

Chapter 4

Duality

How to prove optimality from scratch ?

Example

Consider the problem

$$\begin{aligned} \max \quad & x_1 + x_2 + 2x_3 \\ & x_1 + 2x_2 + 3x_3 \leq 7 & (1) \\ & x_1 + x_2 - x_3 \leq 2 & (2) \\ & -x_1 + 2x_3 \leq 3 & (3) \\ & x_1 - x_2 \leq 0 & (4) \end{aligned}$$

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Claim 1

The optimal value is **not** $+\infty$!

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Claim 1

The optimal value is **not** $+\infty$!

Proof :

Do (1)+(2)+(3)+(4), this yields

$$2x_1 + 2x_2 + 4x_3 \leq 12$$

$$x_1 + x_2 + 2x_3 \leq 6$$

The optimal value is also less or equal to 6 !

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Claim 2

If I am able to provide a solution that has a value of 6, it proves that it is optimal!
Unfortunately. . .

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Finding the best possible bound

$$\begin{aligned} (p_1 + p_2 - p_3 + p_4)x_1 + (2p_1 + p_2 - p_4)x_2 + (3p_1 - p_2 + 2p_3)x_3 &\leq (7p_1 + 2p_2 + 3p_3) \\ 1x_1 + & 1x_2 + & 2x_3 &\leq (7p_1 + 2p_2 + 3p_3) \end{aligned}$$

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$$\min 7p_1 + 2p_2 + 3p_3$$

$$p_1 + p_2 - p_3 + p_4 = 1$$

$$2p_1 + p_2 - p_4 = 1$$

$$3p_1 - p_2 + 2p_3 = 2$$

$$p_1, p_2, p_3, p_4 > 0$$

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Claim 3

The solution $x_1 = \frac{13}{9}, x_2 = \frac{13}{9}, x_3 = \frac{8}{9}$ is optimal! Indeed, it satisfies all constraints (**feasibility**) and makes $\frac{14}{3}$ for the objective function. Furthermore

$$\frac{2}{3}(1) + \frac{1}{3}(4) \equiv x_1 + x_2 + 2x_3 \leq 14$$

which proves that we can not do better! It also is the best bound that we can obtain.

Duality in matrix format

The **primal** reads

$$\begin{aligned} z_P &= \max c^T x \\ \text{s.t. } Ax &\leq b \end{aligned}$$

The **dual** reads

$$\begin{aligned} z_D &= \min b^T p \\ \text{s.t. } A^T p &= c \\ p &\geq 0. \end{aligned}$$

If the primal has an optimal solution x^* , then the dual has an optimal solution p^* such that

$$c^T x^* = b^T p^*$$

Another view towards duality : the Lagrange multipliers

Consider the primal linear programming problem in standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

We can **dualize** the equality constraints and solve

$$\begin{aligned} g(p) = \min \quad & c^T x + p^T (b - Ax) \\ \text{s.t.} \quad & x \geq 0. \end{aligned}$$

Note $g(p) \leq c^T x^*$ (where x^* is the optimal solution of the primal) and

$$\begin{aligned} \min_{x \geq 0} (c^T - p^T A)x &= 0 \quad \text{if } c^T - p^T A \geq 0 \\ &= -\infty \quad \text{if } (c^T - p^T A)_i < 0 \text{ for some } i \end{aligned}$$

The dual problem is therefore

$$\begin{aligned} \max \quad & b^T p \\ \text{s.t.} \quad & p^T A \leq c^T \end{aligned}$$

