Linear Control Systems
Lecture #3 - State Feedback

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Control principle

- We want to design a controller such that the output of the closed-loop system tracks a specific reference value ("stabilization").

- Tools that we will use in the next two lectures: systems modeling in the time-domain and state-space representation.
Control principle

- We want to design a controller such that the output of the closed-loop system **tracks a specific reference value** ("stabilization").

- Tools that we will use in the next two lectures: systems modeling in the time-domain and **state-space representation**.
Outline

- State-space representation (reminder)
- Controllability/Reachability
- Stabilization by state feedback and eigenvalue assignment/pole placement
- State feedback design (second order systems)
- Integral action
Outline

- State-space representation (reminder)
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State-space representation of LTI systems

- A LTI system can be represented as follows

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

where \( x \) is the state vector whose dimension gives the dimension of the system.

- In a single-input/single-output (SISO) system, we have (ex: 4 states)

\[
\begin{align*}
A &= \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{pmatrix} & B &= \begin{pmatrix}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{pmatrix} & C &= (\cdot \cdot \cdot \cdot) \\
D &= (\cdot)
\end{align*}
\]
State-space representation of LTI systems: canonical form

- Let’s consider the general case

\[ \sum_{k=0}^{N} a_k y^{(k)} = b_0 u, \quad a_N = 1 \]

- Which can be written

\[ y^{(N)} = - \sum_{k=0}^{N-1} a_k y^{(k)} + b_0 u \]
State-space representation of LTI systems: canonical form

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- States are outputs of integrators:
  \[ x_1 = y, \ x_2 = y^{(1)}, \ x_3 = y^{(2)}, \ldots, \ x_n = y^{(N-1)} \]
State-space representation of LTI systems: canonical form

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\[ x_1 = y, \ x_2 = y^{(1)}, \ x_3 = y^{(2)}, \ldots, \ x_n = y^{(N-1)} \]

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
& \vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{N-1} x_n + b_0 u \\
y &= x_1
\end{align*}
\]
State-space representation of LTI systems: canonical form

Let’s now consider the general case

\[ \sum_{k=0}^{N} a_k y^{(k)} = \sum_{k=0}^{N} b_k u^{(k)}, \quad a_N = 1 \]

We have to consider an intermediate signal \( \nu \) where

\[ \sum_{k=0}^{N} a_k \nu^{(k)} = u \quad \text{and} \quad y = \sum_{k=0}^{N} b_k \nu^{(k)} \]

Justification:

\[ P(D)y = Q(D)u \iff P(D)\nu = u \quad \text{and} \quad y = Q(D)\nu \]
State-space representation of LTI systems: canonical form

\[ \sum_{k=0}^{N} a_k \nu^{(k)} = u \quad \text{and} \quad y = \sum_{k=0}^{N} b_k \nu^{(k)} \]
State-space representation of LTI systems: canonical form

\[
\sum_{k=0}^{N} a_k u^{(k)} = u \quad \text{and} \quad y = \sum_{k=0}^{N} b_k u^{(k)}
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State-space representation of LTI systems: canonical form

$$\sum_{k=0}^{N} a_k \nu^{(k)} = u \quad \text{and} \quad y = \sum_{k=0}^{N} b_k \nu^{(k)}$$

- The **update function** is the same: states are outputs of integrators:

$$x_1 = y, x_2 = y^{(1)}, x_3 = y^{(2)}, \ldots, x_n = y^{(N-1)}$$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{N-1} x_n + u
\end{aligned}$$
State-space representation of LTI systems: canonical form

\[ \sum_{k=0}^{N} a_k \nu^{(k)} = u \quad \text{and} \quad y = \sum_{k=0}^{N} b_k \nu^{(k)} \]

- The **output function** is more complex:

\[ x_1 = y, \quad x_2 = y^{(1)}, \quad x_3 = y^{(2)}, \ldots, \quad x_n = y^{(N-1)} \]

\[ y = b_0x_1 + b_1x_2 + b_2x_3 + \cdots + b_{N-1}x_n \]
\[ + b_N(u - a_0x_1 - a_1x_2 - \cdots - a_{N-1}x_n) \]
State-space representation of LTI systems: canonical form

\[ \sum_{k=0}^{N} a_k u^{(k)} = u \quad \text{and} \quad y = \sum_{k=0}^{N} b_k u^{(k)} \]

- It gives the following **canonical state-space representation**:

\[ x_1 = y, \ x_2 = y^{(1)}, \ x_3 = y^{(2)}, \ldots, \ x_n = y^{(N-1)} \]

\[ A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{N-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
b_0 - b_N a_0 & b_1 - b_N a_1 & \cdots & b_{N-1} - b_N a_{N-1}
\end{bmatrix}, \quad D = b_N. \]
The dynamics and stability of a LTI system are determined by the **eigenvalues of the dynamics matrix** $A$ (i.e. the poles of the transfer function).
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In this course, we want to control the dynamics and stability of a system.

