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Chapter 6

Conic Programming

From Linear to Conic Programming

Initial Question : Which **class of problems** does keep the **good properties** of linear programming, namely **strong duality** ?

Conic Programming

Replace the $x \geq 0$ constraint by another type of **nonnegativity** $x \succeq 0$.

$$\min c^T x$$

$$\text{s.t. } Ax = b$$

$$x \geq 0$$

$$\min c^T x$$

$$\text{s.t. } Ax = b$$

$$x \succeq 0$$

Question : which important properties **on the ordering** $\succeq 0$ do we need to keep ?

Properties of the ordering that are carried over

(1) Reflexivity

$$a \succeq a$$

(2) Antisymmetry

If both $a \succeq b$ and $b \succeq a$ then $a = b$

(3) Transitivity

If $a \succeq b$ and $b \succeq c$ then $a \succeq c$

(4) Compatibility with linear operations

- ▶ If $a \succeq b$ then $\lambda a \succeq \lambda b$ for all $\lambda > 0$
- ▶ If $a \succeq b$ and $c \succeq d$ then $a + b \succeq c + d$

An ordering that satisfies (1)-(4) keeps the good properties of linear programming.

Properties of the set $K = \{x \mid x \succeq 0\}$

We now write equivalently

$$x \succeq 0 \quad \Leftrightarrow \quad x \in K$$

What are the properties of K ?

(1) K is a cone

We must have $a \in K, \lambda \geq 0 \Rightarrow \lambda a \in K$

(2) K is a convex cone

We must have $a, b \in K \Rightarrow a + b \in K$

(3) K is a pointed cone

$a \in K$ and $-a \in K \Rightarrow a = 0$

Two additional useful properties are

(4) K is closed

This way, if we have a sequence (x_i) with $x_i \in K$ for all i , and if $\lim_{i \rightarrow \infty} x_i = \bar{x}$ then $\bar{x} \in K$ as well.

(5) K has a nonempty interior

Three useful cones

The following three cones are the ones that are mostly used.

- The **nonnegative orthant**

$$K_1 = \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}.$$

Optimizing over this cone corresponds to **linear programming**.

- The **second-order cone**

... also called the Lorentz cone or the ice-cream cone.

$$K_2 = \mathbb{L}^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_n^2 \leq x_0^2, x_0 \geq 0\}$$

- The **semidefinite cone**

This cone is defined over the square matrices.

$$K_3 = \mathbb{S}_+^n = \{X \in \mathbb{R}^{n \times n} \mid X \text{ is symmetric and positive semidefinite} \\ \text{i.e. all eigenvalues of } X, \lambda_i(X) \geq 0. \quad \}$$

Conic programming

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in K \end{aligned}$$

where K is a **closed, convex, pointed cone** for example $\mathbb{R}_+^n, \mathbb{L}^n, \mathbb{S}_+^n$.

Conic duality

Consider the conic program

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & (Ax - b) \in K \end{aligned}$$

Question : what **combinations of the rows** allow us to derive a **valid consequence** concerning the objective function ?

We look for $p \in \mathbb{R}^m$ such that

$$\text{for all } x \in K \subseteq \mathbb{R}^m (x \succeq 0) \Rightarrow p^T x \geq 0 \text{ (in the traditional sense)}$$

If p is of that type then we can obtain the implications $Ax \succeq b \Rightarrow p^T Ax \geq p^T b$.

The dual cone

Let $K \subseteq \mathbb{R}^n$ be a closed convex pointed cone, the **dual cone** is defined by

$$K_* = \{p \in \mathbb{R}^n \mid p^T a \geq 0 \text{ for all } a \in K\}.$$

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Properties of the dual cone

- $\text{int}(K) \neq \emptyset \Rightarrow K_*$ is pointed
- If K is a closed convex pointed cone then $\text{int}(K_*) \neq \emptyset$
- If K is closed then K_* is closed
- If K is closed then $(K_*)_* = K$

Deriving conic duality

The primal problem

$$\begin{aligned} \min c^T x \\ \text{s.t. } (Ax - b) \in K \end{aligned}$$

If $p \in K_*$ and $p^T A = c^T$, we have

$$\begin{aligned} p^T (Ax - b) &\geq 0 \\ p^T Ax &\geq p^T b \\ c^T x &\geq p^T b \end{aligned}$$

If we want to obtain the **best bound** possible, we solve the problem

$$\begin{aligned} \max p^T b \\ \text{s.t. } p^T A = c^T \\ p \in K_* \end{aligned}$$

This is the **dual conic problem**

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The conic duality theorem

Consider the primal-dual pair

$$\begin{aligned}c^* &= \min c^T x \\ \text{s.t. } &(Ax - b) \in K\end{aligned}$$

PRIMAL

$$\begin{aligned}b^* &= \max p^T b \\ \text{s.t. } &p^T A = c^T \\ &p \in K_*\end{aligned}$$

DUAL

Theorem

- (i) The value $p^T b$ for a feasible solution for the dual is less or equal to the value $c^T x$ for a feasible solution of the primal.
The duality gap $c^T x - p^T b \geq 0$ for every primal-dual pair (x, p) .
- (ii) If the primal is strictly feasible ($Ax - b \succ_K 0$) and bounded below then the dual is feasible and $b^* = c^*$.
- (iii) If the dual is strictly feasible and bounded above then the primal is feasible and $b^* = c^*$.
- (iv) Assume one of the problems is bounded and strictly feasible then (x^*, p^*) is an optimal primal-dual pair.

