual cones of the three main cones

- $K^1 = \mathbb{R}^n_+$, $K^1_* = \mathbb{R}^n_+$
  $\{x \in \mathbb{R}^n | x^T y \geq 0 \text{ for all } y \in \mathbb{R}^n_+\} = \mathbb{R}^n_+$
- $K^2 = \mathbb{L}^n$, $K^2_* = \mathbb{L}^n$
  Proof : $\mathbb{L}^n \subseteq \mathbb{L}^n_*$
  $x_0, y_0 \geq 0$ and

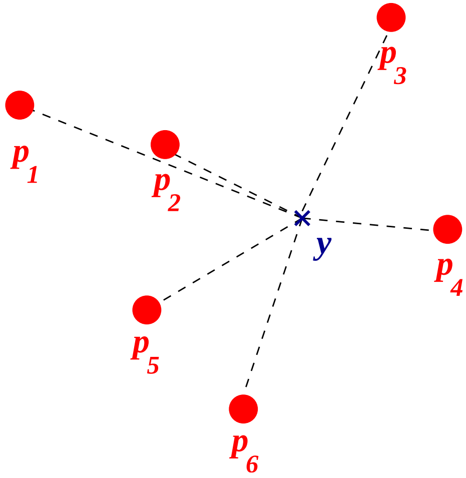
$$
\begin{aligned}
-x_1y_1 - \cdots - x_ny_n &\leq \|x_{1:n}\|\|y_{1:n}\| \qquad \text{Cauchy-Schwarz inequality} \\
-x_1y_1 - \cdots - x_ny_n &\leq x_0y_0 \\
0 &\leq x_0y_0 + x_1y_1 + \cdots + x_ny_n.
\end{aligned}
$$

- $K^3 = \mathbb{S}^n_+$, $K^3_* = \mathbb{S}^n_+$

The three main cones are self-dual.
But it is not the case in general.

A first application of the second order cone

Location Problem Find the point y that minimizes the sum of the distances to all given points p_i.



$$\min \eta_1 + \cdots + \eta_6$$
$$\text{s.t. } \eta_i \geq \|y - p_i\|_2 \quad i = 1, \ldots, 6$$

A first application of semidefinite programming

A spring-mass system is explained by the ODE

$$M\frac{d^2x(t)}{dt^2} = -Ax(t).$$

The main motion is given by

$$x(t) = \sum_{j=1}^{n} \left(a_j \cos(\omega_j t) + b_j \sin(\omega_j t)\right) e_j$$

where e_j are the eigenvalues of (M, A) namely

$$(\lambda_j M - A)e_j = 0 \qquad \text{and} \qquad \omega_j = \sqrt{\lambda_j}.$$

For a structure, a typical requirement is that the resonance frequencies cannot be of the same magnitude as the typical loads of the structure.

Typical Requirement : $\lambda_{\min}(A : M) \geq \lambda_*$
Type of constraint :

$$A - \lambda^* M \succeq 0$$

Well solved by semidefinite programming

SQL-representability

Convex programming can be solved in polynomial time theoretically.
In practice, not all convex programs can be solved efficiently.
In practice optimizing over semidefinite ($\mathbb{S}_+^n$), conic quadratic ($\mathbb{L}^n$) or linear ($\mathbb{R}_+^n$) can be done efficiently.

There are however functions that do not seem to correspond to easy cases that can be solved using $\mathbb{S}_+^n, \mathbb{L}^n, \mathbb{R}_+^n$ by some trick.

Definition

A set $S \subseteq \mathbb{R}^n$ is SQL-representable if S is the projection of a set $\bar{S} \subseteq \mathbb{R}^{n+m}$ that is the feasible domain of a conic program that uses $\mathbb{S}_+^n, \mathbb{L}^n, \mathbb{R}_+^n$.

A function is SQL-representable if its epigraph is SQL-representable.

Remark about the objective function

Optimizing over a convex function is like optimizing over a linear function over its epigraph.

Definition

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function. The epigraph of f is

$$\text{epi}(f) := \{(x, t) \in \mathbb{R}^{n+1} \mid f(x) \leq t\}.$$

Optimizing over a linear function is general

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in X \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & t \\ \text{s.t.} & f(x) \leq t \\ & x \in X \end{array}$$

Remark : f is convex iff epi(f) is convex. The second program remains convex !
The same about conic problems !

CQ-representable functions

We now focus on functions that can be expressed using the the second order cone (Conic quadratic representable functions)

Analogy with continuity

Is the function $f(x) = e^{2x^2+3} + \cosh(x^3 + 2^x)\sin(x)$ continuous ?
Yes because we know that $e^x, \sin x, x^m$ are continuous...
We know that addition, multiplication, composition preserve continuity.

CQ-representable functions

1. $f(x) = a$
2. $f(x) = a^T x + b$
3. $f(x) = \|x\|_2 \qquad \|x\|_2 \leq t$
4. $f(x) = x^T x$
   Because

$$x^T x \leq t$$

$$x^T x + \frac{(t-1)^2}{4} \leq \frac{(t+1)^2}{4}$$

$$\left(\frac{t+1}{2}, x, \frac{t-1}{2}\right) \in \mathbb{L}^2$$

CQ-representable functions

- The fractional quadratic function

$$g(x,s) = \begin{cases} \frac{x^T x}{s}, & s > 0 \\ 0 & s = 0 \\ +\infty & \text{otherwise} \end{cases}$$

  Equivalent to $\left(\frac{t+s}{2}, x, \frac{t-s}{2}\right) \in \mathbb{L}^2$.

- A branch of hyperbola

$$\left\{(t,s) \in \mathbb{R}^2 \mid ts \geq 1, t \geq 0\right\}$$

$$\left(\frac{t+s}{2}, \frac{t-s}{2}, 1\right) \in \mathbb{L}^2.$$

Elementary operations on cones that preserve CQ-representability

- Intersection : $X_1 \cap X_2 \cap \cdots$
- Direct Product : $x_1 \times X_2 \times \cdots$
- Affine image

CQ-representable functions

Elementary operations on functions that preserve the CQ-representability :

- Taking maximum : $g(x) = \max_{i=1,\dots,m} g_i(x)$
- Summation with nonnegative weights : $g(x) = \sum_{i=1}^{n} \alpha_i g_i(x)$.
- Affine substitution of argument :
  If $g(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is CQ-representable and $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$ then

$$h : \mathbb{R}^m \mapsto \mathbb{R},$$
$$h(y) = g(Ay + b) \quad \text{is CQ-representable}$$