In other words, we want to assign the values of the eigenvalues of the dynamic matrix/poles of the transfer function.
State feedback

In this course, we want to control the dynamics and stability of a system.

To do so, we need to

I. Analyze if the system can be controlled by feedback (controllability).

II. If yes, define a control role that will stabilize the system to a reference value.

III. Tune the feedback gains to increase the performance and/or robustness of the closed-loop system.
Illustration: we want to stabilize a satellite on a specific orbit at a specific speed.

\[
\begin{align*}
\ddot{r}(t) &= r(t)\dot{\theta}^2(t) - \frac{k}{r^2(t)} + u_1(t) \\
\ddot{\theta}(t) &= -2\frac{\dot{r}(t)}{r(t)}\dot{\theta}(t) + \frac{1}{r(t)}u_2(t)
\end{align*}
\]

Goal: \( r(t) = \sigma, \ \theta(t) = \omega t \)

Inputs: radial force \( u_1(t) \)  
   tangential force \( u_2(t) \)
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Controllability/reachability

- A state $x_1$ is **reachable** if there exists an input that transfers the state from $x_0$ to $x_1$ in some finite time interval $[t_0, T)$.

- A **linear system** is **reachable** if every state is reachable.

- In general, we want to reach equilibrium points (or steady-states), i.e. $\dot{x} = 0$. 
Can we assess the reachability of a linear system?

- For illustration, we will consider a general discrete LTI system:

\[ x[n + 1] = Ax[n] + Bu[n] \]
\[ y[n] = Cx[n] + Du[n] \]

- We want to reach the state \( x[n] \) starting from \( x[0] \) using specific values of the input \( u[n] \).
Can we assess the reachability of a linear system?

- We want to reach the state $x[n]$ starting from $x[0]$ using specific values of the input $u[n]$:

$$x[1] = Ax[0] + Bu[0]$$
Can we assess the reachability of a linear system?

- We want to reach the state \( x[n] \) starting from \( x[0] \) using specific values of the input \( u[n] \):

\[
\begin{align*}
x[1] &= Ax[0] + Bu[0] \\
&= A(Ax[0] + Bu[0]) + Bu[1] \\
&= A^2 x[0] + ABu[0] + Bu[1]
\end{align*}
\]
Can we assess the reachability of a linear system?

We want to reach the state \( x[n] \) starting from \( x[0] \) using specific values of the input \( u[n] \):

\[
\begin{align*}
    x[1] &= Ax[0] + Bu[0] \\
         &= A(Ax[0] + Bu[0]) + Bu[1] \\
         &= A^2x[0] + ABu[0] + Bu[1] \\
         &= A(A^2x[0] + ABu[0] + Bu[1]) + Bu[2] \\
\end{align*}
\]
Can we assess the reachability of a linear system?

We want to reach the state $x[n]$ starting from $x[0]$ using specific values of the input $u[n]$:

\[
\begin{align*}
x[1] &= Ax[0] + Bu[0] \\
&= A(Ax[0] + Bu[0]) + Bu[1] \\
&= A^2x[0] + ABu[0] + Bu[1] \\
&= A(A^2x[0] + ABu[0] + Bu[1]) + Bu[2] \\
&\vdots \\
x[n] &= A^nx[0] + \sum_{k=0}^{n-1} A^{n-1-k} Bu[k], \quad n > 0
\end{align*}
\]
Can we assess the reachability of a linear system?

The relationship between the state values $x[0]$ and $x[n]$ is given by

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{n-1-k} B u[k], \quad n > 0$$

where $A^n = AAA \cdots A$ is the transition matrix of the system.
Can we assess the reachability of a linear system?

- The relationship between the state values $x[0]$ and $x[n]$ is given by

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where $A^n = AAA \cdots A$ is the **transition matrix** of the system.

- So the input values that will bring the state values from $x[0]$ to $x[n]$ is given by

$$x[n] - A^n x[0] = \sum_{k=0}^{n-1} A^{n-k-1} B u[k], \ n > 0$$

$$= \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}$$
Can we assess the reachability of a linear system?

- The relationship between the state values $x[0]$ and $x[n]$ is given by

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{n-1-k} Bu[k], \quad n > 0$$

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$$x[n] - A^n x[0] = \sum_{k=0}^{n-1} A^{n-k-1} Bu[k], \quad n > 0$$

$$= \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Reachability matrix $W_r$
Testing for reachability: the reachability matrix

$$x[n] - A^nx[0] = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Reachability matrix $W_r$

- A system is reachable if all the rows of the reachability matrix are linearly independent, i.e. the reachability matrix is full row rank, i.e. is invertible.