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The **duality gap** $c^T x - p^T b \geq 0$ for every **primal-dual pair** (x, p) .
- (ii) If the primal is **strictly feasible** ($Ax - b \succ_K 0$) and bounded below then the dual is feasible and $b^* = c^*$.
- (iii) If the dual is **strictly feasible** and bounded above then the primal is feasible and $b^* = c^*$.
- (iv) Assume one of the problems is bounded and strictly feasible then (x^*, p^*) is an **optimal primal-dual pair**
 - ▶ if and only if $c^T x^* = (p^*)^T b$
 - ▶ if and only if $p^T (Ax - b) = 0$ (**complementarity slackness**)

Dual cones of the three main cones

- $K^1 = \mathbb{R}_+^n$, $K_*^1 = \mathbb{R}_+^n$
 $\{x \in \mathbb{R}^n \mid x^T y \geq 0 \text{ for all } y \in \mathbb{R}_+^n\} = \mathbb{R}_+^n$
- $K^2 = \mathbb{L}^n$, $K_*^2 = \mathbb{L}^n$
Proof : $\mathbb{L}^n \subseteq \mathbb{L}_*^n$
 $x_0, y_0 \geq 0$ and

$$-x_1 y_1 - \cdots - x_n y_n \leq \|x_{1:n}\| \|y_{1:n}\| \quad \text{Cauchy-Schwarz inequality}$$

$$-x_1 y_1 - \cdots - x_n y_n \leq x_0 y_0$$

$$0 \leq x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$$

- $K^3 = \mathbb{S}_+^n$, $K_*^3 = \mathbb{S}_+^n$

The three main cones are self-dual.

But it is not the case in general.

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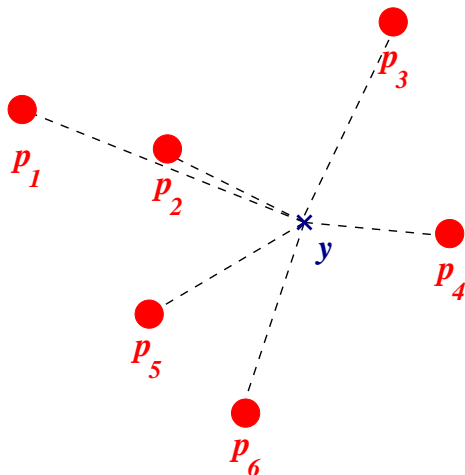
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A first application of the second order cone

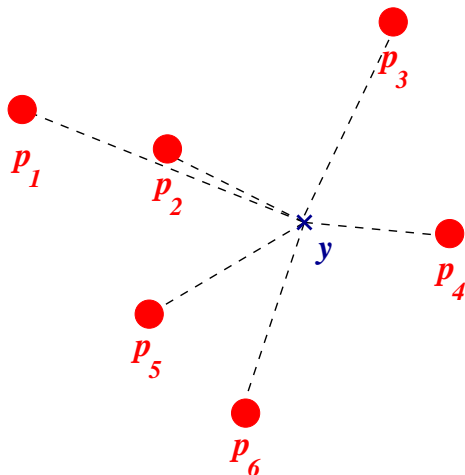
Location Problem Find the point y that minimizes the sum of the distances to all given points p_i .



$$\begin{aligned} \min \quad & \eta_1 + \dots + \eta_6 \\ \text{s.t.} \quad & \eta_i \geq \|y - p_i\|_2 \quad i = 1, \dots, 6 \end{aligned}$$

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A first application of semidefinite programming

A spring-mass system is explained by the ODE

$$M \frac{d^2 x(t)}{dt^2} = -Ax(t).$$

The main motion is given by

$$x(t) = \sum_{j=1}^n (a_j \cos(\omega_j t) + b_j \sin(\omega_j t)) e_j$$

where e_j are the eigenvalues of (M, A) namely

$$(\lambda_j M - A)e_j = 0 \quad \text{and} \quad \omega_j = \sqrt{\lambda_j}.$$

For a structure, a typical requirement is that the **resonance frequencies** cannot be of the same magnitude as the typical loads of the structure.

