Introduction à l'Optimisation Numérique

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Repartition of the students for the exercises

Room	Assistent	Orientation
I.94	Bertrand Cornélusse	Mécanique, Ingénieur Informatique
I.97	David Detry	Electricité, Sciences Informatiques
I.123	Laurent Poirrier	Biomédical, Physique

Chapter 2
Linear Programming : different forms and geometry

Different forms of linear programming

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ \text{s.t. } & 3x_1 + x_2 \leq 3 \\ & x_1 - x_2 = 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min\ & -2x_1 + 3x_2 \\ \text{s.t. } & 3x_1 + 4x_2 \geq 3 \\ & x_1 - 2x_2 \leq 2 \\ & x_1 \geq 0, \\ & -3 \leq x_2 \leq 0 \end{aligned}$$

$$\begin{aligned} \min\ & 2x_1 - 3x_2 \\ \text{s.t. } & 7x_1 - x_2 \leq 3 \\ & x_1 + 2x_2 = 5 \\ & x_1 \geq 0, x_2 \in \mathbb{R} \end{aligned}$$

Objective : min ou max
Constraints : $\geq, \leq, =$
Bounds : $\geq 0, \leq 0, [l, u], \mathbb{R}$

Different forms of linear programming

We can go equivalently from one form to the other

Objective :

$$\max\ f(x) \equiv -\min\ -f(x)$$

$$\max\ 2x_1 - 7x_2 \equiv -\min -2x_1 + 7x_2$$

Constraints :

$$f(x) \leq b \equiv -f(x) \geq -b \qquad 2x_1 - x_2 \leq 1 \equiv -2x_1 + x_2 \geq -1$$

$$f(x) = b \equiv f(x) \leq b \text{ et } f(x) \geq b \qquad 3x_1 - x_2 = 3 \equiv 3x_1 - x_2 \leq 3 \text{ et } 3x_1 - x_2 \geq 3$$

$$f(x) \leq b \equiv f(x) + s = b,\ \text{with } s \geq 0 \qquad 3x_1 - 2x_2 \geq 0 \equiv 3x_1 - 2x_2 - s \geq 0 \text{ with } s \geq 0$$

Bounds :

$$x \leq 0 \quad \equiv \quad \hat{x} := -x \ \text{ et } \ \hat{x} \geq 0$$

$$y \in \mathbb{R} \quad \rightarrow \quad y = y^+ - y^- \ \text{ and } \ y^+, y^- \geq 0 \text{ !Not equivalent !}$$

Standard form

The standard form consists in

- Objective : minimization
- Constraints : equalities
- Bounds : Nonnegative variables

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \in \mathbb{R}^n_+ \end{aligned}$$

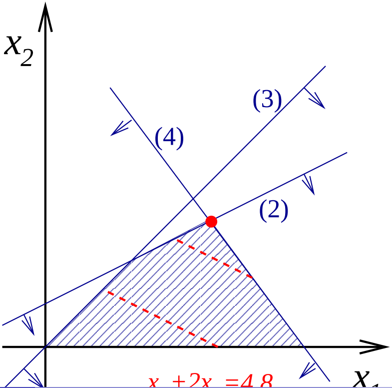
Exercise : Reduce a given problem into standard form

Graphic representation

We can represent a problem in two dimensions graphically.

Example :

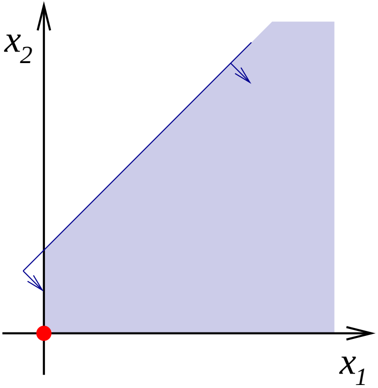
$$\max \; x_1 + 2x_2 \tag{1}$$
$$-x_1 + 2x_2 \leq 1 \tag{2}$$
$$-x_1 + x_2 \leq 0 \tag{3}$$
$$4x_1 + 3x_2 \leq 12 \tag{4}$$
$$x_1, \quad x_2 \geq 0 \tag{5}$$