- If the reachability matrix is not full row rank, it means that two states are dependent on each other and we cannot reach independent values for these states via specific values of the input.

- This rule extends to continuous-time systems as well.
Testing for reachability: the reachability matrix

☐ In practice in this course, the reachability is a square matrix, and the condition for reachability is that its determinant is non-zero.

☐ **Exercise**: show that the position of a satellite, whose state-space representation is given below, cannot be controlled via a radial force alone \((u_1)\), but can be controlled via a tangential force alone \((u_2)\).

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 \\
3\omega^2 & 0 & 0 & 2\omega \\
0 & 0 & 0 & 1 \\
0 & -2\omega & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}
\]

with \(x_1 = r - \sigma\), \(x_2 = \dot{r}\), \(x_3 = \sigma(\theta - \omega t)\), \(x_4 = (\dot{\theta} - \omega)\)
Can we derive a **general rule** to assess reachability and design a state feedback controller for LTI systems?

We can use the **canonical state-space representation** of LTI systems:

\[
\frac{dz}{dt} = \begin{pmatrix}
-a_1 & -a_2 & -a_3 & \cdots & -a_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & \\
\end{pmatrix} z + \begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
0 \\
\end{pmatrix} u,
\]

\[
y = \begin{pmatrix}
b_1 & b_2 & b_3 & \cdots & b_n \\
\end{pmatrix} z + du.
\]

The characteristic polynomial (i.e. eigenvalues) is given by

\[
\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n
\]
Reachable canonical form

- The **reachability matrix** of the canonical form is given by

\[
W_r = \begin{pmatrix} B & AB & \ldots & A^{n-1}B \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 & \cdots & * \\ 0 & 1 & -a_1 & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}
\]

- This matrix is **full rank** (triangular form).

- If a system can be put in canonical form, it is reachable and conversely.
Change of coordinate to the canonical form

- We want to represent the LTI system using the canonical form:
  \[ \dot{x} = Ax + Bu \rightarrow \dot{z} = \tilde{A}z + \tilde{B}u \]

- We use the change of coordinate \( z = Tx \), which gives
  \[ \tilde{A} = TAT^{-1} \quad \tilde{B} = TB \]

- The reachability matrix writes
  \[ \tilde{W}_r = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \ldots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix} \]

  with
  \[ \tilde{A}\tilde{B} = TAT^{-1}TB = TAB \]
  \[
  \ldots
  \]
  \[ \tilde{A}^n\tilde{B} = TAT^{-1}TB = TA^nB \]
Change of coordinate to the canonical form

□ The reachability matrix writes

\[ \widetilde{W}_r = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix} \]

with

\[ \tilde{A}\tilde{B} = TAT^{-1}TB = TAB \]

\[ \cdots \]

\[ \tilde{A}^n\tilde{B} = TAT^{-1}TB = TA^nB \]

□ It gives

\[ \widetilde{W}_r = T \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = TW_r \]

□ And thus

\[ T = \widetilde{W}_r W_r^{\text{-1}} \]

is a transformation that will take the system into the canonical form.
Change of coordinate to the canonical form

If a system is reachable, there exists a transformation $z = T x$ such that the dynamics and control matrices are in canonical form. It is given by

$$T = \widetilde{W}_r W_r^{-1}$$

Example: put this system into canonical form:

$$\frac{dx}{dt} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$
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State feedback: structure

Performances:
- Static: **stability**.
- Dynamic: step response (**rise-time**), frequency response (**bandwidth**).
- Disturbance attenuation (**robustness**).
Stabilization by state feedback

Let’s consider the system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

We also consider that we can measure the value of all states. We can therefore use the (proportional) state feedback controller

\[u = -Kx + k_r r\]

This gives the closed-loop system

\[
\dot{x} = (A - BK)x + Bk_r r
\]
Closed-loop system for \( u = -Kx + kr \)

\[
\dot{x} = (A - BK)x + Bkr
\]

We know that the **dynamics** of the system depends on the **eigenvalues of the dynamics matrix** (\( \text{poles of the transfer function} \)).

