Introduction à l'Optimisation Numérique

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Chapter 3

The simplex method

The simplex algorithm

Principle

- Start at a feasible basic solution
- Check whether the current basis is optimal
- If not, find a direction of improvement
- The direction of improvement leads to either . . . a better feasible basic solution
- or . . . proving that the problem is unbounded

Fundamental question

Let x be a feasible basic solution.
Find a direction d such that $x + \theta d$ is feasible for some $\theta \geq 0$

Finding a feasible direction from a basic feasible solution

Given : $x = (x_B \; x_N) \in P$
Find d such that $x + d\theta \in P$

- We want to change at least one nonbasic solution.
  Select one index $j \in N$

$$d_j = 1 \quad d_i = 0 \text{ for all } i \neq j$$

- We have $Ax = b$ and $A(x + \theta d) = b \Rightarrow Ad = 0$
-

$$0 = Ad = \sum_{i=1}^{n} A_{B(i)} d_{B(i)} + A_j$$
$$Bd_B + A_j$$

- $d_B = -B^{-1}A_j$ is the j^{th} basic direction

Taking care of the nonnegativity constraints

- Nonbasic variables : x_i $(i \neq j)$: $= 0$: $\Rightarrow$ OK !
  x_j : goes in the positive direction $\Rightarrow$ OK !
- Basic Variables
  Nondegenerate case : $x_B > 0$
  Therefore $x_B + \theta d_B \geq 0$ for θ sufficiently small
- Degenerate case : $x_{B(i)} = 0$ for some i
  2 cases
  (1) $(d_B)_i = (-B^{-1}A_j)_i \geq 0 \Rightarrow$ feasible for θ small enough
  (2) $(d_B)_i = (-B^{-1}A_j)_i < 0 \Rightarrow$ not feasible for any θ

Checking optimality

Essential question : Is the current basis optimal or can we find a basic direction that improves the objective function ?

Definition : $c_B = (c_{B(1)}, \dots, c_{B(m)})$

Let us compute the rate of objective change for the j^{th} basic direction.

$d = (d_B \; d_N)$ with $d_B = -B^{-1}A_j \quad d_N = e_j$

$$\begin{aligned} c^T d &= c^T d_B + c^T d_N \\ &= -c_B^T B^{-1} A_j + c_j \end{aligned}$$

Definition

The reduced cost of the j^{th} nonbasic variable is

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j.$$

Optimality Criterion

Theorem

Consider a basic feasible solution x

(i) If $\bar{c} \geq 0$ then x is optimal

(ii) If x is optimal and nondegenerate then $\bar{c} \geq 0$.

Definition

A basis matrix B is said to be optimal if

(i) $B^{-1}b \geq 0$ and

(ii) $\bar{c} = -c_B^T B^{-1} A \geq 0$

Simplex algorithm

Iterative step : From a basis B do

If $\bar{c}_j \geq 0$ for all j

then the current basis is optimal

else select a nonbasic variable j with $\bar{c}_j < 0$

We bring j into the basis

We now look for θ maximal such that $x + \theta d \in P$

If $d_i \geq 0$ for all $i \in B$

then $x + \theta d \geq 0$ for all $\theta \geq 0$

The problem is unbounded and $OPT = -\infty$

If $d_i <$ for some $i \in B$

then $\theta^* = \min_{\{i \in B | d_{B(i)} < 0\}} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right).$

Variable i achieving the minimum goes out of the basis

We move to the next vertex (basic feasible solution)

$B \leftarrow B \cup \{j\} \setminus \{i\}$

New point:$= x + \theta^* d$

The operation of moving to another basis

In an iteration of the simplex algorithm, in the basis, we replace the variable i such that

$$i = \arg\min\{-\frac{x_{B(i)}}{d_{B(i)}} \mid i \in B \text{ with } d_{B(i)} < 0\}$$

by the variable j entering the basis (that was chosen with $\bar{c}_j < 0$).

Theorem

(i) The columns $A_{B(k)}$, $k \neq i$ and A_j are linearly independent

(ii) The vector $y = x + \theta^* d$ is a basic feasible solution associated with the new basis matrix.

Finiteness of the simplex algorithm

Theorem

If all bases of P are nondegenerate then the simplex algorithm terminates in a finite number of iterations. At termination, there are two possibilities

- We have an optimal basis and an associated basic feasible solution which is optimal
- We have found a vector d such that $Ad = 0$ and $c^T d < 0$ implying that the optimal cost is $-\infty$

The degenerate case

We may do a pivot (j enters the basis and i goes out) and not changing the corresponding basic feasible solution. This is called a degenerate pivot. Using degenerate pivots, two possibilities occur

- After some degenerate pivots, we find a direction of improvement and move to a basis with a different corresponding solution
- After some degenerate pivots, we come back to the same basis $\rightarrow$ the algorithm cycles.
- For a good pivot rule, we can avoid cycling (see next section).

Implementation issues

Implementing a good linear software is not a trivial task.
We will see that the way the simplex method is implemented is not the most obvious one.

Full Tableau Implementation

We do not compute the inverse matrix B^{-1} but maintain at each iteration the simplex tableau

$$B^{-1}(b \mid A).$$

A pivot = elementary row operations
Complexity :
Memory $\mathcal{O}(mn)$
Number of operations $\mathcal{O}(mn)$.
Drawback : possibility of fill in

Implementation issues

The revised simplex

Update the inverse matrix B^{-1} instead of the full tableau at each iteration.