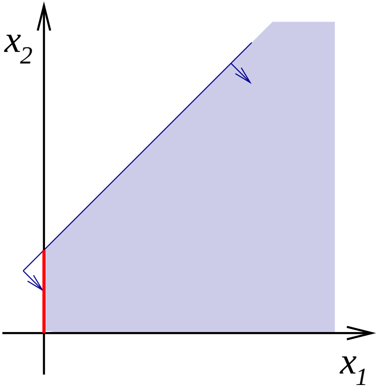
Degenerate cases

In the example we had a unique solution at a vertex of the polyhedron.
Some degenerate cases can lead to different solutions.

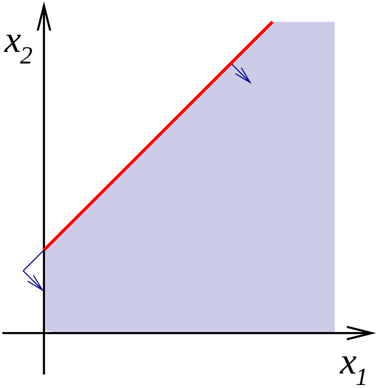
$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



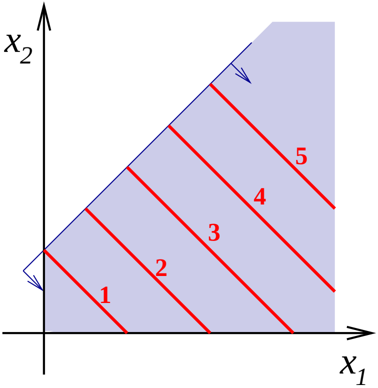
$$\begin{aligned} \min\ & x_1 \\ \text{s.t.}\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



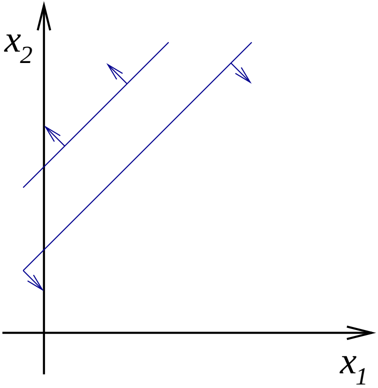
$$
\begin{aligned}
\max \ & -x_1 + x_2 \\
\text{s.t.} \ & -x_1 + x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t.}\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



$$\begin{aligned} \max\ & x_1 + 2x_2 \\ \text{s.t. } & -x_1 + x_2 \leq 1 \\ & -x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Polyhedra

Definition

A polyhedron is a set $\{x \in \mathbb{R}^n | Ax \geq b\}$

A set of the form $Ax \leq b$ is also a polyhedron.
A set $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ is a polyhedron in standard form.

Definition

Let $a \in \mathbb{R}^n \setminus \{0\}$.

(a) The set $\{x \in \mathbb{R}^n | a^T x = b\}$ is a hyperplane

(b) The set $\{x \in \mathbb{R}^n | a^T x \geq b\}$ is a halfspace

Convex Sets

Definition

A set $S \subseteq \mathbb{R}^n$ is convex if for all $x, y \in S$ and all $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in S$.

Definition

Let $x^1, \dots, x^k$ be vectors of $\mathbb{R}^n$.

(i) $\lambda_1 x^1 + \cdots + \lambda_k x^k$ is a conic combination si $\lambda_1, \dots, \lambda_k \geq 0$

(ii) $\lambda_1 x^1 + \cdots + \lambda_k x^k$ is a convex combination si $\lambda_1, \dots, \lambda_k \geq 0$ et $\lambda_1 + \cdots + \lambda_k = 1$

(iii) The convex hull of $x^1, \dots, x^k$ is the set of all convex combinations of $x^1, \dots, x^k$.

