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1 Characteristic lines

1.1 Characteristic line 1 [Strauss 1.2, Ex.3]

Consider the following equation: \((1 + x^2)u_x + u_y = 0\).

(a) Find the general solution of this equation.

(b) Draw some of the characteristic lines.

Solution

(a) The directional derivative of \(u\) in the direction of \((1 + x^2, 1)\) must always be 0. Therefore, the characteristic lines, along which \(u\) is constant, satisfy the relation

\[
\frac{dy}{dx} = \frac{1}{1 + x^2},
\]

which can be rewritten as

\[
dy = \frac{1}{1 + x^2} dx.
\]

Direct integration of both sides yields, with the integration constant \(C \in \mathbb{R}\),

\[
y = \arctan(x) + C
\]

\[
\Leftrightarrow C = y - \arctan(x).
\]

Knowing that \(u\) is constant on these characteristic lines, we have the following general solution:

\[
u(x, y) = f(y - \arctan(x)),
\]

where \(f\) is an arbitrary function.

(b) Some characteristic lines are represented in Fig. 1.

1.2 Characteristic line 2 [Strauss 1.2, Ex.7]

Consider the following problem: \(yu_x + xu_y = 0\) with \(u(0, y) = \exp(-y^2)\).

(a) Solve this problem.

(b) Where is the solution uniquely determined in the \(xy\)-plane?

Solution

(a) The directional derivative of \(u\) along the vector \((y, x)\) must be 0. The equation of the characteristic lines is therefore

\[
\frac{dy}{dx} = \frac{x}{y}.
\]
The characteristic lines satisfy
\[ y^2 = x^2 + C \] (1.2.2)
such that
\[ C = y^2 - x^2. \] (1.2.3)
Along these lines \( u \) is constant. Thus \( u \) is only a function of the parameter \( C \). Then using Eq. (1.2.3) yields
\[ u(x, y) = f(y^2 - x^2). \] (1.2.4)
Thanks to the additional condition, it is possible to determine the shape of the function \( f \). Indeed, at \( x = 0 \)
\[ f(y^2) = \exp\{-y^2\}. \] (1.2.5)
Thus one can write
\[ f(C) = \exp\{-C\} \] (1.2.6)
and finally, from Eq. (1.2.3), one has
\[ u(x, y) = \exp\{x^2 - y^2\}. \] (1.2.7)

(b) The additional condition is valid for \( x = 0 \). The characteristic lines satisfying \( C < 0 \) never go through \( x = 0 \), therefore the condition does not apply in that case. The two following cases must be distinguished in the final answer
\[
\begin{cases}
  \text{if } y^2 - x^2 < 0 \text{ then } u(x, y) = f(y^2 - x^2) \\
  \text{if } y^2 - x^2 \geq 0 \text{ then } u(x, y) = \exp\{x^2 - y^2\}.
\end{cases}
\] (1.2.8)

1.3 Characteristic line 3 [Strauss 1.5, Ex. 6]

Find the general solution of the equation \( u_x + 2xy^2u_y = 0 \).
Solution

The equation can be rewritten as

\[ \mathbf{a} \cdot \nabla u = 0 \quad (1.3.1) \]

where \( \mathbf{a} = (1, 2xy^2) \).

The characteristic curves thus satisfy the equation

\[ \frac{dy}{dx} = \frac{2xy^2}{1} \quad (1.3.2) \]

Therefore

\[ \frac{dy}{y^2} = 2xdx, \quad (1.3.3) \]

*i.e.,* upon integration,

\[ -\frac{1}{y} + C = x^2 \quad (1.3.4) \]

\[ \Rightarrow \quad y = \frac{1}{C - x^2}, \quad (1.3.5) \]

\[ \Rightarrow \quad C = x^2 + \frac{1}{y}. \quad (1.3.6) \]

\( u(x, y) \) is constant on the characteristic curves

\[ y = \frac{1}{C - x^2} \quad (1.3.7) \]

Indeed

\[ \frac{du}{dx} \left( x, \frac{1}{C - x^2} \right) = \frac{\partial u}{\partial x} + \frac{2x}{(C - x^2)^2} \frac{\partial u}{\partial y} = u_x + 2xy^2u_y = 0. \quad (1.3.8) \]

Hence, \( u(x, y) = f(C) \), *i.e.*

\[ u(x, y) = f \left( x^2 + \frac{1}{y} \right), \quad (1.3.9) \]

where \( f \) is an arbitrary function.

### 1.4 Transport with decay [Strauss 1.2, Ex. 8]

Find the general solution of the equation \( au_x + bu_y + cu = 0 \).

**Solution**

The equation can be written as

\[ \mathbf{a} \cdot \nabla u = -cu \quad (1.4.1) \]

where \( \mathbf{a} = (a, b) \). The directional derivative along \((a, b)\) is not 0. Therefore \( u \) is not constant along the lines

\[ \frac{dy}{dx} = \frac{b}{a} \quad (1.4.2) \]

\[ \Rightarrow y = \frac{b}{a}x + C. \quad (1.4.3) \]
Along these lines however, \( u \) satisfies
\[
\frac{du}{dx} = -\frac{c}{a} u. \tag{1.4.4}
\]
Indeed
\[
\frac{d}{dx} \left( u \left( x, \frac{b}{a} x + C \right) \right) = u_x + \frac{b}{a} u_y = -\frac{c}{a} u. \tag{1.4.5}
\]
Solving the Eq.(1.4.4) for \( u \) yields
\[
u = f(C) \exp\left\{ -\frac{c}{a} x \right\}, \tag{1.4.6}
\]
where the function \( f(C) \) comes from the integration of Eq.(1.4.4). From Eq.(1.4.3), the final solution is
\[
u(x, y) = f \left( y - \frac{b}{a} x \right) \exp \left( -\frac{c}{a} x \right). \tag{1.4.7}
\]

1.5 **Characteristic line 4 [Olver 2.2, Example 2.4]**

Find the general solution of the equation \( u_t + \frac{1}{x^2 + 1} u_x = 0 \).

**Solution**

The general solution is
\[
u(x, t) = f \left( \frac{1}{3} x^3 + x - t \right). \tag{1.5.1}
\]

1.6 **Characteristic line 5 [Olver 2.2, Example 2.5]**

Consider the equation
\[
u_t + (x^2 - 1) u_x = 0, \quad \forall (x, t) \in \mathbb{R} \times [0, \infty[.
\]
with the initial condition
\[
u(x, 0) = \exp \left( -x^2 \right).
\]

(a) Find the characteristic curves. Draw these curves.

(b) Which of these curves intersect the \( x \)-axis?

(c) Does the initial condition determine uniquely the solution of the problem? Where is the solution not defined?

(d) Give the solution to the problem in the region where it is uniquely defined.
Solution

(a) The characteristic curves are solutions of

\[ \frac{dx}{dt} = x^2 - 1 \]  \hspace{1cm} (1.6.1)

i.e.

\[ \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| = t + C, \quad C \in \mathbb{R} \quad \text{(or} \quad x = \pm 1) \]. \hspace{1cm} (1.6.2)

The characteristic curves can be expressed explicitly as

\[
x(t) = \begin{cases} 
\frac{1 + A \exp(2t)}{1 - A \exp(2t)} & \text{and} \quad x < -1 \\
\frac{1 - A \exp(2t)}{1 + A \exp(2t)} & \text{and} \quad 1 < x < -1 \\
\frac{1 + A \exp(2t)}{1 - A \exp(2t)} & \text{and} \quad x > 1 
\end{cases}
\]  \hspace{1cm} (1.6.3)

where \( A \triangleq \exp(2C) \) and thus \( A > 0 \).

These curves are given in Figure 2.

(b) For \( x < -1 \), the possible intersections are given by

\[ x_0 = \frac{1 + A}{1 - A}. \]  \hspace{1cm} (1.6.4)

Because \( x_0 < -1 \), only the curves for which \( A > 1 \) intersect the axis.
• For $-1 < x < 1$, the possible intersections are given by

$$x_0 = \frac{1 - A}{1 + A}$$  \hspace{1cm} (1.6.5)

All $(\forall A > 0)$ these curves intersect the axis.

• For $x > 1$, the possible intersections are given by

$$x_0 = \frac{1 + A}{1 - A}$$  \hspace{1cm} (1.6.6)

Because $x_0 > 1$, only the curves for which $A < 1$ intersect the axis.

(c) The solution is constant along each characteristic curve. The solution is therefore completely known provided the initial condition allows to determine this constant for each curve. The initial condition specifies the solution on the $x$-axis. Therefore, the constant value is fixed for the curves that crosses the $x$-axis. For the other curves, the value of the constant can not be determined and thus the solution is thus not unique.

The curves

$$x(t) = \frac{1 + A \exp(2t)}{1 - A \exp(2t)} \quad \text{and} \quad x < -1$$

(1.6.7)

do not cross the $x$-axis when $A < 1$. From Figure 2, it appears that this region is equivalent to

$$x \leq \frac{1 + \exp(2t)}{1 - \exp(2t)}, \quad t > 0.$$  \hspace{1cm} (1.6.8)

Similarly, the curves

$$x(t) = \frac{1 + A \exp(2t)}{1 - A \exp(2t)} \quad \text{and} \quad x > 1$$

(1.6.9)

do not cross the $x$-axis when $A > 1$. From Figure 2, it appears that this region is equivalent to

$$x \geq \frac{1 + \exp(2t)}{1 - \exp(2t)}, \quad t < 0.$$  \hspace{1cm} (1.6.10)

(d) On the regions where the characteristic curves intersect the $x$-axis, the constant $A$ can be expressed in terms of the intersection coordinate $x_0$, i.e.

• For $x < -1$, $A = \frac{x_0 - 1}{x_0 + 1}$

• For $-1 < x < 1$, $A = -\frac{x_0 - 1}{x_0 + 1}$

• For $x > 1$, $A = \frac{x_0 - 1}{x_0 + 1}$

thus

$$x(t) = \frac{(x_0 + 1) + (x_0 - 1) \exp(2t)}{(x_0 + 1) - (x_0 - 1) \exp(2t)}$$  \hspace{1cm} (1.6.11)

and

$$x_0 = \frac{(x + 1) + (x - 1) \exp(-2t)}{(x + 1) - (x - 1) \exp(-2t)}.$$  \hspace{1cm} (1.6.12)
Along the characteristic curves, the equation writes
\[ \frac{d}{dt} u(x(t), t) = 0 \quad (1.6.13) \]

thus
\[ u(x(t), t) = k = u(x(0), 0) = \exp(-x_0^2) \quad (1.6.14) \]

and finally
\[ u(x, t) = \exp \left( - \left[ \frac{(x + 1) + (x - 1) \exp(-2t)}{(x + 1) - (x - 1) \exp(-2t)} \right]^2 \right) \text{ for } x > \frac{1 + \exp(2t)}{1 - \exp(2t)}, \quad t > 0. \quad (1.6.15) \]

### 1.7 Characteristic line 6 [Strauss 14.1, Ex.2]

Solve \((1 + t)u_t + xu_x = 0\). Then solve it with the auxiliary condition \(u(x, 0) = x^5\) for \(t > 0\).