The dual problem

$$\begin{array}{ll}
 \min c^T x & \max p^T b \\
 \text{s.t. } a_i^T x \geq b_i & i \in M_1 \quad \text{s.t. } p_i \geq 0 \quad i \in M_1 \\
 a_i^T x \leq b_i & i \in M_2 \quad \quad \quad p_i \leq 0 \quad i \in M_2 \\
 a_i^T x = b_i & i \in M_3 \quad \quad \quad p_i \text{ free} \quad i \in M_3 \\
 x_j \geq 0 & j \in N_1 \quad \quad \quad p^T A_j \leq c_j \quad j \in N_1 \\
 x_j \leq 0 & j \in N_2 \quad \quad \quad p^T A_j \geq c_j \quad j \in N_2 \\
 x_j \text{ free} & j \in N_3 \quad \quad \quad p^T A_j = c_j \quad j \in N_3
 \end{array}$$

PRIMAL	minimize	maximize	DUAL
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
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The dual of the dual is the primal

If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original problem

Duality Theorems

Weak duality

If x is a feasible solution to the primal and p is a feasible solution to the dual then

$$p^T b \leq c^T x.$$

Corollary

- If the optimal cost of the primal is $-\infty$ then the dual is infeasible.
- If the optimal cost of the dual is $+\infty$ then the primal is infeasible.

Corollary

If x and p are feasible solutions of the primal and the dual respectively and $p^T b = c^T x$, then x is **optimal** for the primal and p is **optimal** for the dual.

Strong Duality

If a linear programming problem has an optimal solution, then the dual also has an optimal solution and the **respective costs are equal**.

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Possible Cases

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	Case 1		
Unbounded			Case 2
Infeasible		Case 2	Case 3

Interpretation of duality : mechanics, price fixing, geometric representation.

Theorem

Let x and p be feasible solutions to the primal and the dual respectively. The vectors x and p are **optimal** if and only if

$$\begin{aligned} p_i(a_i^T x - b_i) &= 0 && \text{for all } i \\ (c_j - p^T A_j)x_j &= 0 && \text{for all } j. \end{aligned}$$

Dual Simplex Method

We maintain a tableau **as usual**.

We start with a basis such that

- the reduced costs all are **nonnegative**
→ $p^T A \leq c^T$, we have a **feasible solution for the dual**
- the right-hand-side may be **negative**
→ $B^{-1}b \not\geq 0$, we **do not have a primal feasible solution**

At each iteration

- Select a **row** i such that $(B^{-1}b)_i < 0$
- A variable j with $a_{ij} < 0$ will enter the basis
- Select the variable such that **all reduced costs remain nonnegative**
- Perform a pivot operation

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Dual Simplex Method

We start with a **dual feasible basis** B and the corresponding tableau \bar{A}

An iteration of the dual simplex

- If $(B^{-1}b)_i \geq 0$ for all i , then B is **optimal** else choose l such that $x_{B(l)} < 0$.
- If $\bar{a}_{lj} \geq 0$ for all j , then the dual is **unbounded** and the primal **infeasible**
- Otherwise select

$$j = \arg \min_{i | \bar{a}_{li} < 0} \frac{\bar{c}_i}{|\bar{a}_{li}|}.$$

The variable x_j enters the basis and the variable $B(l)$ exits the basis

- Perform a pivot operation in order to obtain $\bar{a}_{lj} = 1$ and $\bar{a}_{ki} = 0$ for all $k \neq l$.

The Farkas Lemma

Showing the **infeasibility** of a linear system by the **feasibility** of another system !

Example

$$(1) \quad 10x_1 + 14x_2 \leq 35$$

$$(2) \quad -x_1 + x_2 \leq 0$$

$$(3) \quad -x_2 \leq -2$$

$$x_1, x_2 \geq 0$$

A certificate of infeasibility

$$y = (1 \quad 8 \quad 21)^T$$

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$$8(2) \quad -8x_1 + 8x_2 \leq 0$$

$$21(3) \quad -21x_2 \leq -42$$

$$2x_1 + x_2 \leq -7$$

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Theorem

Exactly one of the following systems is feasible

(i)

$$Ax = b$$

$$x \geq 0$$

(ii)

$$p^T A \geq 0$$

$$p^T b < 0$$

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