Introduction à l'Optimisation Numérique

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Organisation du cours

Horaire

- Cours théorique tous les vendredis de 14h00 à 16h00 (R3)
- Répétitions les vendredis de 16h00 à 18h00 (I.94, I.97, I.123)
- Assistants : Bertrand Cornélusse, David Detry, Laurent Poirrier

Evaluation

- Projet de modélisation et d'utilisation d'un logiciel de programmation linéaire par groupes de 2 (25 % de la note finale)
- Examen d'exercices à livre ouvert (75 % de la note finale)
- ... ce qui ne veut pas dire qu'il ne faut pas étudier !

Notes de cours et références

- Transparents disponibles sur le web
  `http ://www.montefiore.ulg.ac.be/˜louveaux/optim.html`
- Livres de référence
  D. Bertsimas, J. Tsitsiklis. *Introduction to Linear Programming*.
  D. de Werra, J.F. Heche, T. Liebling. *Recherche opérationnelle pour ingénieurs I*.
  Attention : certaines parties déborderont du cadre des livres.

What will we be doing ?

Modeling

- Various problems from practical applications
- Modeling by determining
  decision variables, constraints, and an objective function to minimize or maximize
- Often very complicated in practice

Resolution

Generic problem

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. }\ & g(x) \leq 0 \\ & h(x) = 0 \\ & x \in X, \end{aligned}$$

où $X \subseteq \mathbb{R}^n$.
The problems are more or less difficult depending on the structure of f, g, h and X.

Complexity overview

Linear Programming

f, g, h are linear.
$X = \mathbb{R}^n$ or $X = \mathbb{R}^n_+$.

$$\max c^T x$$
$$\text{s. t. } Ax = b$$
$$x \in \mathbb{R}^n_+$$

Polynomial algorithm
Structured : strong duality

Convex programming

f, g are convex and h linear.
$X = \mathbb{R}^n$ or $X = \mathbb{R}^n_+$.

$$\min f(x)$$
$$\text{s. t. } Ax = b$$
$$g(x) \leq 0$$
$$x \in \mathbb{R}^n_+$$

Polynomial in theory

Conic programming

$$\min c^T x$$
$$\text{s.t. } b - Ax \in K$$

where K is a cone.
Polynomial algorithm
Structured : strong duality

Integer Programming

f, g, h are linear. $X = \mathbb{Z}^n_+$

$$\max c^T x$$
$$\text{s. t. } Ax = b$$
$$x \in \mathbb{Z}^n_+$$

NP-hard

Non convex programming

f, g, h arbitrary
$X = \mathbb{R}^n$ or $X = \mathbb{R}^n_+$.
Local minimum is hard to find and global as well ...

Outline of the lecture

- Modeling
  Linear, Convex, Nonlinear, Integer, Stochastic,. . .
- Linear Programming
  Simplex algorithm, duality, sensitivity, interior point methods , network problems
- Non linear programming
  Optimality conditions in general, Conic programming

Modeling of some practical problems

Production problem

A furniture company produces two types of chairs from beech and oak wood.

The first type (basic) requires 9 boards of beech wood and 2 boards of oak wood. A basic chair is easy to construct and requires one worker's hour.

The second type (classic) requires 7 boards of beech wood and 5 boards of oak wood. Due to the more important completion time, the classic chair requires three worker's hours.

The price of a basic chair is 30 euros whereas a classic chair's price is 70 euros.

The stock of boards of the entreprise is 800 boards of beech wood and 200 boards of oak wood. There are 4 workers working each 40 hours per week.

What is the number of basic and classic chairs that the entreprise must produce for this week ?

Modeling of the problem

Choice of the decision variables

x_B = Number of basic chairs to construct this week
x_C = Number of classic chairs to construct this week

Objective to optimize

$$\max 30x_B + 70x_C$$

Constraints

Beech : $9x_B + 7x_C \leq 800$
Oak : $2x_B + 5x_C \leq 200$
Work : $x_B + 3x_C \leq 160$
Other : $x_B \geq 0, x_C \geq 0$ (and x_B, x_C integer).

$$\begin{aligned} \max\ & 30x_B + 70x_C \\ \text{s.t.}\ & 9x_B + 7x_C \leq 1000 \\ & 2x_B + 5x_C \leq 200 \\ & x_B + 3x_C \leq 160 \\ & x_B, \quad x_C \geq 0 \end{aligned}$$

Alloy production

The company Steel has received an order for 500 tons of steel to be used in shipbuilding. This steel must have the following characteristics

Chemical element	Minimum grade	Maximum grade
Carbon (C)	2	3
Copper (Cu)	0.4	0.6
Manganese (Mn)	1.2	1.65

The company has seven different raw materials in stock that may be used for the production of this steel. The following Table lists the grades, available amounts and prices for all raw materials.