Typical Requirement : $\lambda_{\min}(A : M) \geq \lambda_*$

Type of constraint :

$$A - \lambda^* M \succeq 0$$

Well solved by semidefinite programming

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SQL-representability

Convex programming can be solved in **polynomial time theoretically**.

In practice, not all convex programs can be solved **efficiently**.

In practice optimizing over **semidefinite** (\mathbb{S}_+^n), **conic quadratic** (\mathbb{L}^n) or **linear** (\mathbb{R}_+^n) can be done efficiently.

There are however functions that **do not seem to correspond to easy cases** that can be solved using $\mathbb{S}_+^n, \mathbb{L}^n, \mathbb{R}_+^n$ by some trick.

Definition

A set $S \subseteq \mathbb{R}^n$ is **SQL-representable** if S is the **projection** of a set $\tilde{S} \subseteq \mathbb{R}^{n+m}$ that is **the feasible domain** of a conic program that uses $\mathbb{S}_+^n, \mathbb{L}^n, \mathbb{R}_+^m$.

A **function** is **SQL-representable** if its **epigraph** is SQL-representable.

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Remark about the objective function

Optimizing over a **convex function** is like optimizing over a **linear function** over its **epigraph**.

Definition

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a function. The epigraph of f is

$$\text{epi}(f) := \{(x, t) \in \mathbb{R}^{n+1} \mid f(x) \leq t\}.$$

Optimizing over a linear function is general

$$\begin{array}{ll} \min f(x) & \\ \text{s.t. } x \in X & \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min t & \\ \text{s.t. } f(x) \leq t & \\ & x \in X \end{array}$$

Remark : f is convex iff $\text{epi}(f)$ is convex. The second program remains convex!
The same about conic problems!

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CQ-representable functions

We now focus on functions that can be expressed using the **the second order cone** (Conic quadratic representable functions)

Analogy with continuity

Is the function $f(x) = e^{2x^2+3} + \cosh(x^3 + 2^x) \sin(x)$ continuous?

Yes because we know that e^x , $\sin x$, x^m are continuous. . .

We know that **addition**, **multiplication**, **composition** preserve continuity.

CQ-representable functions

- $f(x) = a$
- $f(x) = a^T x + b$
- $f(x) = \|x\|_2 \quad \|x\|_2 \leq t$
- $f(x) = x^T x$

Because

$$\begin{aligned} x^T x &\leq t \\ x^T x + \frac{(t-1)^2}{4} &\leq \frac{(t+1)^2}{4} \\ \left(\frac{t+1}{2}, x, \frac{t-1}{2} \right) &\in \mathbb{L}^2 \end{aligned}$$

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- The fractional quadratic function

$$g(x, s) = \begin{cases} \frac{x^T x}{s}, & s > 0 \\ 0 & s = 0 \\ +\infty & \text{otherwise} \end{cases}$$

Equivalent to $(\frac{t+s}{2}, x, \frac{t-s}{2}) \in \mathbb{L}^2$.

- A branch of hyperbola

$$\left\{ (t, s) \in \mathbb{R}^2 \mid ts \geq 1, t \geq 0 \right\}$$
$$\left(\frac{t+s}{2}, \frac{t-s}{2}, 1 \right) \in \mathbb{L}^2.$$

Elementary operations on **cones** that preserve CQ-representability

- Intersection : $X_1 \cap X_2 \cap \dots$
- Direct Product : $x_1 \times X_2 \times \dots$
- Affine image

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- The fractional quadratic function

$$g(x, s) = \begin{cases} \frac{x^T x}{s}, & s > 0 \\ 0 & s = 0 \\ +\infty & \text{otherwise} \end{cases}$$

Equivalent to $(\frac{t+s}{2}, x, \frac{t-s}{2}) \in \mathbb{L}^2$.

- A branch of hyperbola

$$\left\{ (t, s) \in \mathbb{R}^2 \mid ts \geq 1, t \geq 0 \right\}$$
$$\left(\frac{t+s}{2}, \frac{t-s}{2}, 1 \right) \in \mathbb{L}^2.$$

Elementary operations on **cones** that preserve CQ-representability

- Intersection : $X_1 \cap X_2 \cap \dots$
- Direct Product : $x_1 \times X_2 \times \dots$
- Affine image

CQ-representable functions

Elementary operations on **functions** that preserve the CQ-representability :

- Taking maximum : $g(x) = \max_{i=1,\dots,m} g_i(x)$
- Summation with nonnegative weights : $g(x) = \sum_{i=1}^n \alpha_i g_i(x)$.
- Affine substitution of argument :
If $g(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is CQ-representable and $A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n$ then

$$h : \mathbb{R}^m \mapsto \mathbb{R},$$

$$h(y) = g(Ay + b) \quad \text{is CQ-representable}$$

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