**Solution**

The PDE can be rewritten as
\[ u_t + \frac{x}{1+t} u_x = 0. \quad (1.7.1) \]

Therefore, the characteristic curves satisfy the equation \(^1\)
\[ \frac{dx}{dt} = \frac{x}{1+t}, \quad (1.7.2) \]
\[ \Rightarrow \quad x = C(1 + t) \quad C \in \mathbb{R}, \quad (1.7.3) \]
\[ \Rightarrow \quad C = \frac{x}{1+t}. \quad (1.7.4) \]

The solution thus has the following form
\[ u(x, t) = f \left( \frac{x}{1+t} \right). \quad (1.7.5) \]

If the initial condition is
\[ u(x, 0) = f(x) = x^5, \quad (1.7.6) \]

therefore the solution is
\[ u(x, t) = \left( \frac{x}{1+t} \right)^5. \quad (1.7.7) \]

### 1.8 Characteristic line in 2D

Find the solution of the equation \(yu_x + (-x)u_y + u_t = 0\) with \(u(x, y, 0) = \exp \left( -((x-1)^2 + 4y^2) \right)\).

---

\(^1\)Direct integration yields the solution \(|x| = C|1 + t|\). This solution is correct but incomplete.
Solution

The characteristic curves are defined as

\[
\begin{align*}
\frac{dx}{ds} &= y, \\
\frac{dy}{ds} &= -x, \\
\frac{dt}{ds} &= 1
\end{align*}
\]  

(1.8.1)

with

\[
x(0) = x_0, \quad y(0) = y_0 \quad \text{and} \quad t(0) = 0
\]  

(1.8.2)

thus

\[
\begin{align*}
x(s) &= x_0 \cos s + y_0 \sin s, \\
y(s) &= y_0 \cos s - x_0 \sin s, \\
t(s) &= s.
\end{align*}
\]  

(1.8.3)

The equation can then be written

\[
\begin{align*}
(y &- x \quad 1) \cdot (u_x \quad u_y \quad u_t) = 0 \\
\Leftrightarrow \quad (\frac{dx}{ds} \quad \frac{dy}{ds} \quad \frac{dt}{ds}) \cdot (u_x \quad u_y \quad u_t) &= 0 \\
\Leftrightarrow \quad \frac{d}{ds} u(x(s), y(s), t(s)) &= 0 \\
\Leftrightarrow \quad u(x(s), y(s), t(s)) &= C.
\end{align*}
\]  

(1.8.4) (1.8.5) (1.8.6) (1.8.7)

The field \( u \) is therefore constant along any characteristic curve. With the constant \( C \) of course depending on the curve. From the initial data, one finds that the constant \( C \) for the curve passing through the point \((x_0, y_0)\) at \( t = 0 \) is

\[
C = u(x_0, y_0, 0) = \exp \left( - \left( (x_0 - 1)^2 + 4 y_0^2 \right) \right).
\]  

(1.8.8)

Finally using the form

\[
\begin{align*}
x_0 &= x \cos t - y \sin t, \\
y_0 &= y \cos t + x \sin t
\end{align*}
\]  

(1.8.9)

of the characteristic curve, the solution is given by

\[
u(x, y, t) = C = \exp \left( - \left[ (x \cos t - y \sin t - 1)^2 + 4(y \cos t + x \sin t)^2 \right] \right).
\]  

(1.8.10)
2 Classification

2.1 Maxwell’s equations

Consider the continuum media Maxwell’s equations

\[ \nabla \times (h) = j + d_t \quad \text{(Ampère-Maxwell), (\diamond)} \]
\[ \nabla \times (e) = -b_t \quad \text{(Faraday), (\ast)} \]
\[ \nabla \cdot (b) = 0 \quad \text{(Gauss), (\dagger)} \]
\[ \nabla \cdot (d) = \rho \quad \text{(Coulomb), (\ddagger)} \]

where \( h \) [A/m] is the magnetic field, \( e \) [V/m] is the electric field, \( b \) [W/m\(^2\)] is the magnetic flux density, \( d \) [C/m\(^2\)] is the electric displacement field, \( j \) [A/m\(^2\)] is the current density and \( \rho \) [C/m\(^3\)] is the charge density. (Because these equations have too many unknowns (i.e. \( h, e, b, d, j \) and \( \rho \)), they must be complemented with constitutive equations.)

(a) Consider empty space \( i.e \)
\[ d = \epsilon_0 e, \quad b = \mu_0 h, \quad \rho = 0 \]
where \( \epsilon_0, \mu_0 \) are constants and that \( j \) is a known function.

- Making the assumption of transverse electric (TE) fields, \( i.e \quad e = u(x, y, t) \hat{z} \) and \( j = j(x, y) \hat{z} \), show that the electric field verifies
\[ u_{tt} - c^2 u_{xx} = f. \]
Determine \( c \) and \( f \). Classify this equation.

- Making the assumption of transverse magnetic (TM) fields, \( i.e \quad h = u(x, y) \hat{z} \) and \( \nabla \times (j) = m(x, y) \hat{z} \), show that the electric field verifies
\[ u_{tt} - c^2 u_{xx} = f. \]
Determine \( c \) and \( f \). Classify this equation.

(b) At steady state (\( i.e \quad \frac{\partial}{\partial t} = 0 \)) in linear media (\( b = \mu h \) and \( d = \epsilon e \), \( \epsilon \) and \( \mu \) being constants), show that the vector potential \( a \), defined such that \( \nabla \times (a) = b \) (with \( \nabla \cdot (a) = 0 \) imposed) and the scalar potential \( \phi \), defined such that \( -\nabla (\phi) = e \) verifies
\[ \Delta (\phi) = f_\phi, \]
and
\[ \Delta (a) = f_a. \]
Determine \( f_a \) and \( f_\phi \). Classify these equations.

(c) Using the magnetodynamics hypothesis \( d_t = 0 \) in linear media (\( b = \mu h, d = \epsilon e \) and \( j = \sigma e, \epsilon, \mu \) and \( \sigma \) being constants) show that the magnetic field verifies
\[ h_t - \alpha \Delta (h) = f. \]
Determine \( f \) and \( \alpha \). Classify this equation.

The magnetic constitutive law is replaced \( b_t = \mu (h_t + \omega_0 h) \). Update the previous equations. Classify.
Solution

(a) The derivative w.r.t $t$ of the Ampère-Maxwell equation Eq. ($\diamond$) is

$$\nabla \times (h_t) = j_t + \epsilon_0 e_{tt}. \quad (2.1.1)$$

Using the Maxwell-Faraday equation Eq. ($\ast$) yields

$$e_{tt} + \frac{1}{\epsilon_0 \mu_0} \nabla \times (\nabla \times (e)) = -\frac{1}{\epsilon_0} j_t. \quad (2.1.2)$$

The double curl simplifies because of the Gauss law as

$$\nabla \times (\nabla \times (e)) = \nabla (\nabla \cdot (e)) - \nabla \cdot (\nabla \times (e))$$

$$= -\nabla \cdot (\nabla (e)) \quad (2.1.3)$$

$$= -\Delta (e) \quad (2.1.4)$$

$$= -\Delta (u) \hat{z}. \quad (2.1.5)$$

The Ampère-Maxwell equation Eq. ($\diamond$) then finally writes

$$u_{tt} - \frac{1}{\epsilon_0 \mu_0} \Delta (u) = -\frac{1}{\epsilon_0} j_t \quad (2.1.7)$$

thus

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \text{and} \quad f = -\frac{1}{\epsilon_0} j_t \quad (2.1.8)$$

(b) The derivative w.r.t $t$ of the Maxwell-Faraday equation Eq. ($\ast$) is

$$\nabla \times (e_t) = -\mu_0 h_{tt}. \quad (2.1.9)$$

Using the Ampère-Maxwell equation Eq. ($\diamond$) yields

$$h_{tt} + \frac{1}{\epsilon_0 \mu_0} \nabla \times (\nabla \times (h)) = \frac{1}{\mu_0 \epsilon_0} \nabla \times (j). \quad (2.1.10)$$

The Maxwell-Faraday equation Eq. ($\ast$) then finally writes

$$u_{tt} - \frac{1}{\epsilon_0 \mu_0} \Delta (u) = \frac{1}{\mu_0 \epsilon_0} m \quad (2.1.11)$$

thus

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \text{and} \quad f = \frac{1}{\mu_0 \epsilon_0} m. \quad (2.1.12)$$

These equations are

- non homogeneous
- linear
- second order in space and time
- hyperbolic
Indeed, they can be written (for a single spatial dimension)

\[ Au_{tt} + B u_{tx} + C u_{xx} + D u_t + E u_x + F u = G \]  
(2.1.13)

\[ 1 u_{tt} + 0 u_{tx} - c^2 u_{xx} + 0 u_t + 0 u_x + 0 u = f \]  
(2.1.14)

thus

\[ B^2 - 4AC = 4c^2 > 0. \]  
(2.1.15)

(b) The Maxwell-Faraday Eq.(\(\ast\)) and the magnetic Gauss' law Eq.(\(\dagger\)) are trivially satisfied by the potentials

indeed

\[ \nabla \times (e) = -\nabla \times (\nabla (\phi)) = 0 \]  
(2.1.16)

and

\[ \nabla \cdot (b) = \nabla \cdot (\nabla \times (a)) = 0 \]  
(2.1.17)

while the electric Gauss' law Eq.(\(\dagger\)) becomes

\[ \nabla \cdot (d) = \rho \]  
(2.1.18)

\[ \nabla \cdot (\varepsilon e) = \rho \]  
(2.1.19)

\[ \nabla \cdot (-\nabla (\phi)) = \frac{\rho}{\varepsilon} \]  
(2.1.20)

\[ \Delta (\phi) = -\frac{\rho}{\varepsilon} \]  
(2.1.21)

and the Ampère-Maxwell Eq.(\(\diamond\)) becomes

\[ \nabla \times (h) = j \]  
(2.1.22)

\[ \nabla \times (\nabla \times (a)) = \mu j \]  
(2.1.23)

\[ \nabla (\nabla \cdot (a)) - \nabla \cdot (\nabla (a)) = \mu j \]  
(2.1.24)

\[ \Delta (a) = -\mu j \]  
(2.1.25)