Raw material	C %	Cu %	Mn %	Availability in t	Cost in €/t
Iron alloy 1	2.5	0	1.3	400	200
Iron alloy 2	3	0	0.8	300	250
Iron alloy 3	0	0.3	0	600	150
Copper alloy 1	0	90	0	500	220
Copper alloy 2	0	96	4	200	240
Aluminum alloy 1	0	0.4	1.2	300	200
Aluminum alloy 2	0	0.6	0	250	165

The objective is to determine the composition of the steel that minimizes the production cost.

Formulation

Decision Variables

use_i : Quantity of alloy i used ($i \in I$)

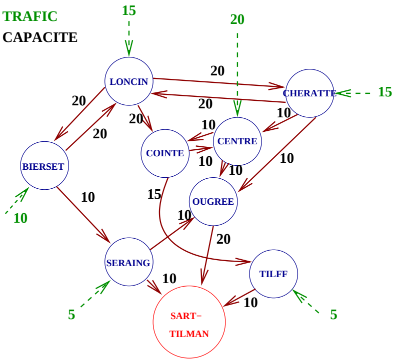
Objective to optimize

$$\min \sum_{i \in I} price_i \; use_i$$

Constraints

Carbon : $LB_C \leq \sum_{i \in I} C_i \; use_i \leq UB_C$
Copper : $LB_{Cu} \leq \sum_{i \in I} Cu_i \; use_i \leq UB_{Cu}$
Manganese :$LB_{Mn} \leq \sum_{i \in I} Mn_i \; use_i \leq UB_{Mn}$
Availability : $0 \leq \mathrm{use}_i \leq Stock_i$
Production : $\sum_{i \in I} use_i = Demand$

Traffic problem



Questions :

What is the toal capacity of the network ?

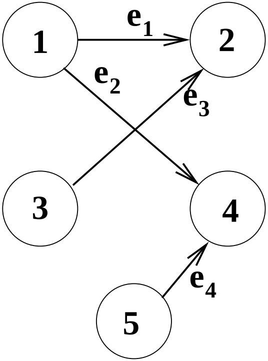
Which roads are saturated ? Which new roads need to be constructed ?

Maximum flow problem

The previous problem can be seen as a maximum flow problem in a directed graph

Definition of a graph

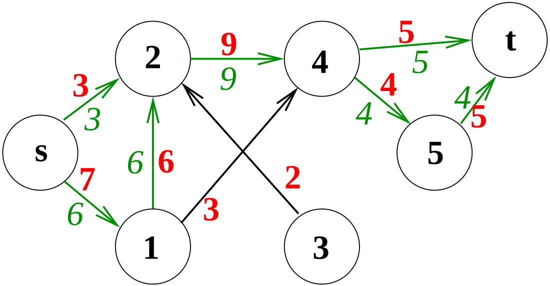
A graph $G = (V, E)$ is a set of nodes V and a set of edges $e = (i, j)$ linking the node i to the node j.



A directed graph or digraph

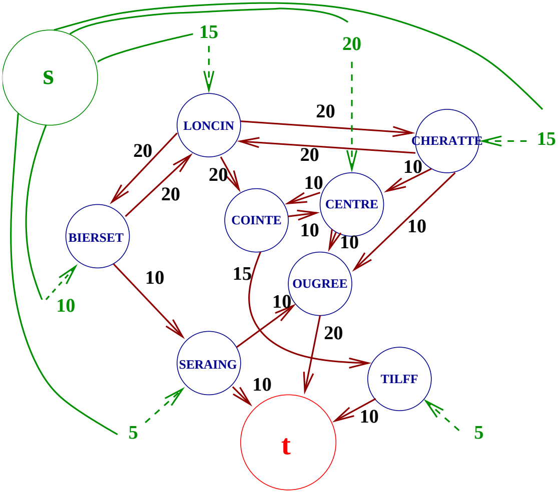
The max flow problem

Send the maximum flow from a node s (source) to a node t (sink).
Each edge has a capacity
Example of a feasible flow



Retour à Liège. . .

Can be seen as a max flow problem !