(i) Start with a basis $A_{B(1)}, \ldots, A_{B(m)}$ and the inverse B^{-1}

(ii) Compute the reduced costs $\bar{c}_j = c_j - c_B^T B^{-1} A_j$.
Choose an index j with $\bar{c}_j < 0$ (if none exists, return optimal).

(iii) Compute $u = B^{-1} A_j$

(iv) Compute

$$\theta^* = \min_{i=1,\ldots,m | u_i > 0} \frac{x_{B(i)}}{u_i}.$$

(v) Let l be the index achieving the minimum, the new basic solution is

$$y_j = \theta^*, y_{B(i)} = x_{B(i)} - \theta^* u_i, i \neq l.$$

(vi) Perform the corresponding row operations on B^{-1} in order to compute the new inverse matrix.

Complexity :
Memory : $\mathcal{O}(m^2)$ (because A is sparse most of the times)
Operations : $\mathcal{O}(m^2)$ for the update of the inverse and $\mathcal{O}(mn)$ for the computation of the reduced cost $\rightarrow$ this can be reduced if we find a good variable quickly !

Anticycling rules

Lexicographic order

We define a total ordering on vectors of $\mathbb{R}^n$. We say that $a \in \mathbb{R}^n$ is lexicographically smaller than $b \in \mathbb{R}^n$, $a \prec^L b$ if the first nonzero component of $b - a$ is positive.

Example

$$(1;\ 2\ 3\ 4) \prec^L; (1\ 3;\ 0\ 0)$$
$$(3;\ 1;\ -2;\ 7) \prec^L; (7\ 0\ -3;\ 1)$$
$$(1;\ 2\ 3) \prec^L (1\ 2\ 7)$$

Lexicographic pivoting rule

(i) Choose a variable x_i ($i \in N$) with $\bar{c}_i < 0$ arbitrarily

(ii) For each i with $\frac{x_{B(i)}}{u_i} (u_i > 0)$ minimum select the variable with the corresponding tableau row which is lexicographically minimal to leave the basis.

Lexicographic pivoting rule

In the tableau form, we use the following convention

- the 0^{th} row corresponds to the reduced costs
- the 0^{th} column corresponds to the right-hand-sides
- the element in the 0^{th} row and 0^{th} column corresponds to the opposite of the objective.

Theorem

Suppose that all rows (except the 0^{th} row) start being lexicographically positive. If we use the lexicographic pivoting rule then

(i) Every row (except the 0^{th} row) remain lexicographically positive during the simplex algorithm

(ii) The zeroth row strictly increases lexicographically during the simplex algorithm

(iii) The simplex algorithm terminates after a finite number of steps.

Finite pivoting rule

The following pivoting rule is extremely simple to use with the revised simplex algorithm

Bland's rule

(i) The variable i that enters the basis corresponds to the smallest subscript such that $\bar{c}_i < 0$

(ii) If there is a tie between variables to exit the basis, select the one with the smallest subscript

Using Bland's rule, the simplex algorithm is guaranteed to terminate after a finite number of steps.

Finding an intial basis

2 issues : finding a feasible solution and finding the corresponding basis.

One easy case

Consider a problem of the type $Ax \leq b$ where $b \geq 0$.
The standard form, if we add the slack variables

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n + s_1 \quad\quad &= b_1 \\ \vdots \quad\quad \ddots \quad &\\ a_{m1}x_1 + \cdots + a_{mn}x_n \quad\quad + s_m &= b_m \end{aligned}$$

The point $(x_1 = 0, \cdots, x_n = 0)$ is a feasible solution and $(s_1 \; \cdots \; s_m)$ is the corresponding basis.

In general

It is not easy to find a feasible basis.
One can try any basis (choice of m variables) but there is no guarantee that it is feasible.
In some cases, the problem is infeasible $\rightarrow$ it might be impossible to find a basic feasible solution.

Using a linear program and the simplex algorithm to find an initial basis

Phase I

Consider a problem of the type $Ax = b, x \in \mathbb{R}^n_+$ where we assume (after suitable multiplication of rows by -1) that $b \geq 0$.

$$
\begin{array}{llllll}
\min & & \xi_1 + \cdots + & & \xi_m & \\
& a_{11}x_1 + \cdots + a_{1n}x_n & + \xi_1 & & & = b_1 \\
& & \vdots & \ddots & \vdots & \\
& a_{m1}x_1 + \cdots + a_{mn}x_n & & & + \xi_m & = b_m
\end{array}
$$

- If the optimal solution of the phase 1 problem is $> 0 \Rightarrow$ the initial problem is infeasible.
- If the initial problem is feasible $\Rightarrow$ the phase 1 problem has 0 as optimal solution
- One more issue : how to obtain an initial basis for the original problem without ξ_i variables in it

The big-M method

Phase I and Phase II together

Consider a problem of the type $Ax = b, x \in \mathbb{R}^n_+$ where we assume (after suitable multiplication of rows by -1) that $b \geq 0$.

$$
\begin{array}{llll}
\min c_1x_1 + \cdots + c_nx_n & + M\xi_1 + \cdots + & M\xi_m & \\
a_{11}x_1 + \cdots + a_{1n}x_n & + \xi_1 & & = b_1 \\
 & \vdots \ddots & & \vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n & & + \xi_m & = b_m
\end{array}
$$

- M is always considered as very big
- When an auxiliary variable ξ_i exits the basis, we can get rid of the full column.