These equations are

– non homogeneous
– linear
– second order in space and independent of time
– elliptic

Indeed, it can be written (for a two spatial dimensions) (either \(u = \phi\), either \(u = a\))

\[ Au_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \]  
(2.1.26)

\[ 1 u_{xx} + 0 u_{xy} + 1 u_{yy} + 0 u_x + 0 u_y + 0 u = f \]  
(2.1.27)

thus

\[ B^2 - 4AC = -4 < 0. \]  
(2.1.28)
(c) The Ampère-Maxwell law Eq. 8 becomes
\[ \nabla \times (h) = \sigma e \quad (2.1.29) \]
and the Maxwell-Faraday’s law Eq. 10 can then be written
\[ \frac{1}{\sigma} \nabla \times (\nabla \times (h)) = -\mu (h_t + \omega_0 h) \quad (2.1.30) \]
or
\[ h_t + \frac{1}{\mu \sigma} \nabla \times (\nabla \times (h)) + \omega_0 h = 0 \quad (2.1.31) \]
\[ h_t - \frac{1}{\mu \sigma} \Delta (h) + \omega_0 h = 0 \quad (2.1.32) \]

This equation is
- homogeneous
- linear
- second order in space and first order in time
- parabolic

Indeed, it can be written (for a single spatial dimension)
\[ Ah_{tt} + Bh_{tx} + Ch_{xx} + Dh_t + Eh_x + Fu = G \quad (2.1.33) \]
\[ 0h_{tt} + 0h_{tx} - \frac{1}{\mu \sigma}h_{xx} + h_t + 0h_x + 0u = 0 \quad (2.1.34) \]
thus
\[ B^2 - 4AC = 0. \quad (2.1.35) \]

2.2 Navier’s equation

Consider the linear momentum conservation (for small displacements)
\[ \rho u_{tt} - \nabla \cdot (\sigma) = \rho b \]
with the isotropic Hooke’s law
\[ \sigma = \lambda \nabla \cdot (u) I + 2\mu e, \quad e = \frac{1}{2} \left( \nabla (u) + \nabla (u)^T \right) \]
where \( u \) [m] is the displacement field, \( \rho \) [kg/m\(^3\)] is the density, \( \sigma \) [N/m\(^2\)] is the stress tensor, \( b \) [N/kg] are the external forces and \( \lambda \) and \( \mu \) [N/m\(^2\)] are the Lamé parameters.

(a) Show that for constant \( \rho, \lambda \) and \( \mu \), the conservation of momentum can be written
\[ u_{tt} - \frac{\lambda + 2\mu}{\rho} \nabla \cdot (u) + \frac{\mu}{\rho} \nabla \times (\nabla \times (u)) = b. \]

(b) Making the assumptions that \( \nabla \cdot (u) = 0 \) (shear only) or that \( \nabla \times (u) = 0 \) (pressure only), show that the conservation of momentum writes
\[ u_{tt} - c^2 \Delta (u) = f. \]
Determine \( c^2 \) and \( f \) for both assumptions. Classify this equation.
Solution

(a) The divergence of the stress tensor is given by (using the summation convention)(\( \mathbf{u} = (u^x \ u^y \ u^z) \))

\[
\nabla \cdot (\lambda \nabla \cdot (\mathbf{u}) \mathbf{I} + 2\mu \mathbf{e}) = \partial_i (\lambda u^k_i \delta^{ij} + \mu (u^i_j + u^j_i)) \\
= \lambda u^k_{ij} + \mu (u^i_{ji} + u^j_{i}) \\
= \lambda u^k_{ij} + \mu (u^i_{ji} + u^j_{i}) \\
= \lambda u^k_{ij} + \mu (u^i_{ji} + u^j_{i}) \\
= (\lambda + \mu) u^k_{ij} + \mu u^i_{ji} \\
= (\lambda + \mu) \nabla (\nabla \cdot (\mathbf{u})) + \mu \Delta (\mathbf{u})
\]

(2.2.1)

(2.2.2)

(2.2.3)

(2.2.4)

(2.2.5)

(2.2.6)

then using the vector identity

\[
\nabla \times (\nabla \times (\mathbf{u})) = \nabla (\nabla \cdot (\mathbf{u})) - \nabla \cdot (\nabla \Delta (\mathbf{u}))
\]

(2.2.7)

one finds

\[
\nabla \cdot (\sigma) = (\lambda + 2\mu) \nabla (\nabla \cdot (\mathbf{u})) - \mu \nabla \times (\nabla \times (\mathbf{u}))
\]

(2.2.8)

such that the linear momentum conservation writes

\[
\mathbf{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \nabla (\nabla \cdot (\mathbf{u})) + \frac{\mu}{\rho} \nabla \times (\nabla \times (\mathbf{u})) = \mathbf{b}.
\]

(2.2.9)

(b) If \( \nabla \cdot (\mathbf{u}) = 0 \) then the momentum conservation writes (using the identity Eq.(2.2.7))

\[
\mathbf{u}_{tt} + \frac{\mu}{\rho} \nabla \times (\nabla \times (\mathbf{u})) = \mathbf{b}
\]

(2.2.10)

\[
\mathbf{u}_{tt} - \frac{\mu}{\rho} \nabla \cdot (\nabla (\mathbf{u})) = \mathbf{b}
\]

(2.2.11)

\[
\mathbf{u}_{tt} - \frac{\mu}{\rho} \Delta (\mathbf{u}) = \mathbf{b}
\]

(2.2.12)

thus \( c^2 = \frac{\mu}{\rho} \) and \( f = \mathbf{b} \).

Similarly, if \( \nabla \times (\mathbf{u}) = 0 \) then the momentum conservation writes (using the identity Eq.(2.2.7))

\[
\mathbf{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \nabla (\nabla \cdot (\mathbf{u})) = \mathbf{b}
\]

(2.2.13)

\[
\mathbf{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \nabla \cdot (\nabla (\mathbf{u})) = \mathbf{b}
\]

(2.2.14)

\[
\mathbf{u}_{tt} - \frac{\lambda + 2\mu}{\rho} \Delta (\mathbf{u}) = \mathbf{b}
\]

(2.2.15)

thus \( c^2 = \frac{\lambda + 2\mu}{\rho} \) and \( f = \mathbf{b} \).

2.3 Stokes’ equation

Consider the mass and linear momentum conservation

\[
\begin{cases}
\frac{D\rho}{Dt} + \rho \nabla \cdot (\mathbf{v}) = 0 \\
\frac{D\mathbf{v}}{Dt} + \frac{\rho}{\rho} \nabla \cdot (\sigma) = \rho \mathbf{b}
\end{cases}
\]
Exercise statements and solutions

\[ \frac{D}{Dt} \triangleq \partial_t + v \cdot \nabla (\cdot) \] (is the material derivative)

for Newtonian fluids

\[ \sigma = -pI + 2\mu e, \quad e = \frac{1}{2} \left( \nabla (v) + \nabla (v)^T \right) \]

where \( v \) [m/s] is the velocity field, \( p \) is the pressure [N/m\(^2\)], \( \rho \) [kg/m\(^3\)] is the density, \( \sigma \) [N/m\(^2\)] is the stress tensor, \( b \) [N/kg] are the external forces and \( \mu \) [Ns/m\(^2\)] is the viscosity.

(a) For incompressible fluids, i.e. \( \frac{D\rho}{Dt} = 0 \), show that these equations become

\[
\begin{align*}
\nabla \cdot (v) &= 0 \\
\rho v_t + \rho v \cdot \nabla (v) - \mu \Delta (v) + \nabla (p) &= \rho b 
\end{align*}
\]

Classify this system.

(b) The stream function \( \psi \) is such that \( v = \nabla \times (\psi) \). Determine \( v \) for \( \psi = \psi(x, y) \hat{z} \).

(c) At low Reynolds numbers, the terms \( \rho v_t \) and \( \rho v \cdot \nabla (v) \) can be neglected. In that case and if \( b = 0 \), show that the curl of the momentum conservation can then be written

\[
\Delta (\Delta (\psi)) = 0.
\]

Classify this equation.

Solution

(a) The divergence of the stress tensor is given by (using the sommation convention)(\( v = (v^x \ v^y \ v^z) \))

\[
\nabla \cdot (-pI + 2\mu e) = \partial_i \left( -p \delta^{ij} + \mu \left( v^j_i + v^i_j \right) \right) = -p_j + \mu \left( v^j_i + v^i_j \right) = -p_j + \mu \left( v^j_i + v^i_j \right) = -\nabla (p) + \mu \Delta (v) + \mu \nabla (\nabla \cdot (v)) = -\nabla (p) + \mu \Delta (v). \tag{2.3.1}
\]

The system is

- non homogeneous
- non linear
- second order in space and first order in time.

(b)

\[
v = \nabla \times (\psi) = \nabla \times (\psi(x, y) \hat{z}) = \psi_y \hat{x} - \psi_x \hat{y} \tag{2.3.6}
\]

Thus \( v \) is in the \( x - y \) plane and does not depend on \( z \).
(c) The curl of the momentum conservation is

\[ \nabla \times (\mu \Delta (v)) = \nabla \times (\nabla (p)) \]

(2.3.7)

\[ \Rightarrow \hat{z} \left( v_{yyx} + v_{xxx} - v_{xxy} - v_{yyy} \right) = 0 \]

(2.3.8)

\[ \Rightarrow - \psi_{xyyx} - \psi_{xxxx} - \psi_{yxxy} - \psi_{yyyy} = 0 \]

(2.3.9)

\[ \Rightarrow (\partial_{xx} + \partial_{yy})(\psi_{xx} + \psi_{yy}) = 0 \]

(2.3.10)

\[ \Rightarrow \Delta (\Delta (\psi)) = 0 \]

(2.3.11)

The system is

- homogeneous
- linear
- fourth order in space and zeroth order in time.