Mathematical programming formulation

There are very good combinatorial algorithms to solve the max flow problem.
We can also model it as a mathematical program

Decision variables

x_{ij} = flow in the edges linking node i to node j (if existing)

Objective to optimize

$$\max \sum_{i \in \delta^+(s)} x_{si}$$

Constraints

For each node i (other than s or t), we have the flow conservation constraint :

$$\sum_{j \in \delta^+(i)} x_{ij} = \sum_{l \in \delta^-(i)} x_{li} \qquad \text{for all } i \neq s, t.$$

Bound on the flow in the edges : $0 \leq x_{ij} \leq u_{ij}$ where u is the capacity

Max Flow problem

Some unexpected problems can be modeled as a max flow problem.

Shipment of perishable goods by boat

An entreprise transports perishable goods by boat.
It owns a set of different loads situated in different harbours. These loads are perishable which means that they have to be transported at a given date in another harbour otherwise they are lost.
The goal of the problem is to find the minimum number of boats needed in order to achieve the operation

Example :

Load	Origin	Destination	Due Date (in weeks)
1	Port A	Port C	3
2	Port A	Port C	8
3	Port B	Port D	3
4	Port B	Port C	6

Duration of the trip (fully loaded)

	C	D
A	3	2
B	2	3

Duration of the trip (empty)

	A	B
C	2	1
D	1	2

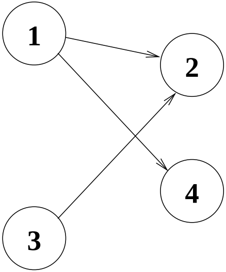
Shipment of perishable goods by boat

1	A	C	3
2	A	C	8
3	B	D	3
4	B	C	6

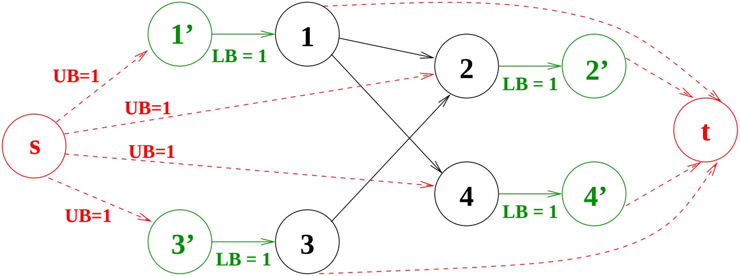
	C	D
A	3	2
B	2	3

	A	B
C	2	1
D	1	2

Construction of a precedence graph
A node = one load
One arc (i, j) = one boat can ship both load i and j



Shipment of perishabe goods by boat



Precedence graph

Each node is doubled.

An arc is also created to represent the actual shipment of a given load

Creation of source and sink nodes and lower and upper bounds on the arcs

Objective : Minimize the flow in the graph !

Modeling with integer variables

The airline company *SafeFlight* uses Roissy as a hub. This implies that all domestic flights land in Roissy between 11h and 12h30. Then all european flights take off from Roissy from 12h30. The number of passengers coming from one destination and going to another is listed below :

	Berlin	Bern	Brussels	London	Rome	Vienna
Bordeaux	35	12	16	38	5	2
Cl.-Ferrand	25	8	9	24	6	8
Marseille	12	8	11	27	3	2
Nantes	38	15	14	30	2	9
Nice	-	9	8	25	10	5
Toulouse	-	-	-	14	6	7

How to assign the planes to the destinations in order to minimize the number of passengers that must change of plane ?

Modeling the matching problem

Decision Variables

$$x_{ij} = 1 \quad \text{if the plane coming from } i \text{ flies to } j$$
$$= 0 \quad \text{otherwise}$$

Objective

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} pass_{ij} x_{ij}$$

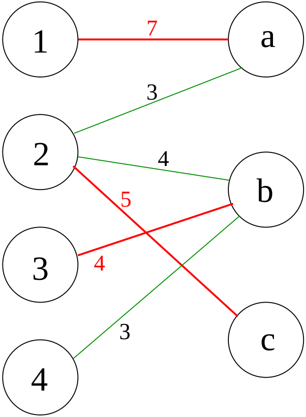
where $pass_{ij}$ is the number of passengers coming from i and going to j.

Constraints

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{for all } j \qquad 1 \text{ destination} = 1 \text{ plane}$$
$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for all } i \qquad 1 \text{ origin} = 1 \text{ plane}$$
$$x_{ij} \in \{0, 1\}$$

The matching problem in a bipartite graph

A graph is bipartite if all edges go from one node-set V_1 to a node-set V_2.
Weights on the edges
A matching is a set of edges where no two edges have one common incident node.



