Chapter 5
Sensitivity Analysis

Sensitivity analysis

- We assume that we have an optimal solution and the corresponding basis.
- In particular,

$$B^{-1}b \geq 0, \qquad \bar{c}^T = c^T - C_B^T B^{-1} A \geq 0.$$

- We now consider a slight change in the data :
  - An entry of the matrix A
  - An entry of the right-hand-side b
  - An entry of the cost function c
  - One constraint is removed or added
  - One variable is added
- Initial question : does the basis remain the same ?
- If the basis changes, can we take advantage of the already computed optimal solution ?

A new variable is added

The new problem is now

$$\begin{aligned} \min\ & c^T x + c_{n+1}x_{n+1} \\ & Ax + A_{n+1}x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{aligned}$$

The vector x^* is optimal for $\min\{c^T x \mid Ax = b, x \geq 0\}$.

- The vector $(x^*, 0)$ is feasible for the new problem
- The basis B is a feasible basis.
- The basis is optimal for the new problem if

$$\bar{c}_{n+1} = c_{n+1} - C_B^T B^{-1} A \geq 0.$$

- If the reduced cost is negative, we can perform simplex iterations from the current basis. In particular, at the first step, x_{n+1} enters the basis.

A new inequality is added

The new problem can be written in matrix format as

$$\begin{pmatrix} A & 0 \\ a_{m+1}^T & -1 \end{pmatrix} \begin{pmatrix} x \\ s_{m+1} \end{pmatrix} = \begin{pmatrix} b \\ b_{m+1} \end{pmatrix}$$

- If x^* satisfies the constraint, then it is also optimal for the new problem.
- Suppose x^* does not satisfy the new constraint.
- A possible (infeasible) basis is $B \cup \{s_{m+1}\}$.
- Remark that the reduced costs of the nonbasic variables remain unchanged.
- The new basis is therefore feasible for the dual !
- We can apply the dual simplex algorithm from the current basis $B \cup \{s_{m+1}\}$ which is better than starting from scratch.
- Remark that it is as adding a variable to the dual problem $\Rightarrow$ natural that we resort to the dual simplex.

Change in the right-hand-side vector

b_i is changed to $b_i + \delta$
Question : $B^{-1}(b + \delta e_i) \geq 0$?
Let $g = (\beta_{1i}, \beta_{2i}, \ldots, \beta_{mi})$ be the i^{th} column of B^{-1}. In the range

$$\max_{\{j|\beta_{ji}>0\}} \left(-\frac{x_{B(j)}}{\beta_{ji}}\right) \leq \delta \leq \min_{\{j|\beta_{ji}<0\}} \left(-\frac{x_{B(j)}}{\beta_{ji}}\right)$$

the optimal basis remains the same.

- If δ is outside the range
- The current basis (previously optimal) becomes primal infeasible
- The current basis (previously optimal) remains dual feasible
- We can reoptimize using the dual simplex

Change in the objective function

c_j becomes $c_j + \delta$

- The feasibility is not affected $\Rightarrow$ the basis remains primal feasible.
- What about optimality ?
- For a nonbasic variable

$$\bar{c}_j = c_j - C_B^T B^{-1} A_j \to \bar{c}_j + \delta$$

If $\delta \geq -\bar{c}_j$, the basis remains optimal.
Otherwise, we apply the primal simplex algorithm.

- For a basic variable $x_{B(l)}$,

$$(c_B + \delta e_l)^T B^{-1} A_i \leq c_i \quad \text{for all } i$$
$$\delta q_{li} \leq \bar{c}_i,$$

where $q_{:,i}$ is the i^{th} column of the simplex tableau.

Change in a nonbasic entry of the matrix A

We suppose a_{ij} changes to $a_{ij} + \delta$ and the j^{th} column corresponds to a nonbasic variable.

- The basis matrix B does not change $\Rightarrow$ feasibility is not affected
- The reduced cost of only the j^{th} variable is modified.

$$\begin{aligned} c_j - p^T(A_j + \delta e_i) &\geq 0 \\ \bar{c}_j &\geq \delta p_i \end{aligned}$$

- If the condition is violated, then we can reoptimize using the primal simplex algorithm.

Global dependence on the right-hand-side

$$P(b) = \{x \mid Ax = b, x \geq 0\}$$
$$S = \{b \mid P(b) \neq \emptyset\}$$

S is a convex set !

$$F(b) = \min_{x \in P(b)} c^T x$$

Theorem

The optimal cost $F(b)$ is a convex function of b on S.

If we consider the cost functions, we obtain a concave function.