### 2.4 Heat equation

Consider the energy conservation

\[ \rho \frac{D e}{D t} - \sigma : \nabla (v) + \nabla \cdot (q) - \rho s = 0 \]

(2.4.1)

\( \frac{D}{D t} \triangleq \partial_t + v \cdot \nabla (\cdot) \) is the material derivative
where \( v \) [m/s] is the velocity field, \( \rho \) [kg/m\(^3\)] is the density, \( \sigma \) [N/m\(^2\)] is the stress tensor, \( s \) [W/kg] are the external sources, \( e \) [J/kg] is the internal energy and \( q \) [W/m\(^2\)] is the heat flux.

(a) Considering that the internal energy is proportional to the temperature, \( i.e \ e = cT \) while the heat flux is proportional to the temperature gradient, \( i.e \ q = -k\nabla (T) \) (\( c \) and \( k \) are constants) and that the velocity field and the stress tensor are known. Show that the conservation of energy becomes

\[ T_t - \alpha \Delta (T) + \beta \cdot \nabla (T) = f. \]

(2.4.2)

Determine \( \alpha, \beta \) and \( f \). Classify this equation.

**Solution**

(a) With the assumptions, the conservation of energy becomes

\[ \rho \frac{D e}{D t} - \sigma : \nabla (v) + \nabla \cdot (q) - \rho s = 0 \]

(2.4.3)

\[ \rho c T_t + \rho c v \cdot \nabla (T) - \sigma : \nabla (v) - k\nabla \cdot (\nabla (T)) - \rho s = 0 \]

(2.4.4)

\[ T_t - \frac{k}{\rho c} \Delta (T) + v \cdot \nabla (T) = \frac{\sigma : \nabla (v)}{\rho c} + \frac{s}{c} \]

(2.4.5)

thus \( \alpha = \frac{k}{\rho c}, \beta = v \) and \( f = \frac{\sigma : \nabla (v)}{\rho c} + \frac{s}{c} \).

This equation is

- non homogeneous
Indeed, it can be written (for a single spatial dimension)

\[ AT_{tt} + BT_{tx} + C T_{xx} + DT_t + ET_x + Fu = G \]  

(2.4.6)

\[ 0T_{tt} + 0 T_{tx} - \frac{k}{\rho c} T_{xx} + 1 T_t + v^2 T_x + 0 u = G \]  

(2.4.7)

thus

\[ B^2 - 4AC = 0. \]  

(2.4.8)
3 Wave equation

3.1 Spherical wave equation [Strauss 2.1, Ex. 8]

The spherical wave equation reads

\[ u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right). \]

(a) Using the substitution \( v = ru \), show that the spherical wave equation can be rewritten as \( v_{tt} = c^2 v_{rr} \).

(b) Find the general solution of the spherical wave equation.

(c) Give an expression for the solution corresponding to the initial conditions \( u(r, 0) = \phi(r) \) and \( u_t(r, 0) = \psi(r) \), for given functions \( \phi \) and \( \psi \).

Solution

(a) Using the change of variables \( v = ru \) yields

\[ v_t = ru_t, \quad v_{tt} = ru_{tt}, \quad v_r = u + ru_r, \quad \text{and} \quad v_{rr} = 2u_r + ru_{rr}. \] (3.1.1)

The equation thus becomes

\[ \frac{v_{tt}}{r} = c^2 \frac{v_{rr}}{r}, \] (3.1.2)

\[ \Rightarrow v_{tt} = c^2 v_{rr}. \] (3.1.3)

(b)

\[ v = f(r + ct) + g(r - ct), \] (3.1.4)

\[ \Rightarrow u = \frac{v}{r} = \frac{1}{r} \left[ f(r + ct) + g(r - ct) \right]. \] (3.1.5)

(c) Let us define \( \Phi(r) = r\phi(r) \) and \( \Psi(r) = r\psi(r) \). Using the general solution of this initial value problem is

\[ v(r, t) = \frac{1}{2} \left[ \Phi(r + ct) + \Phi(r - ct) \right] + \frac{1}{2c} \int_{r-ct}^{r+ct} \Psi(s) \, ds. \] (3.1.6)

Since \( v = ru \), \( \Psi(r) = r\psi(r) \) and \( \Phi(r) = r\phi(r) \), the solution for \( u(r, t) \) is given by

\[ u(r, t) = \frac{1}{2r} \left[ (r + ct)\phi(r + ct) + (r - ct)\phi(r - ct) \right] + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi(s) \, ds. \] (3.1.7)

3.2 Equation factorization [Strauss 2.1, Ex. 10]

Solve the equation \( u_{xx} + u_{xt} - 20u_{tt} = 0 \) using the initial conditions \( u(x, 0) = \phi(x) \) and \( u_t(x, 0) = \psi(x) \).
Solution

The equation can be written

\[(\partial_{xx} + \partial_{xt} - 20\partial_{tt}) u = 0.\]  \hspace{1cm} (3.2.1)

Factoring the operator yields

\[(\partial_x + 5\partial_t) (\partial_x - 4\partial_t) u = 0.\]  \hspace{1cm} (3.2.2)

Let \(v(x, t) = (\partial_x - 4\partial_t) u\), the problem can be written as a system first order equations

\[
\begin{cases}
(\partial_x + 5\partial_t) v = 0, \\
(\partial_x - 4\partial_t) u = v, \\
u(x, 0) = \phi(x), \\
u_t(x, 0) = \psi(x).
\end{cases}
\]  \hspace{1cm} (3.2.3) - (3.2.6)

From Eq.(3.2.3), it can be shown that \(v\) is constant along the characteristic lines of equations \(t - 5x = C\). Therefore

\[v(x, t) = f(t - 5x).\]  \hspace{1cm} (3.2.7)

Using Eq.(3.2.7), Eq.(3.2.4) can be written as:

\[(\partial_x - 4\partial_t) u = f(t - 5x).\]  \hspace{1cm} (3.2.8)

It can be verified by differentiation that a particular solution of Eq.(3.2.8) is given by \(w^p(x, t) = h(t - 5x)\), where \(h'(s) = -\frac{1}{9}f(s)\).

Besides this, using the same reasoning as the one used for \(v\), one can show that the general solution of the homogeneous part of Eq.(3.2.8) can be written as \(w^h(x, t) = g(t + 4x)\).

As Eq.(3.2.8) is linear, the solution can be written as the sum of a particular solution and the general solution of the homogeneous PDE i.e.

\[u(x, t) = h(t - 5x) + g(t + 4x).\]  \hspace{1cm} (3.2.9)

Then using the auxiliary conditions Eq.(3.2.5) and Eq.(3.2.6) yields

\[
\begin{cases}
h(-5x) + g(4x) = \phi(x), \\
h'(-5x) + g'(4x) = \psi(x).
\end{cases}
\]  \hspace{1cm} (3.2.10) - (3.2.11)

Deriving Eq.(3.2.10) with respect to \(x\) and using the chain rule, successively gives

\[
\begin{cases}
-5h'(-5x) + 4g'(4x) = \phi'(x), \\
h'(-5x) + g'(4x) = \psi(x).
\end{cases}
\]  \hspace{1cm} (3.2.12)

\[
\begin{cases}
9g'(4x) = \phi'(x) + 5\psi(x), \\
h'(-5x) + g'(4x) = \psi(x).
\end{cases}
\]  \hspace{1cm} (3.2.13)

\[
\begin{cases}
g'(4x) = \frac{1}{9}\phi'(x) + \frac{5}{9}\psi(x), \\
h'(-5x) = -\frac{1}{9}\phi'(x) + \frac{4}{9}\psi(x).
\end{cases}
\]  \hspace{1cm} (3.2.14)
Let solve this system one equation at a time. Integrating both sides of the first equation of Eqs. (3.2.14) successively gives

\[ \int_0^x g'(4s)ds = \int_0^x \frac{1}{9} \phi'(s)ds + \int_0^x \frac{5}{9} \psi(s)ds, \]

(3.2.15)

\[ \iff \int_0^{4x} \frac{1}{4} g'(\tilde{s})d\tilde{s} = \frac{1}{9} \int_0^x \phi'(s)ds + \frac{5}{9} \int_0^x \psi(s)ds, \]

(3.2.16)

\[ \iff g(4x) = \frac{4}{9} \phi(x) + \frac{20}{9} \int_0^x \psi(s)ds + A, \]

(3.2.17)

where the change of variable \( 4s = \tilde{s} \) was used and where \( A \) is an integration constant. Posing \( 4x = y \) in (3.2.17) yields

\[ g(y) = \frac{4}{9} \phi \left( \frac{y}{4} \right) + \frac{20}{9} \int_0^{\frac{y}{4}} \psi(s)ds + A. \]

(3.2.18)

Then solving the second equation of Eqs. (3.2.14) gives

\[ \int_0^x h'(-5s)ds = -\int_0^x \frac{1}{9} \phi'(s)ds + \int_0^x \frac{4}{9} \psi(s)ds, \]

(3.2.19)

\[ \iff \int_0^{-5x} -\frac{1}{5} h'(s)d\tilde{s} = -\frac{1}{9} \int_0^x \phi'(s)ds + \frac{4}{9} \int_0^x \psi(s)ds, \]

(3.2.20)

\[ \iff h(-5x) = \frac{5}{9} \phi(x) - \frac{20}{9} \int_0^x \psi(s)ds + B, \]

(3.2.21)

where the change of variable \( -5s = \tilde{s} \) was used and where \( B \) is an integration constant. Posing \( -5x = y \) in (3.2.21) yields

\[ h(y) = \frac{5}{9} \phi \left( \frac{-y}{5} \right) - \frac{20}{9} \int_0^{\frac{-y}{5}} \psi(s)ds + B. \]

(3.2.22)

From Eq. (3.2.9), Eq. (3.2.18) and Eq. (3.2.22) it follows

\[ u(x,t) = \frac{5}{9} \phi \left( \frac{-(t - 5x)}{5} \right) + \frac{4}{9} \phi \left( \frac{t + 4x}{4} \right) - \frac{20}{9} \int_0^{\frac{-(t - 5x)}{5}} \psi(s)ds + \frac{20}{9} \int_0^{\frac{t + 4x}{4}} \psi(s)ds + A + B. \]

(3.2.23)

From the first equation in Eqs. (3.2.10), Eq. (3.2.17) and Eq. (3.2.21), it can be shown that \( A + B = 0 \). Therefore, the solution is

\[ u(x,t) = \frac{1}{9} \left[ 5\phi \left( x - \frac{t}{5} \right) + 4\phi \left( x + \frac{t}{4} \right) + 20 \int_{x - \frac{t}{5}}^{x + \frac{t}{4}} \psi(s)ds \right]. \]

(3.2.24)

### 3.3 Equation factorization [Olver Ex. 2.4.19] 🌟

Solve the equation \( u_{tt} - 2u_{tx} - 3u_{xx} = 0 \) using the initial conditions \( u(x,0) = x^2 \) and \( u_t(x,0) = \exp(x) \).

