Chapter 4
Duality

How to prove optimality from scratch ?

Example

Consider the problem

$$\begin{array}{rrrrl} \max & x_1 + & x_2 + & 2x_3 & \\ & x_1 + & 2x_2 + & 3x_3 & \leq 7 \quad (1) \\ & x_1 + & x_2 - & x_3 & \leq 2 \quad (2) \\ & -x_1 & & +2x_3 & \leq 3 \quad (3) \\ & x_1 - & x_2 & & \leq 0 \quad (4) \end{array}$$

Claim 1

The optimal value is not $+\infty$!

Proof :

Do (1)+(2)+(3)+(4), this yields

$$2x_1 + 2x_2 + 4x_3 \leq 12$$

$$x_1 + x_2 + 2x_3 \leq 6$$

The optimal value is also less or equal to 6 ! Claim 2

If I am able to provide a solution that has a value of 6, it proves that it is optimal !

Unfortunately. . . Finding the best possible bound

Duality

The duality theorem for linear programming states that, if there is an optimal solution, it is always possible to prove that it is optimal by a suitable combination of the rows of the problem !

The dual problem for a maximization, $\leq$ and free variables

Consider the primal problem with optimal value z_P^*

$$\begin{aligned} z_P = \max \quad & c_1x_1 + \cdots + c_nx_n \\ & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ & \vdots \qquad\qquad \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m. \end{aligned}$$

The following corresponding dual problem has optimal value $z_D^* = z_P^*$

$$\begin{aligned} z_D = \min \quad & b_1p_1 + \cdots + b_mp_m \\ & a_{11}p_1 + \cdots + a_{m1}p_m = c_1 \\ & \vdots \qquad\qquad \vdots \\ & a_{1n}p_1 + \cdots + a_{mn}p_m = c_n \\ & p_1, \ \ldots, \quad p_m \geq 0 \end{aligned}$$

Duality in matrix format

The primal reads

$$\begin{aligned} z_P = \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned}$$

The dual reads

$$\begin{aligned} z_D = \min\ & b^T p \\ \text{s.t. } & A^T p = c \\ & p \geq 0. \end{aligned}$$

If the primal has an optimal solution x^*, then the dual has an optimal solution p^* such that

$$c^T x^* = b^T p^*$$

Another view towards duality : the Lagrange multipliers

Consider the primal linear programming problem in standard form

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0. \end{aligned}$$

We can dualize the equality constraints and solve

$$\begin{aligned} g(p) = \min\ & c^T x + p^T(b - Ax) \\ \text{s.t. } & x \geq 0. \end{aligned}$$

Note $g(p) \leq c^T x^*$ (where x^* is the optimal solution of the primal) and

$$\begin{aligned} \min_{x \geq 0} (c^T - p^T A)x \quad & = 0 \qquad \text{if } c^T - p^T A \geq 0 \\ & = -\infty \qquad \text{if } (c^T - p^T A)_i < 0 \text{ for some } i \end{aligned}$$

The dual problem is therefore

$$\begin{aligned} \max\ & b^T p \\ \text{s.t. } & p^T A \leq c^T \end{aligned}$$

The dual problem

$$
\begin{array}{lllllll}
\min & c^T x & & \qquad \max & & p^T b & \\
\text{s.t.} & a_i^T x \geq b_i & i \in M_1 & & \text{s.t.} & p_i \geq 0 & i \in M_1 \\
& a_i^T x \leq b_i & i \in M_2 & & & p_i \leq 0 & i \in M_2 \\
& a_i^T x = b_i & i \in M_3 & & & p_i \text{ free} & i \in M_3 \\
& x_j \geq 0 & j \in N_1 & & & p^T A_j \leq c_j & j \in N_1 \\
& x_j \leq 0 & j \in N_2 & & & p^T A_j \geq c_j & j \in N_2 \\
& x_j \text{ free} & j \in N_3 & & & p^T A_j = c_j & j \in N_3
\end{array}
$$

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	**variables**
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	**constraints**

Duality

The dual of the dual is the primal

If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original problem

Duality Theorems

Weak duality

If x is a feasible solution to the primal and p is a feasible solution to the dual then

$$p^T b \leq c^T x.$$

Corollary

- If the optimal cost of the primal is $-\infty$ then the dual is infeasible.
- If the optimal cost of the dual is $+\infty$ then the primal is infeasible

Corollary

If x and p are feasible solutions of the primal and the dual respectively and $p^T b = c^T x$, then x is optimal for the primal and p is optimal for the dual.

Strong Duality

If a linear programming problem has an optimal solution, then the dual also has an optimal solution and the respective costs are equal.

Possible Cases

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	Case 1		
Unbounded			Case 2
Infeasible		Case 2	Case 3

Interpretation of duality : mechanics, price fixing, geometric representation.

Complementarity Slackness

Theorem

Let x and p be feasible solutions to the primal and the dual respectively. The vectors x and p are optimal if and only if

$$p_i(a_i^T x - b_i) = 0 \qquad \text{for all } i$$
$$(c_j - p^T A_j)x_j = 0 \qquad \text{for all } j.$$

Dual Simplex Method

We maintain a tableau as usual.
We start with a basis such that

- the reduced costs all are nonnegative
  $\rightarrow p^T A \leq c^T$, we have a feasible solution for the dual
- the right-hand-side may be negative
  $\rightarrow B^{-1}b \ngeq 0$, we do not have a primal feasible solution

At each iteration

- Select a row i such that $(B^{-1}b)_i < 0$
- A variable j with $a_{ij} < 0$ will enter the basis
- Select the variable such that all reduced costs remain nonnegative
- Perform a pivot operation

Dual Simplex Method

We start with a dual feasible basis B and the corresponding tableau $\bar{A}$

An iteration of the dual simplex

- If $(B^{-1}b)_i \geq 0$ for all i, then B is optimal else choose l such that $x_{B(l)} < 0$.
- If $\bar{a}_{lj} \geq 0$ for all j, then the dual is unbounded and the primal infeasible
- Otherwise select
$$j = \arg \min_{i | \bar{a}_{li} < 0} \frac{\bar{c}_i}{|\bar{a}_{li}|}.$$
The variable x_i enters the basis and the variable $B(l)$ exits the basis
- Perform a pivot operation in order to obtain $\bar{a}_{li} = 1$ and $\bar{a}_{ki} = 0$ for all $k \neq l$.

The Farkas Lemma

Showing the infeasibility of a linear system by the feasibility of another system !

Example

$$\begin{array}{rrrrl} (1) & 10x_1 & +14x_2 & \leq & 35 \\ (2) & -x_1 & + \quad x_2 & \leq & 0 \\ (3) & & - \quad x_2 & \leq & -2 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

A certificate of infeasibility

$$y = (1 \quad 8 \quad 21)^T$$

$$\begin{array}{rrrrl} (1) & 10x_1 & +14x_2 & \leq & 35 \\ 8(2) & -8x_1 & + \; 8x_2 & \leq & 0 \\ 21(3) & & -21x_2 & \leq & -42 \end{array}$$

$$2x_1 + \quad x_2 \leq -7$$

Farkas Lemma

Theorem

Exactly one of the following systems is feasible

(i)

$$Ax = b$$
$$x \geq 0$$

(ii)

$$p^T A \geq 0$$
$$p^T b < 0$$