Theorem

(a) The intersection of two convex sets is convex

(b) Every polyhedron is convex

(c) The convex hull of a finite number of points is a polyhedron.

Extreme points and vertices

Definition

Let P be a polyhedron. A point $x \in P$ is an extreme point of P if there do not exist two points $y, z \in P$ such that x is a convex combination of y and z.

Definition

Let P be a polyhedron. A point $x \in P$ is a vertex of P if there exists $c \in \mathbb{R}^n$ such that $c^T x < c^T y$ for all $y \in P$ and $y \neq x$.

Bases of a polyhedron

We subdivide the equalities and inequalities into three categories :

$$\begin{aligned} a_i^T x &\geq b_i & i &\in M_{\geq} \\ a_i^T x &\leq b_i & i &\in M_{\leq} \\ a_i^T x &= b_i & i &\in M_{=} \end{aligned}$$

Definition

Let $\bar{x}$ be a point satisfying $a_i^T \bar{x} = b_i$ for some $i \in M_{\geq}, M_{\leq}$ or $M_{=}$. The constraint i is said to be active or tight.

Theorem

Let $\bar{x} \in \mathbb{R}^n$ and let I be the set of active constraints for $\bar{x}$. The three following statements are equivalent.

(i) There exist n linearly independent vectors in $\{a_i | i \in I\}$

(ii) $\text{span}\{a_i | i \in I\} = \mathbb{R}^n$

(iii) The system $a_i^T x = b_i$, $i \in I$ has a unique solution.

Bases of a polyhedron

Definition

Let P be a polyhedron and let $\bar{x} \in \mathbb{R}^n$.

(a) $\bar{x}$ is a basic solution if

- all equalities ($i \in M_=$) are active
- among the active constraints, there are n linearly independent

(b) if $\bar{x}$ is a basic solution that satisfies all constraints, then $\bar{x}$ is a feasible basic solution.

Theorem

Let P be a polyhedron and let $\bar{x} \in P$. The three following statements are equivalent.

(i) $\bar{x}$ is a vertex

(ii) $\bar{x}$ is an extreme point

(iii) $\bar{x}$ is a basic feasible solution

Polyhedra in standard form

Consider $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$.
We assume that the rows of A are linearly independent.

Theorem

A point $\bar{x}$ is a basic feasible solution if $A\bar{x} = b$ and if there exist m indices $B(1), \ldots, B(m)$ such that

(i) The columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent

(ii) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$

Explanation :

$$\begin{array}{c} m \text{ rows} \\ n \text{ rows} \end{array} \qquad \begin{pmatrix} A \\ I \end{pmatrix} x \begin{array}{c} = \\ \geq \end{array} \begin{pmatrix} b \\ 0 \end{pmatrix}$$

We have $n + m$ constraints and n variables.
A basic solution $\Rightarrow$ n constraints satisfied with equality.
The m equalities are automatically satisfied.
There are $n - m$ inequalities $x_i \geq 0$ that are active (the nonbasic variables).
There are m inequalities $x_i \geq 0$ that are possibly not active (basic variables).

Construction of a basis

Procedure ($\neq$ Algorithm)

(i) Choose m linearly independent columns $A_{B(1)}, \ldots, A_{B(m)}$

(ii) $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$

(iii) Solve $Ax = b$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$

If the solution $x \geq 0$, then x is a basic feasible solution.
We construct the basic matrix as

$$B = \begin{pmatrix} A_{B(1)} & A_{B(2)} & \cdots & A_{B(m)} \end{pmatrix}$$

The nonbasic matrix N corresponds to nonbasic indices.
The basic vector is $x_B = (x_{B(1)}, \ldots, x_{B(m)})$ and the nonbasic vector corresponds to the other indices.
We have