### 3.4 Inhomogeneous wave equation [Olver 2.4, Ex. 11] 🌟

(a) Solve the initial value problem

\[ \partial_t^2 u - c^2 \partial_{xx} u = 0, \quad u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x), \quad -\infty < x < \infty, \quad t \geq 0. \]

(3.4.1)
(b) Is \( u(x, t) \) a periodic function of \( t \)?

(c) Solve the forced initial value problem

\[
\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = \cos(\omega t), \quad u(x, 0) = \sin(x), \quad \frac{\partial u}{\partial t}(x, 0) = \cos(x), \quad -\infty < x < \infty, \quad t \geq 0.
\]

(3.4.2)

(d) Does the solution exhibits resonance?

(e) What would happen if the forcing function is \( \sin(\omega t) \) instead of \( \cos(\omega t) \)?

**Solution**

(a) Starting from

\[
(\partial_t - c \partial_x)(\partial_t + c \partial_x) u = 0,
\]

(3.4.3)

define the characteristic coordinates

\[
\xi = x + ct \quad \text{and} \quad \eta = x - ct.
\]

(3.4.4)

The chain rule gives

\[
\partial_x = \partial_\xi + \partial_\eta \quad \text{and} \quad \partial_t = c \partial_\xi - c \partial_\eta
\]

(3.4.5)

such that the equation simplifies as

\[
-4c^2 (\partial_\xi)(\partial_\eta) u = 0
\]

(3.4.6)

\[
\Rightarrow \quad \partial_\xi \partial_\eta u = 0
\]

(3.4.7)

\[
\Rightarrow \quad u(x, t) = f(\xi) + g(\eta) = f(x + ct) + g(x - ct).
\]

(3.4.8)

The initial conditions yields

\[
\begin{aligned}
\left\{\begin{aligned}
\quad u(x, 0) &= f(x) + g(x) = \sin(x), \\
\quad u'(x, 0) &= cf'(x) - cg'(x) = \cos(x).
\end{aligned}\right.
\]

(3.4.9)

(3.4.10)

Differentiating the first equation yields

\[
f'(x) + g'(x) = \cos(x)
\]

(3.4.11)

Then solving for \( f'(x) \) and \( g'(x) \) gives

\[
\begin{aligned}
\quad f'(x) &= \frac{c + 1}{2c} \cos(x) \\
\quad g'(x) &= \frac{c - 1}{2c} \cos(x)
\end{aligned}
\]

(3.4.12)

(3.4.13)

Integrating Eq.(3.4.12) and Eq.(3.4.13) gives

\[
\begin{aligned}
\quad f(x) &= \frac{c + 1}{2c} \sin(x) + A \\
\quad g(x) &= \frac{c - 1}{2c} \sin(x) + B
\end{aligned}
\]

(3.4.14)

(3.4.15)
Since \( f(x) + g(x) = \sin(x) \), \( A + B = 0 \). Finally, the solution \( u(x, t) = f(x + ct) + g(x - ct) \) to the problem is given by
\[
    u(x, t) = \frac{c + 1}{2c} \sin(x + ct) + \frac{c - 1}{2c} \sin(x - ct).
\] (3.4.16)

(b) The function \( u(x, t) \) is a periodic function of \( t \) if it is possible to find \( T \) such that
\[
    u(x, t + nT) = u(x, t)
\] (3.4.17)
for any \( n \in \mathbb{N} \).
It can be checked that choosing \( T = \frac{2\pi}{c} \) this condition is satisfied so that \( u(x, t) \) is a periodic function of \( t \).

(c) The equation can be written as
\[
    (\partial_t + c\partial_x)(\partial_t - c\partial_x) u = \cos(\omega t).
\] (3.4.18)

Define the characteristic coordinates
\[
    \xi = x + ct \quad \text{and} \quad \eta = x - ct.
\] (3.4.19)
The chain rule gives
\[
    \partial_x = \partial_\xi + \partial_\eta \quad \text{and} \quad \partial_t = c\partial_\xi - c\partial_\eta,
\] (3.4.20)
such that the equation simplifies as
\[
    -4c^2 (\partial_\xi)(\partial_\eta) u = \cos\left(\frac{\omega}{2c}(\xi - \eta)\right)
\] (3.4.21)

\[
    \Rightarrow \quad \partial_\xi \partial_\eta u = -\frac{1}{4c^2} \cos\left(\frac{\omega}{2c}(\xi - \eta)\right)
\] (3.4.22)

\[
    \Rightarrow \quad u(\xi, \eta) = -\frac{1}{\omega^2} \cos\left(\frac{\omega}{2c}(\xi - \eta)\right) + f(\xi) + g(\eta)
\] (3.4.23)

\[
    \Rightarrow \quad u(x, t) = -\frac{1}{\omega^2} \cos(\omega t) + f(x + ct) + g(x - ct).
\] (3.4.24)

The initial conditions then give
\[
\begin{align*}
    u(x, 0) &= -\frac{1}{\omega^2} + f(x) + g(x) = \sin(x), \\
    u'(x, 0) &= cf'(x) - cg'(x) = \cos(x).
\end{align*}
\] (3.4.25) (3.4.26)

Differentiating the first equation yields
\[
    f'(x) + g'(x) = \cos(x).
\] (3.4.27)

Then solving for \( f'(x) \) and \( g'(x) \) yields
\[
\begin{align*}
    f'(x) &= \frac{c + 1}{2c} \cos(x) \quad \text{and} \quad g'(x) = \frac{c - 1}{2c} \cos(x).
\end{align*}
\] (3.4.28) (3.4.29)
Integrating Eq. (3.4.29) gives
\[
\begin{aligned}
f(x) &= \frac{c + 1}{2c} \sin(x) + A \\
g(x) &= \frac{c - 1}{2c} \sin(x) + B
\end{aligned}
\] (3.4.30) (3.4.31)

Since \(f(x) + g(x) = \sin(x) + \frac{1}{\omega^2}\), \(A + B = \frac{1}{\omega^2}\) and the final solution is
\[
u(x, t) = \frac{1}{\omega^2}(1 - \cos(\omega t)) + \frac{c + 1}{2c} \sin(x + ct) + \frac{c - 1}{2c} \sin(x - ct).
\] (3.4.32)

(d) The solution Eq. (3.4.32) is bounded \(\forall \omega > 0\), hence there is no resonance.

(e) In that case, the solutions is
\[
u(x, t) = \frac{1}{\omega^2}(\omega t - \sin(\omega t)) + \frac{c + 1}{2c} \sin(x + ct) + \frac{c - 1}{2c} \sin(x - ct).
\] (3.4.33)

\(u(x, t)\) grows with \(t\) which indicates resonance.
4 Conservation laws

A conservation law is an equation of the form, see [Olver, Definition 2.7],
\[ \frac{\partial c}{\partial t} + \nabla \cdot (f) = 0. \]

The function \( c \) is known as the conserved density, while \( f \) is the associated flux. Indeed, let \( \Omega \) be a subset of \( \mathbb{R}^3 \) whose boundary is denoted by \( \partial \Omega \) then
\[ \int_{\Omega} \left( \frac{\partial c}{\partial t} + \nabla \cdot (f) \right) \, d\Omega = 0 \]
\[ \Rightarrow \frac{\partial}{\partial t} \int_{\Omega} c \, d\Omega + \int_{\Omega} \nabla \cdot (f) \, d\Omega = 0 \]
\[ \Rightarrow \frac{\partial}{\partial t} \int_{\Omega} c \, d\Omega + \int_{\partial\Omega} f \cdot \hat{n} \, d\partial\Omega = 0. \]

Defining
\[ C \triangleq \int_{\Omega} c\, d\Omega \quad \text{and} \quad F \triangleq \int_{\partial\Omega} f \cdot \hat{n} \, d\partial\Omega \]
gives
\[ \frac{\partial C}{\partial t} = -F \]
such that when the flux is, on average, going outside (resp. inside) of the volume \( \Omega \) then the conserved quantity \( C \) decreases (resp. increases).

4.1 Nonlinear transport [Olver, Section 2.3]

Find the conserved density and the flux for the nonlinear transport equation
\[ u_t + uu_x = 0. \]

Solution
\[ c = u \quad \text{and} \quad f = \frac{1}{2} u^2. \quad (4.1.1) \]

4.2 Third order equation [Olver, Exercise 8.5.7]

Consider the third order equation
\[ u_t + u_{xxx} = 0. \]

(a) Find a trivial conserved quantity and its associated flux.

(b) Show that \( c = u^2 \) is a conserved quantity. Give the associated flux.

(c) For both couple, give a boundary condition such that \( C \) is constant.
Solution

\[ c = u \quad \text{and} \quad f = u_{xx} \quad \text{or} \quad c = u^2 \quad \text{and} \quad f = 2uu_{xx} - u_x^2. \tag{4.2.1} \]

A suitable boundary condition is that the flux cancels on the boundaries, \( i.e \)

\[ f = 0 \quad \text{on} \quad \partial \Omega. \tag{4.2.2} \]

Another suitable boundary condition is that the ingoing and the outgoing part of the flux cancels, \( i.e \)

\[ f = \text{Cst} \quad \text{on} \quad \partial \Omega. \tag{4.2.3} \]

Indeed for both cases, one has

\[ F = \int_{\partial \Omega} f \hat{x} \cdot \hat{n} d\partial \Omega = 0 \quad \Rightarrow \quad \frac{\partial C}{\partial t} = 0. \tag{4.2.4} \]

### 4.3 Korteweg-de Vries equation [Olver, Exercice 8.5.18]

Consider the Korteweg-de Vries equation

\[ u_t + u_{xxx} + uu_x = 0. \]

Show that \( c = u, \ c = u^2, \ c = u_x^2 + \mu u^3 \) (for a suitable constant \( \mu \)) are conserved densities.

