

# Chapter 5

## Sensitivity Analysis

# Sensitivity analysis

- We assume that we have an **optimal** solution and the corresponding **basis**.
- In particular,

$$B^{-1}b \geq 0, \quad \bar{c}^T = c^T - C_B^T B^{-1}A \geq 0.$$

- We now consider a **slight change** in the **data** :
  - ▶ An entry of the matrix  $A$
  - ▶ An entry of the right-hand-side  $b$
  - ▶ An entry of the cost function  $c$
  - ▶ One constraint is **removed** or **added**
  - ▶ One variable is **added**
- Initial question : does the **basis** remain the same ?
- If the basis changes, can we **take advantage** of the **already computed optimal solution** ?

## A new variable is added

The new problem is now

$$\begin{aligned} \min \quad & c^T x + c_{n+1} x_{n+1} \\ & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{aligned}$$

The vector  $x^*$  is **optimal** for  $\min\{c^T x \mid Ax = b, x \geq 0\}$ .

- The vector  $(x^*, 0)$  is **feasible** for the new problem
- The basis  $B$  is a feasible basis.
- The basis is optimal for **the new problem** if

$$\bar{c}_{n+1} = c_{n+1} - C_B^T B^{-1} A \geq 0.$$

- If the reduced cost is negative, we can perform simplex iterations from the current basis. In particular, at the first step,  $x_{n+1}$  **enters the basis**.

## A new inequality is added

The new problem can be written in matrix format as

$$\begin{pmatrix} A & 0 \\ a_{m+1}^T & -1 \end{pmatrix} \begin{pmatrix} x \\ s_{m+1} \end{pmatrix} = \begin{pmatrix} b \\ b_{m+1} \end{pmatrix}$$

- If  $x^*$  satisfies the constraint, then it is also optimal for the new problem.
- Suppose  $x^*$  does not satisfy the new constraint.
- A possible (infeasible) basis is  $B \cup \{s_{m+1}\}$ .
- Remark that the reduced costs of the nonbasic variables remain unchanged.
- The new basis is therefore feasible for the dual!
- We can apply the dual simplex algorithm from the current basis  $B \cup \{s_{m+1}\}$  which is better than starting from scratch.
- Remark that it is as adding a variable to the dual problem  $\Rightarrow$  natural that we resort to the dual simplex.

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## Change in the right-hand-side vector

$b_i$  is changed to  $b_i + \delta$

Question :  $B^{-1}(b + \delta e_i) \geq 0$ ?

Let  $g = (\beta_{1i}, \beta_{2i}, \dots, \beta_{mi})$  be the  $i^{\text{th}}$  column of  $B^{-1}$ . In the range

$$\max_{\{j|\beta_{ji}>0\}} \left( -\frac{x_{B(j)}}{\beta_{ji}} \right) \leq \delta \leq \min_{\{j|\beta_{ji}<0\}} \left( -\frac{x_{B(j)}}{\beta_{ji}} \right)$$

the **optimal basis** remains the same.

- If  $\delta$  is outside the range
- The current basis (previously optimal) becomes **primal infeasible**
- The current basis (previously optimal) remains **dual feasible**
- We can reoptimize using the **dual simplex**

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## Change in the objective function

$c_j$  becomes  $c_j + \delta$

- The feasibility is not affected  $\Rightarrow$  the basis remains **primal feasible**.
- What about optimality?
- For a **nonbasic variable**

$$\bar{c}_j = c_j - C_B^T B^{-1} A_j \rightarrow \bar{c}_j + \delta$$

If  $\delta \geq -\bar{c}_j$ , the basis remains optimal.

Otherwise, we apply the **primal simplex algorithm**.

- For a **basic variable**  $x_{B(l)}$ ,

$$(c_B + \delta e_l)^T B^{-1} A_i \leq c_i \quad \text{for all } i$$
$$\delta q_{li} \leq \bar{c}_i,$$

where  $q_{:,i}$  is the  $i^{\text{th}}$  column of the simplex tableau.

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## Change in a nonbasic entry of the matrix $A$

We suppose  $a_{ij}$  changes to  $a_{ij} + \delta$  and the  $j^{\text{th}}$  column corresponds to a nonbasic variable.

- The basis matrix  $B$  does not change  $\Rightarrow$  feasibility is not affected
- The reduced cost of only the  $j^{\text{th}}$  variable is modified.

$$c_j - p^T(A_j + \delta e_i) \geq 0$$
$$\bar{c}_j \geq \delta p_i$$

- If the condition is violated, then we can reoptimize using the **primal simplex algorithm**.

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## Global dependence on the right-hand-side

$$P(b) = \{x \mid Ax = b, x \geq 0\}$$

$$S = \{b \mid P(b) \neq \emptyset\}$$

$S$  is a convex set!

$$F(b) = \min_{x \in P(b)} c^T x$$

### Theorem

The optimal cost  $F(b)$  is a **convex** function of  $b$  on  $S$ .

If we consider the cost functions, we obtain a **concave** function.