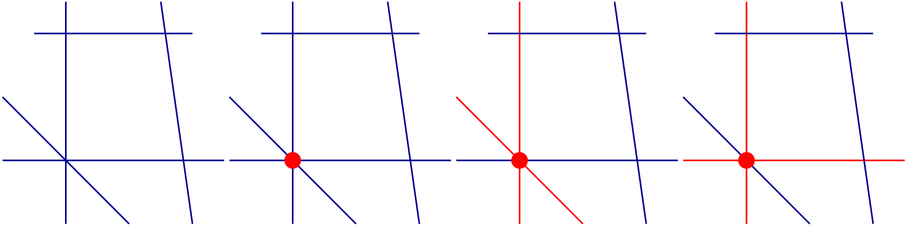
$$\begin{aligned} Bx_B &= b \\ x_N &= 0 \\ Bx_B + Nx_N &= b \end{aligned}$$

Example

Some important remarks

Correspondence between the base and the basic solution

Two different bases could lead to the same solution x.



Adjacent Bases

Two bases are adjacent they differ by only one index.

Differently stated they have $n-1$ indices in common !

Degenerescence

Definition

A basic solution $x \in \mathbb{R}^n$ is degenerate if more than n constraints are active at the solution.

Degenerescence for a standard form

Let $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$, with $A \in \mathbb{R}^{m \times n}$. A basic solution x is degenerate if x has more than $n - m$ zero elements.

Remark : Degenerescence may be representation-dependent.
A non degenerate basis can be degenerate in another representation of the problem and conversely.
Example :

Existence of extreme points

Definition

A polyhedron $P \subseteq \mathbb{R}^n$ contains a straight line if there exist $x \in P$ and a vector $d \in \mathbb{R}^n$ such that $x + \lambda d \in P$ for all $\lambda \in \mathbb{R}$.

Theorem

Let $P = \{x \in \mathbb{R}^n | a_i^T x \leq b_i, i = 1, \ldots, m\}$ be a nonempty polyhedron. The three following statements are equivalent.

(i) P has at least one extreme point

(ii) P does not contain a straight line

(iii) There are n linearly independent vectors a_i

Corollary

Every bounded polyehdron has at least one extreme point
Every polyhedron in standard form has at least one extreme point.

The extreme points are the *candidates* for being optimal points

Theorem

Consider

$$\max c^T x$$
$$\text{s.t. } x \in P.$$

If P has at least one extreme point and if there exists an optimal solution, then there exists one optimal solution that is an extreme point.

Generalization of the previous theorem : either the optimal solution is $+\infty$ or there exists an extreme point which is an optimal solution.

Theorem

If P has at least one extreme point, then if we consider the problem

$$\max c^T x$$
$$\text{s.t. } x \in P.$$

it either has $+\infty$ as optimal value, or there exists an extreme point which is optimal.

Representation of polyhedra

Fundamental theorem of Minkowski-Weyl

Every polyhedron $P = \{x \in \mathbb{R}^n | Ax \geq b\}$ can be represented as

$$P = \text{conv}\{v^1, \dots, v^p\} + \text{cone}\{r^1, \dots, r^q\}.$$

The v^i are the extreme points.
The r^j are the extreme rays.
Equivalently,

$$\begin{aligned} P = \{x \in \mathbb{R}^n | x = \lambda_1 v^1 + \cdots + \lambda_p v^p + \mu_1 r^1 + \cdots + \mu_q r^q \\ \lambda_1 + \cdots + \lambda_p = 1 \\ \lambda_i, \mu_j \geq 0 \text{ for all } i, j. \quad \} \end{aligned}$$

Fourier-Motzkin elimination

Oldest method to solve linear programming.

- Main drawback : Theoretical and practical complexity is very bad !
- 1st Advantage : Allows us to compute projections of polyhedra
- 2nd Avantage : Allows us to go from the representation $Ax \geq b$ (outer representation) to the representation conv + cone (inner representation)
- Principle : Very similar to the substitution method to solve linear systems of equations

Projections of polyhedra

Definition

$$\pi_k(x_1, \ldots, x_n) := (x_1, \ldots, x_k)$$

The projection of a vector on its first k components is simply the vector built from its first k components.