**Solution**

Easy for \( c = u \), rather easy for \( c = u^2 \). For the third conserved quantity, \( \mu = -1/3 \) and \( c = u_x^2 - u^3/3 \) with the associated flux

\[ f = -\frac{1}{4}u^4 - u^2u_{xx} + 2uu_x^2 + 2u_xu_{xxx} - u_x^2. \tag{4.3.1} \]

### 4.4 Modified Korteweg-de Vries equation [Olver, Exercice 8.5.19]

Find two conserved densities and the associated fluxes for the modified Korteweg-de Vries equation

\[ u_t + u_{xxx} + u^2u_x = 0. \]

### 4.5 Benjamin-Bona-Mahony equation [Olver, Exercice 8.5.19]

Find two conserved densities and the associated fluxes for the Benjamin-Bona-Mahony equation

\[ u_t - u_{xxx} + uu_x = 0. \]

Note that there are actually an infinite number of conserved quantities in this equation.
4.6 Damped string [Strauss 2.2, Ex. 5]

The equation of motion for a damped string is given by

\[ u_{tt} - c^2 u_{xx} + ru_t = 0, \quad r > 0. \]

Show that, in this case, the energy \( E = K + P \) decreases.

The kinetic energy is defined here as

\[ K = \frac{1}{2} \int \mathbb{R} u_t^2 \, dx \]

while the potential is defined as

\[ P = \frac{1}{2} \int \mathbb{R} c^2 \left( u_x^2 \right) \, dx. \]

Solution

The energy decreases with time provided that

\[ \frac{dE}{dt} < 0. \] (4.6.1)

By definition, this derivative is given by

\[
\frac{dE}{dt} = \frac{dK}{dt} + \frac{dP}{dt} \]

(4.6.2)

\[
= \int \mathbb{R} u_t u_{tt} \, dx + \frac{1}{2} \frac{d}{dt} \int \mathbb{R} c^2 u_x^2 \, dx \]

(4.6.3)

\[
= \int \mathbb{R} u_t c^2 u_{xx} \, dx - \int \mathbb{R} ru_t u_t \, dx + \frac{1}{2} \frac{d}{dt} \int \mathbb{R} c^2 u_x^2 \, dx \]

(4.6.4)

\[
= -\frac{1}{2} \frac{d}{dt} \int \mathbb{R} c^2 u_x^2 \, dx + \left[ u_x u_t \right]_{-\infty}^{\infty} - \int \mathbb{R} ru_t^2 \, dx + \frac{1}{2} \frac{d}{dt} \int \mathbb{R} c^2 u_x^2 \, dx \]

(4.6.5)

\[
= -\int \mathbb{R} ru_t^2 \, dx < 0. \] (4.6.6)

4.7 Damped wave equation

Consider the damped wave equation

\[ u_{tt} - c^2 u_{xx} + ru_t = 0, \quad r \in \mathbb{R}. \]

(a) For \( r = 0 \), show that \( e \triangleq \frac{1}{2} \left( u_t^2 + c^2 u_x^2 \right) \) and \( p \triangleq u_t u_x \) are conserved densities.

(b) For \( r \neq 0 \), determine \( d \) such that \( e + d \) is a conserved quantity. Give an interpretation of \( d \) and \( r \).
Solution

\[ d = \int_{t_0}^{t} ru_t^2(x, \tau) d\tau. \]  

(4.7.1)

Indeed

\[
\frac{\partial}{\partial t} (e + d) = \frac{1}{2} \frac{\partial}{\partial t} \left( u_t^2 + c^2 u_x^2 + 2 \int_{t_0}^{t} ru_t^2(x, \tau) d\tau \right) 
\]

(4.7.2)

\[ = u_t u_{tt} + c^2 u_x u_{xt} + r u_t^2 \]  

(4.7.3)

\[ = u_t \left( u_{tt} - c^2 u_{xx} + r u_t \right) + c^2 u_t u_{xx} + c^2 u_x u_{xt} \]  

(4.7.4)

\[ = c^2 u_t u_{xx} + c^2 u_x u_{xt} \]  

(4.7.5)

\[ = \frac{\partial}{\partial x} \left( c^2 u_x u_t \right) \]  

(4.7.6)

and thus

\[ f = -c^2 u_x u_t. \]  

(4.7.7)

4.8 Diffusion equation

Consider the diffusion equation

\[ u_t - \alpha u_{xx} = 0 \quad (x,t) \in [0,1] \times \mathbb{R}^+ \]

(a) Find a trivial conserved density and the associated flux.

(b) Give a physical interpretation of the conservation law when either \( u \) is a temperature field ([K]) or \( u \) a concentration field ([kg/m]).

(c) Consider now the boundary conditions

\[ u_x(0) = -a \quad u_x(1) = b. \]

Give the physical interpretation of \( a \) and \( b \).

4.9 Schrodinger equation

Consider the homogeneous Schrodinger equation for one particle in one-dimension

\[ u_t - iu_{xx} = 0. \]

(a) Show that \( c = u^* u \) is a conserved quantity ( \( ^* \) denotes the complex conjugate). Give the associated flux.

(b) Give a physical interpretation of the conservation law when \( u \) is a quantum wave function ([m^{1/2}]).
Solution

The associated flux is

\[ f = i(u^* u_x - uu_x^*). \quad (4.9.1) \]

Indeed

\[ \partial_t (u^* u) = u_t^* u + u^* u_t \]
\[ = u(-iu_{xx}^*) + u^*(iu_{xx}) \quad (4.9.3) \]
\[ = \partial_x (i(u^* u_x - uu_x^*)). \quad (4.9.4) \]
5 Von Neumann stability analysis

Consider a linear partial differential equation whose solution is approximated with a time-stepping finite difference scheme. The discretization in space and time are on regular grids \( x_j \) with index \( j \), and \( t_n \) with index \( n \), respectively. For example, for the transport equation (see Ex. 5.1), forward in time and right-centered in space,

\[
u(x_j, t_{n+1}) - (1 + \alpha) u(x_j, t_n) + \alpha u(x_{j+1}, t_n) = f(x_j, t_n).
\]

**Sufficient stability condition** Consider that at time step 0, an error \( \epsilon = \epsilon(x, t_0) \in L^1, L^2 \) appears in the numerical solution. Because the problem is linear, the time evolution of this error is the same as the one of the solution for the homogeneous problem. Indeed, for example with the transport equation above,

\[
(u(x_j, t_{n+1}) + \epsilon(x_j, t_{n+1})) - (1 + \alpha) (u(x_j, t_n) + \epsilon(x_j, t_n)) + \alpha (u(x_{j+1}, t_n) + \epsilon(x_{j+1}, t_n)) = f(x_j, t_n),
\]

\[
\Rightarrow \epsilon(x_j, t_{n+1}) - (1 + \alpha) \epsilon(x_j, t_n) + \alpha \epsilon(x_{j+1}, t_n) = 0. \tag{\diamond}
\]

A finite difference scheme is said to be stable if the \( L^2 \)-norm in space \( \| \cdot \| \) of the error does not diverge with time, i.e. if

\[
\exists C \in \mathbb{R} : \lim_{n \to \infty} \| \epsilon(x, t_n) \|^2 \leq C \| \epsilon(x, t_0) \|^2.
\]

Using the spatial Fourier decomposition of the error (it exists because the error is integrable)

\[
\epsilon(x, t_n) = \int \hat{\epsilon}(k, t_n) \exp(-ikx) \, dk,
\]

a sufficient condition for stability is that each mode does not diverge, or that

\[
\exists C \in \mathbb{R} \left| \forall k, \lim_{n \to \infty} |\hat{\epsilon}(k, t_n)|^2 \leq C |\hat{\epsilon}(k, t_0)|^2. \right.
\]

Indeed, we have successively,

\[
\lim_{n \to \infty} \| \epsilon(x, t_n) \|^2 = \lim_{n \to \infty} \int |\hat{\epsilon}(k, t_n)|^2 \, dk \leq C \int |\hat{\epsilon}(k, t_0)|^2 \, dk = C \| \epsilon(x, t_0) \|^2,
\]

using the Parseval identity (it applies because the error is square-integrable) twice, and the permutation of the limit and the integral (we assume uniform convergence with \( n \) of the \( \hat{\epsilon}(k, t_n) \) function suite).

**Stability analysis procedure** By linearity of the equation, each mode \( \epsilon_k(x_j, t_n) \triangleq \hat{\epsilon}(k, t_n) \exp(-ikx_j) \) can be studied individually. Injecting \( \hat{\epsilon}_k(x_j, t_n) \) into the update equation (for example Eq.(\diamond)) and simplifying the exponentials gives rise to a linear recurrence equation for \( \hat{\epsilon}(k, t_n) \), with coefficients depending on \( k \).

**Example (reminder)** Below, we use the notation \( \hat{\epsilon}_n = \hat{\epsilon}(k, t_n) \), implicitly considering the dependence in \( k \). The following degree 2 linear recurrence equation

\[
a\hat{\epsilon}_{n+2} + b\hat{\epsilon}_{n+1} + c\hat{\epsilon}_n = 0,
\]
with \(a, b,\) and \(c\) constant coefficients with respect to \(n\) (but that may depend on \(k\)) admits a solution of the form

\[
\hat{e}_n = A\xi_1^n + B\xi_2^n, \quad \text{or} \quad \hat{e}_n = (A + Bn)\xi^n
\]

if its characteristic polynomial \(a\xi^2 + b\xi + c\) has distinct roots \(\xi_1\) and \(\xi_2\), or degenerate root \(\xi\), respectively, and with \(A\) and \(B\) two constants that depend on the initial errors. The roots are referred to as amplification factors. The solution is stable if no amplification factor is larger than one, whatever \(k\). In the example, the condition is, \(\forall k,\)

\[|\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1, \quad \text{or} \quad |\xi| \leq 1.\]

### 5.1 Advection equation

Consider the advection equation

\[u_t + au_x = 0.\]

(a) Using forward finite differences in time and space, \(i.e.

\[u_x \approx \frac{u^n_{j+1} - u^n_j}{\Delta x} \quad \text{and} \quad u_t \approx \frac{u^{n+1}_j - u^n_j}{\Delta t},\]

show that the scheme is explicit by giving the update equation

\[u^{n+1}_j = f(u^n_{j+1}, u^n_j).\]

(b) Establish a stability criterion for this scheme using Von Neumann analysis.

(c) Why does the stability criterion depend on the sign of \(a\)? What happens if a downward space difference, \(\text{i.e.}

\[u_x \approx \frac{u^n_j - u^n_{j-1}}{\Delta x},\]

is used instead?

Hint: Remember that the general solution of the transport equation is \(f(x - at).\)

(d) Consider now the so-called Lax-Friedrichs scheme, whose update equation is

\[u^{n+1}_j = \frac{1}{2} (u^n_{j+1} + u^n_{j-1}) - \frac{a\Delta t}{2\Delta x} (u^n_{j+1} - u^n_{j-1}).\]

Establish a stability criterion using Von Neumann analysis.