The gain of the feedback \( K \) can therefore be designed to place the eigenvalue of the dynamics matrix of the closed-loop system \( (A - BK) \) where we want:

\[
p(s) = s^n + p_1s^{n-1} + \cdots + p_{n-1}s + p_n
\]

This method is called **eigenvalue assignment** or **pole placement**.
The dynamics and stability of a LTI system are determined by the eigenvalues of the dynamics matrix (i.e. the poles of the transfer function).
We want the system to approach the reference $r$ at equilibrium

$$\dot{x} = 0 = (A - BK)x_e + Bk_r r$$

It gives

$$x_e = -(A - BK)^{-1} Bk_r r$$

$$y_e = Cx_e + Du_e$$

If we consider $D=0$, we have

$$k_r = \frac{-1}{C(A - BK)^{-1} B}$$

The system reaches the reference only if the gain $k_r$ is **perfectly tuned!**
Control of vehicle steering: we want to track and maintain a given reference value of the lateral position of the vehicle.

The system is given by

\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0. \]
Control of vehicle steering: we want to track and maintain a given reference value of the lateral position of the vehicle.

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}, \\
C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 0.
\]

Is the system reachable? Yes:

\[
W_r = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} \gamma & 1 \\ 1 & 0 \end{pmatrix}.
\]
Stabilization by state feedback: illustration

- We can control the system using the feedback rule
  \[ u = -Kx + k_r r = -k_1 x_1 - k_2 x_2 + k_r r, \]

- This gives the close-loop system
  \[
  \frac{dx}{dt} = (A - BK)x + Bk_r r = \begin{pmatrix} -\gamma k_1 & 1 - \gamma k_2 \\ -k_1 & -k_2 \end{pmatrix} x + \begin{pmatrix} \gamma k_r \\ k_r \end{pmatrix} r,
  \]
  \[ y = Cx + Du = \begin{pmatrix} 1 & 0 \end{pmatrix} x. \]

with

\[
\det (sI - A + BK) = \det \begin{pmatrix} s + \gamma k_1 & \gamma k_2 - 1 \\ k_1 & s + k_2 \end{pmatrix} = s^2 + (\gamma k_1 + k_2)s + k_1.
\]
**Dynamic performances**: the gains can be tuned to tune the damping factor and the natural frequency of the system \( p(s) = s^2 + 2\zeta_c \omega_c s + \omega_c^2 \).

(a) Step response for varying \( \omega_c \)

(b) Step response for varying \( \zeta_c \)
Stabilization by state feedback: illustration

- **Static performances**: $k_T$ has to be carefully tuned.

(a) Step response for varying $\omega_c$

(b) Step response for varying $\zeta_c$
State feedback in reachable canonical form

If the system is in canonical form

\[
\frac{dz}{dt} = \tilde{A}z + \tilde{B}u = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u
\]

\[y = \tilde{C}z = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} z.\]

we have the close-loop system

\[
\frac{dz}{dt} = \begin{pmatrix} -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & -a_3 - \tilde{k}_3 & \cdots & -a_n - \tilde{k}_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} k_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} r,
\]

\[y = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} z.\]
State feedback in reachable canonical form

\[ \frac{dz}{dt} = \begin{pmatrix} -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & -a_3 - \tilde{k}_3 & \ldots & -a_n - \tilde{k}_n \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ldots & \vdots \\ 0 & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} k_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} r, \]

\[ y = \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix} z. \]

- The characteristic polynomial is given by
  \[ s^n + (a_1 + \tilde{k}_1)s^{n-1} + (a_2 + \tilde{k}_2)s^{n-2} + \cdots + (a_{n-1} + \tilde{k}_{n-1})s + a_n + \tilde{k}_n. \]

- So for the pole placement, the gain can be chosen such that
  \[ \tilde{k}_1 = p_1 - a_1, \quad \tilde{k}_2 = p_2 - a_2, \quad \ldots \quad \tilde{k}_n = p_n - a_n. \]
Stabilization by state feedback: dynamic performance

- State-feedback design: trade-off (second order systems).

(a) Step response for varying $\omega_c$

(b) Step response for varying $\zeta_c$
Stabilization by state feedback: dynamic performance

- State-feedback design: trade-off (second order systems).

(a) Eigenvalues
(b) Step responses
Stabilization by state feedback: static performance

- To ensure to track the reference without tuning: **integral action!**

\[
\begin{pmatrix}
\dot{x} \\
\dot{z}
\end{pmatrix} = \begin{pmatrix} Ax + Bu \\ y - r \end{pmatrix} = \begin{pmatrix} Ax + Bu \\ Cx - r \end{pmatrix}
\]

\[u = -Kx - k_iz + k_rr\]

- At equilibrium, the derivatives are equal to zero, which ensures that the system will reach the reference value whatever the value of the integrator gain $k_i \neq 0$.

\[\dot{z} = y - r = 0 \quad \Rightarrow \quad y = r\]

- Tuning the integrator gain affects the dynamic performances of the system.
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