Example of a maximum weight matching

Most integer programming problems are difficult

Five tanker ships have arrived at a chemical factory. They are carrying loads of liquid products that must not be mixed : 1200 tonnes of Benzol, 700 tonnes of Butanol, 1000 tonnes of Propanol, 450 tonnes of Styrene, and 1200 tonnes of THF.

Nine tanks are available on site. Their respective capacities are listed below

Tank	1	2	3	4	5	6	7	8	9
Capacity	500	400	600	600	900	800	800	200	200

How to distribute the different chemical products in the different tankers in order to keep the maximum number of them empty ?

Modeling

Decision Variables

$$\begin{aligned} load_{lt} &= 1 \quad \text{if the liquid } l \text{ is filled in the tank } t \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Constraints

$$\sum_{l=1}^{5} load_{lt} \leq 1 \qquad \text{for each tank } t$$

Maximum 1 liquid per tank !

$$\sum_{t=1}^{9} capacity_t \, load_{lt} \geq quantity_l \qquad \text{for each liquid } l$$

The whole quantity must be filled somewhere.

$$load_{lt} \in \{0, 1\}$$

Objective to optimize

$$\min \sum_{l=1}^{5} \sum_{t=1}^{9} load_{lt}$$

Minimize the number of used tanks.

Planification under uncertainty

A farmer owns 100 hectares which he would like to devote to two types of culture : corn and wheat. In February he decides how much land he devotes to which crop.
The weather varies from one year to another and influences the quantity he collects, the demand and the market price. We model the variation of the weather by two scenarios of equal probability : dry weather and rainy weather. Depending on the scenario, the following the following data can be observed.

Scenario	Yield		Demand		Market Price	
	Corn	Wheat	Corn	Wheat	Corn	Wheat
Dry	10 kg/ha	3 kg/ha	900 kg	200 kg	4 €/kg	10 €/kg
Rainy	15 kg/ha	2kg/ha	600 kg	350 kg	3 €/kg	13 €/kg

Finally if the farmer cannot meet the demand, he can buy the two products abroad. Independent of the weather, the corn costs 2 €/kg while the wheat costs 7€/kg.

What does the farmer have to decide in february in order to maximize the expected profit ?

Probleme with recourse

It is a stochastic problem with recourse.

Time 0	Choice of the variables x_0

$\downarrow$

Realization of the random variable

$\swarrow$ $\searrow$

Scenario 1		Scenario 2	
Time 1	Recourse Variables y_1	Time 1	Recourse Variables z_1

Synopsis

$$
\begin{aligned}
\min\ & f(x_0) + \frac{1}{2} g(x_0, y_1) + \frac{1}{2} h(x_0, z_1) \\
\text{s.t. }\ & c_0(x_0) = 0 \\
& c_1(x_0, y_1) = 0 \\
& c_2(x_0, z_1) = 0.
\end{aligned}
$$

Modeling of the stochastic problem

Decision Variables

Variables at time 0

x_{corn}	Number of hectares affected to corn
x_{wheat}	Number of hectares affected to wheat

Variables at time 1

y^{dry}_{corn}	Quantity of bought corn (dry weather)
y^{dry}_{wheat}	Quantity of bought wheat (dry weather)
y^{rainy}_{corn}	Quantity of bought corn (rainy weather)
y^{rainy}_{wheat}	Quantity of bought wheat (rainy weather)

Constraints

Capacity of the field	$x_{corn} + x_{wheat} \leq 100$
Wheat bought (sc. dry)	$3x_{wheat} + y^{dry}_{wheat} \geq 200$
Wheat bought (sc. rainy)	$2x_{wheat} + y^{rainy}_{wheat} \geq 350$
Corn bought (sc. dry)	$10x_{corn} + y^{dry}_{corn} \geq 900$
Corn bought (sc. rainy)	$15x_{corn} + y^{rainy}_{corn} \geq 600$

Modeling of the stochastic problem

Objective

$$\max \ \frac{1}{2}(4 * 900 + 10 * 200 - 2y^{dry}_{corn} - 7y^{dry}_{wheat}) + \frac{1}{2}\left(3 * 600 + 13 * 350 - 2y^{rainy}_{corn} - 7y^{rainy}_{wheat}\right)$$

Bounds on the variables

All the variables in $\mathbb{R}_+$.