(e) Finally, consider the so-called Lax-Wendroff scheme, whose update equation is

\[u^{n+1}_j = u^n_j - \frac{a\Delta t}{2\Delta x} (u^n_{j+1} - u^n_{j-1}) + \frac{1}{2} \left(\frac{a\Delta t}{\Delta x}\right)^2 (u^n_{j+1} - 2u^n_j + u^n_{j-1}).\]

Establish a stability criterion using Von Neumann analysis.
Solution

(a) Using finite differences approximation, the transport equation writes as

\[ u_j^{n+1} = (1 + \alpha) u_j^n - \alpha u_{j+1}^n \]  

(5.1.1)

where \( \alpha = \frac{a \Delta t}{\Delta x} \).

(b) Introducing any error mode

\[ \epsilon(x, t) = \hat{\epsilon}(k, t) \exp(-i k x) \]  

(5.1.2)

in the discrete equation gives

\[ \hat{\epsilon}(k, t_{n+1}) = (1 + \alpha) \hat{\epsilon}(k, t_n) - \alpha \hat{\epsilon}(k, t_n) \exp(-i k \Delta x). \]  

(5.1.3)

Hence the amplification factor is given by

\[ \xi_1 = 1 + \alpha (1 - \exp(-i k \Delta x)) \]  

(5.1.4)

such that the condition on \( \alpha \) is

\[ |\xi_1|^2 \leq 1 \]  

(5.1.5)

\[ \Rightarrow |1 + \alpha (1 - \exp(-i k \Delta x))|^2 \leq 1 \]  

(5.1.6)

\[ \Rightarrow 1 + 2\alpha (\alpha + 1) (1 - \cos(k \Delta x)) \leq 1 \]  

(5.1.7)

\[ \Rightarrow 2\alpha (\alpha + 1) (1 - \cos(k \Delta x)) \leq 0 \]  

(5.1.8)

\[ \Rightarrow 2\alpha (\alpha + 1) \leq 0 \]  

(5.1.9)

\[ \Rightarrow -1 \leq \alpha \leq 0. \]  

(5.1.10)

(c) This is because the numerical domain of dependence must not be disjoint from the real domain of dependance.

(d) Introducing any error mode

\[ \epsilon(x, t) = \hat{\epsilon}(k, t) \exp(-i k x) \]  

(5.1.11)

in the discrete equation gives

\[ \frac{\hat{\epsilon}(k, t_{n+1})}{\hat{\epsilon}(k, t_n)} = \xi_1 = \frac{1}{2} [\exp(-i k \Delta x) + \exp(+i k \Delta x)] - \frac{\alpha}{2} [\exp(-i k \Delta x) - \exp(+i k \Delta x)] \]  

(5.1.12)

\[ = \cos(k \Delta x) + i \alpha \sin(k \Delta x) \]  

(5.1.13)

thus the norm squared is given by

\[ |\xi_1|^2 = \cos^2(k \Delta x) + \alpha^2 \sin^2(k \Delta x) \]  

(5.1.14)

\[ = 1 + [\alpha^2 - 1] \sin^2(k \Delta x). \]  

(5.1.15)

Hence, the Lax-Friedrichs scheme is stable if

\[ -1 \leq \frac{a \Delta t}{\Delta x} \leq 1. \]  

(5.1.16)
(e) Introducing any error mode

\[ \epsilon(x, t) = \hat{\epsilon}(k, t) \exp(-ikx) \]  

in the discrete equation gives

\[ \frac{\hat{\epsilon}(k, t_{n+1})}{\hat{\epsilon}(k, t_n)} = \xi_1 = 1 - \frac{\alpha}{2} (\exp(ik\Delta x) - \exp(-ik\Delta x)) + \frac{\alpha^2}{2} (\exp(ik\Delta x) - 2 + \exp(-ik\Delta x)) \]

\[ = 1 - i\alpha \sin(k\Delta x) + \alpha^2 (\cos(k\Delta x) - 1). \]  

thus the norm squared is given by

\[ |\xi_1|^2 = 1 + \alpha^4 (\cos(k\Delta x) - 1)^2 + 2\alpha^2 (\cos(k\Delta x) - 1) + \alpha^2 \sin(k\Delta x)^2 \]

\[ = 1 + \alpha^4 (\cos(k\Delta x) - 1)^2 + \alpha^2 (2\cos(k\Delta x) - 2) + \alpha^2 (1 - \cos(k\Delta x)^2) \]

\[ = 1 + \alpha^4 (\cos(k\Delta x) - 1)^2 - \alpha^2 (\cos(k\Delta x)^2 - 2\cos(k\Delta x) + 1) \]

\[ = 1 + \alpha^4 (\cos(k\Delta x) - 1)^2 - \alpha^2 (\cos(k\Delta x) - 1)^2 \]

\[ = 1 + \alpha^2 (\alpha^2 - 1) (\cos(k\Delta x) - 1)^2. \]

Hence, the Lax-Wendroff scheme is stable if

\[ -1 \leq \frac{a\Delta t}{\Delta x} \leq 1. \]  

5.2 Diffusion equation in 2D

Consider the 2D diffusion equation

\[ u_t - a(u_{xx} + u_{yy}) = 0. \]

Using forward differences in time and central differences in space, establish a stability criterion using the Von Neumann stability analysis.

Solution

First, discretize the 2D diffusion equation with spatial step size \( \Delta x \), \( \Delta y \) and temporal step size \( \Delta t \).

- Forward in time:

\[ u_t^n = \frac{u^{n+1}_{r,s} - u^n_{r,s}}{\Delta t}. \]  

- Centered in space:

\[ u_{xx}^n = \frac{u^{n+1}_{r+1,s} - 2u^n_{r,s} + u^n_{r-1,s}}{\Delta x^2}, \]

\[ u_{yy}^n = \frac{u^n_{r,s+1} - 2u^n_{r,s} + u^n_{r,s-1}}{\Delta y^2}. \]
The 2D diffusion equation can thus be expressed as

\[
\frac{u_{r,s}^{n+1} - u_{r,s}^{n}}{\Delta t} - a \left( \frac{u_{r+1,s}^{n} - 2u_{r,s}^{n} + u_{r-1,s}^{n}}{\Delta x^2} + \frac{u_{r,s+1}^{n} - 2u_{r,s}^{n} + u_{r,s-1}^{n}}{\Delta y^2} \right) = 0
\]  

(5.2.4)

or as

\[
u_{r,s}^{n+1} = u_{r,s}^{n} + \frac{a\Delta t}{\Delta x^2} (u_{r+1,s}^{n} - 2u_{r,s}^{n} + u_{r-1,s}^{n}) + \frac{a\Delta t}{\Delta y^2} (u_{r,s+1}^{n} - 2u_{r,s}^{n} + u_{r,s-1}^{n}).
\]  

(5.2.5)

Introducing

\[
\beta_x = \frac{a\Delta t}{\Delta x^2} \quad \text{and} \quad \beta_y = \frac{a\Delta t}{\Delta y^2},
\]  

(5.2.6)

the update equation is then

\[
u_{r,s}^{n+1} = u_{r,s}^{n} + \beta_x \left( u_{r+1,s}^{n} - 2u_{r,s}^{n} + u_{r-1,s}^{n} \right) + \beta_y \left( u_{r,s+1}^{n} - 2u_{r,s}^{n} + u_{r,s-1}^{n} \right).
\]  

(5.2.7)

The stability criterion is obtained by injecting an arbitrary error mode

\[
\epsilon_{r,s}^n = \epsilon(x_r, y_s, t_n) = \epsilon(k_x, k_y, t_n) \exp(-i(k_x r \Delta x + k_y s \Delta y))
\]  

(5.2.8)

into the discretized equation. For conciseness, \( \epsilon(k_x, k_y, t_n) \) is written as \( \epsilon_n \).

The evolution of the amplitude of any mode is then governed by

\[
\frac{\epsilon_{n+1}}{\epsilon_n} \exp(-i(k_x r \Delta x + k_y s \Delta y)) = \epsilon_n \exp(-i(k_x r \Delta x + k_y s \Delta y)) \\
+ \beta_x \left[ \epsilon_n \exp(-i(k_x (r + 1) \Delta x + k_y s \Delta y)) - 2\epsilon_n \exp(-i(k_x r \Delta x + k_y s \Delta y)) \\
+ \epsilon_n \exp(-i(k_x (r - 1) \Delta x + k_y s \Delta y)) \right] \\
+ \beta_y \left[ \epsilon_n \exp(-i(k_x r \Delta x + k_y (s + 1) \Delta y)) - 2\epsilon_n \exp(-i(k_x r \Delta x + k_y s \Delta y)) \\
+ \epsilon_n \exp(-i(k_x r \Delta x + k_y (s - 1) \Delta y)) \right].
\]  

(5.2.9)

Dividing both sides by \( \epsilon_n \exp(-i(k_x r \Delta x + k_y s \Delta y)) \) yields

\[
\frac{\epsilon_{n+1}}{\epsilon_n} = \xi_1 = 1 + \beta_x \left( e^{-ik_x \Delta x} - 2 + e^{ik_x \Delta x} \right) + \beta_y \left( e^{-ik_y \Delta y} - 2 + e^{ik_y \Delta y} \right)
\]  

(5.2.10)

\[
= 1 + 2\beta_x (\cos k_x \Delta x - 1) + 2\beta_y (\cos k_y \Delta y - 1)
\]  

(5.2.11)

\[
= 1 - 4\beta_x \sin^2 \frac{k_x \Delta x}{2} - 4\beta_y \sin^2 \frac{k_y \Delta y}{2}.
\]  

(5.2.12)

Ensuring that no mode is divergent, \( i.e. \)

\[
|\xi_1|^2 \leq 1 \quad \forall k_x, k_y
\]  

(5.2.13)

gives

\[
-1 \leq 1 - 4\beta_x \sin^2 \frac{k_x \Delta x}{2} - 4\beta_y \sin^2 \frac{k_y \Delta y}{2} \leq 1
\]  

(5.2.14)

\[
\frac{1}{2} \geq \beta_x \sin^2 \frac{k_x \Delta x}{2} + \beta_y \sin^2 \frac{k_y \Delta y}{2} \geq 0.
\]  

(5.2.15)
These two inequalities must be verified for any $k_x, k_y$. The most restrictive cases for the rightmost inequality are obtained by considering $\left\{ \sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2} \right\} = \{1, 0\}$ and $\left\{ \sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2} \right\} = \{0, 1\}$ which yields

$$\beta_x \geq 0 \quad \text{and} \quad \beta_y \geq 0.$$  \hfill (5.2.16)

Then $\beta_x$ and $\beta_y$ being positive, the most restrictive case for the leftmost inequality is obtained considering $\left\{ \sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2} \right\} = \{1, 1\}$ which yields

$$\frac{1}{2} \geq \beta_x + \beta_y = a \Delta t \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \triangleq \frac{a \Delta t}{\Delta l^2},$$  \hfill (5.2.17)

or

$$\frac{\Delta t}{\Delta l^2} \leq \frac{1}{2a}.$$  \hfill (5.2.18)

5.3 Wave equation [Lecture 5, Slide 15]

Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx}.$$  

Using the centered differences for both time and space, verify the stability of the numerical method using a Von Neumann analysis.