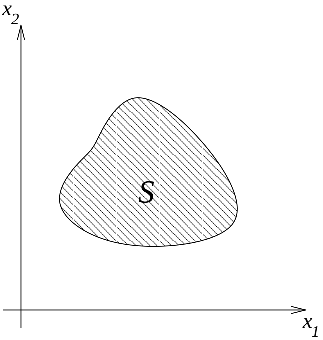
Projections of polyhedra

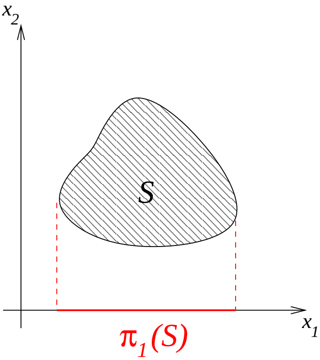
Definition

Let $S \subseteq \mathbb{R}^n$,

$$\pi_k(S) = \{(x_1, \dots, x_k) \in \mathbb{R}^k | \text{there exist } x_{k+1}, \dots, x_n \text{ such that } (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in S \}.$$

The projection of a set on its first k components is the set of vectors in $\mathbb{R}^k$ that can be completed in an element of the initial set.





Fourier-Motzkin Elimination

Algorithm to compute a projection by eliminating a variable.

Principle to eliminate the variable x_n

Rewrite the constraints $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ as

$$x_n \geq \sum_{j=1}^{n-1} \bar{a}_{ij}x_j \quad i \in M_{\geq}$$

$$x_n \leq \sum_{j=1}^{n-1} \bar{a}_{kj}x_j \quad k \in M_{\leq}$$

$$0 \leq \sum_{j=1}^{n-1} \bar{a}_{lj}x_j \quad l \in M_0$$

We can write the projection by combining the inequalities by pairs :

$$\sum_{j=1}^{n-1} \bar{a}_{kj}x_j \geq \sum_{j=1}^{n-1} \bar{a}_{ij}x_j \quad i \in M_{\geq}, k \in M_{\leq}$$

$$0 \leq \sum_{j=1}^{n-1} \bar{a}_{lj}x_j \quad l \in M_0$$

Fourier-Motzkin Elimination

Remark : In the worst case, we create $\frac{m^2}{4}$ constraints in the projection.
It is not easy to detect the redundant ones !
After i steps, we have $\frac{m^{2^i}}{2^{2^i-2}}$ constraints, which grows very quickly !

Principle to optimize or to detect infeasiblity

- A polyhedron P is empty if and only if any of its projection is empty
- To find the optimal solution of $\max\{c^T x | x \in P\}$,
  introduce the variable x_0 and the constraint $x_0 = c^T(x_1, \ldots, x_n)$.
  Then eliminate the n variables by projecting on the variable x_0.
  Solve $\max\{x_0 | x_0 \in \pi_0(P)\}$.

Fourier-Motzkin elimination to change the representation

We can use Fourier-Motzkin elimination to go from

$$P = \text{conv}\{v^1, \ldots, v^p\} + \text{cone}\{r^1, \ldots, r^q\}$$

to the form

$$P = \{x \in \mathbb{R}^n | Ax \leq b\}$$

Method : Write in the extended space :

$$\begin{aligned} P = \{x \in \mathbb{R}^n | x = \lambda_1 v^1 + \cdots + \lambda_p v^p + \mu_1 r^1 + \cdots + \mu_q r^q \\ \lambda_1 + \cdots + \lambda_p = 1 \\ \lambda_1, \lambda_p, \mu_1, \ldots, \mu_q \geq 0\} \end{aligned}$$

Then eliminate the variables $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q$ in order to obtain the desired representation !