Solution

The discrete form of the wave equation can be written as

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2},$$  \hfill (5.3.1)

forming the following update equation

$$u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + 2(1-s)u_j^n - u_j^{n-1},$$  \hfill (5.3.2)

with $s = c^2(\Delta t)^2/(\Delta x)^2$.

Introducing any error mode

$$\epsilon(x, t) = \hat{\epsilon}(k, t) \exp(ikx),$$  \hfill (5.3.3)

$$= \hat{\epsilon}_n \exp(ik\Delta x)^j,$$  \hfill (5.3.4)

in the update equation gives

$$\hat{\epsilon}_{n+1} = s[\exp(ik\Delta x) + \exp(-ik\Delta x)] \hat{\epsilon}_n + 2(1-s) \hat{\epsilon}_n - \hat{\epsilon}_{n-1},$$  \hfill (5.3.5)

or

$$2s \cos(k\Delta x) \hat{\epsilon}_n + 2(1-s) \hat{\epsilon}_n - \hat{\epsilon}_{n-1}.$$  \hfill (5.3.6)

If $p = s[\cos(k\Delta x) - 1]$, the recurrence relation can be rewritten equivalently as

$$\hat{\epsilon}_{n+2} - 2(1+p)\hat{\epsilon}_{n+1} + \hat{\epsilon}_n = 0.$$  \hfill (5.3.7)
The characteristic polynomial of this expression

\[ \xi^2 - 2(1 + p) \xi + 1 = 0, \quad (5.3.8) \]

which has two roots

\[ \xi_1 = 1 + p \pm \sqrt{p^2 + 2p}. \quad (5.3.9) \]

Therefore, the solution of the recurrence relation in Eq. (5.3.7) has the form

\[ \hat{\epsilon}_n = A\xi_1^n + B\xi_2^n, \quad \Delta \ n \text{ is an exponent} \quad (5.3.10) \]

where \( A \) and \( B \) are both constants that depend on the initial error. Hence, the error does not diverge if modulus of the amplification factors verify simultaneously

\[ |\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \quad (5.3.11) \]

Two different cases are taken into account according to the sign of the value inside the square root.

**Reminder:** The modulus of a complex number \(|a + ib| = \sqrt{a^2 + b^2}|.

**Case 1:** \( p^2 + 2p > 0 \quad \Rightarrow p < -2 \)

There are two real roots, therefore both solutions must verify the relation

\[ -1 \leq 1 + p \pm \sqrt{p^2 + 2p} \leq 1, \quad (5.3.12) \]
\[ -2 - p \leq \pm \sqrt{p^2 + 2p} \leq -p. \quad (5.3.13) \]

It can be deduced that the solution \( \xi_2 \) cannot fulfil the condition

\[ -\sqrt{p^2 + 2p} \geq -2 - p. \quad (5.3.14) \]

The left part of this inequality is always negative while the right part can only be positive as \( p < -2 \) in this case. Hence, \(|\xi_2| > 1\) and is unstable. If only one of the modes diverges (in this case \( \xi_2 \)), the whole solution is not stable.

**Case 2:** \( p^2 + 2p \leq 0 \quad \Rightarrow p \geq -2 \)

In both roots, the first term is real while the second one is imaginary and their moduli are equal to

\[ |\xi_1| = (1 + p)^2 + (-p^2 - 2p) \]
\[ = 1. \quad (5.3.15) \]
\[ = 1. \quad (5.3.16) \]

Then, the solution is always stable when \( p \geq -2 \) or equivalently when

\[ s[\cos(k\Delta x) - 1] \geq -2, \quad (5.3.17) \]
\[ \frac{2}{1 - \cos(k\Delta x)} \geq s. \quad (5.3.18) \]

In the most restrictive case, the smallest value of the left part of the previous inequation is obtained when \( \cos(k\Delta x) = -1 \).

Thus, the method is stable when

\[ c^2 \frac{\Delta t^2}{\Delta x^2} \leq 1. \quad (5.3.19) \]
5.4 Schrodinger equation

Consider the 1D and time-dependent Schrödinger equation of a free particle in quantum mechanics

\[ u_t = \frac{i\hbar}{2m} u_{xx}. \]

The equation is similar to a diffusion equation with an imaginary coefficient. Show that this difference induces significant changes in the properties and solutions.

(a) Using the forward difference in time and centered in space, i.e.

\[ u_{xx} \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} \quad \text{and} \quad u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}, \]

show that the scheme is explicit by giving the update equation

\[ u_j^{n+1} = f(u_{j+1}^n, u_j^n, u_{j-1}^n). \] (5.4.1)

(b) Prove using Von Neumann analysis that this method is unstable.

(c) Consider now a new method using a centered difference in time and in space, whose update equation is

\[ u_j^{n+1} = u_j^{n-1} + \frac{i\hbar}{m} \frac{\Delta t}{(\Delta x)^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n). \] (†) (5.4.2)

Establish a stability criterion using Von Neumann analysis.

Solution

(a) The update equation is

\[ u_j^{n+1} = u_j^n + \frac{i\hbar}{2m} \frac{\Delta t}{\Delta x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n). \] (5.4.3)

(b) Introducing any error mode

\[ \epsilon(x, t) = \hat{\epsilon}(k, t) \exp(ikx), \]

\[ = \hat{\epsilon}_n [\exp(ik\Delta x)]^j, \] (5.4.4)

and replace this expression in Eq.(5.4.1).

We obtain the recurrence relation

\[ \hat{\epsilon}_{n+1} = \left[ 1 + \frac{i\hbar}{2m} \frac{\Delta t}{\Delta x^2} (\exp(ik\Delta x) - 2 + \exp(-ik\Delta x)) \right] \hat{\epsilon}_n(k). \] (5.4.5)

The solution of this equation is

\[ \hat{\epsilon}_n = A \xi^n \] (Δ n is an exponent)

with

\[ \xi_1 = 1 + \frac{i\hbar}{2m} \frac{\Delta t}{\Delta x^2} (\exp(ik\Delta x) - 2 + \exp(-ik\Delta x)), \] (5.4.6)

\[ = 1 + \frac{i\hbar}{2m} \frac{\Delta t}{\Delta x^2} (2 \cos(k\Delta x) - 2), \] (5.4.7)

\[ = 1 + i \frac{2\hbar}{m} \frac{\Delta t}{\Delta x^2} \sin^2 \left( \frac{k\Delta x}{2} \right). \] (5.4.8)
The Von Neumann stability is obtained when all modes are not divergent, *i.e.*
\[ |\xi_1| \leq 1. \tag{5.4.9} \]

The modulus of \( \xi_1 \) is given by
\[ |\xi_1| = \left[ 1 + \frac{4\hbar^2}{m^2} \Delta t^2 \sin^4 \left( \frac{k\Delta x}{2} \right) \right]^{1/2}. \tag{5.4.10} \]

Hence, using this expression in Eq.(5.4.9), the following stability condition is obtained
\[ \sin^4 \left( \frac{k\Delta x}{2} \right) \leq 0. \tag{5.4.11} \]

As the right part of the equation is always positive, the method is unstable.

(c) When the error mode in Eq.(5.4.3) is replaced in the discrete method in Eq.(†), it gives
\[ \hat{\epsilon}_{n+1} - \hat{\epsilon}_{n-1} = \frac{i\hbar}{2m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \hat{\epsilon}_n. \tag{5.4.12} \]

This equation can be rewritten equivalently as
\[ \hat{\epsilon}_{n+2} - \hat{\epsilon}_n = \frac{i\hbar}{2m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \hat{\epsilon}_{n+1}, \tag{5.4.13} \]
\[ \Rightarrow \hat{\epsilon}_{n+2} - \hat{\epsilon}_n = \frac{i\hbar}{m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \hat{\epsilon}_{n+1}, \tag{5.4.14} \]
\[ \Rightarrow \hat{\epsilon}_{n+2} + \left[ \frac{i\hbar}{m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \right] \hat{\epsilon}_{n+1} - \hat{\epsilon}_n = 0. \tag{5.4.15} \]

The characteristic polynomial of this recurrence relation
\[ \xi^2 + \left[ \frac{i\hbar}{m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \right] \xi - 1 = 0, \tag{5.4.16} \]

admits two roots
\[ \xi_1 = -\frac{2\hbar}{m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \pm \sqrt{1 - \frac{4\hbar^2}{m^2} \Delta t^2 \sin^4 \left( \frac{k\Delta x}{2} \right)}, \tag{5.4.17} \]
\[ \xi_2 = -\frac{2\hbar}{m} \Delta t^2 \sin^2 \left( \frac{k\Delta x}{2} \right) \pm \sqrt{1 - \frac{4\hbar^2}{m^2} \Delta t^2 \sin^4 \left( \frac{k\Delta x}{2} \right)}, \tag{5.4.18} \]

with \( \rho = 1 - \frac{4\hbar^2}{m^2} \Delta t^2 \sin^4 \left( \frac{k\Delta x}{2} \right) \).

Therefore, the solution of the recurrence relation in Eq.(5.4.15) has the following form
\[ \hat{\epsilon}_n = A\xi_1^n + B\xi_2^n, \tag{5.4.19} \]
where \( A \) and \( B \) are both constants that depend on the initial error.

Hence, the error does not diverge if modulus of the amplification factors verify simultaneously
\[ |\xi_1| \leq 1 \quad \text{and} \quad |\xi_2| \leq 1. \tag{5.4.20} \]

Reminder: The modulus of a complex number \( |a + ib| = \sqrt{a^2 + b^2} \).

That’s why two different situations will be considered as the modulus of \( \xi_1 \) will depend on the sign of \( \rho \).
Case 1: $\rho < 0$

If $\rho$ is negative, both terms of $\xi_2$ are imaginary and its modulus is equal to
\[
|\xi_2| = \left| -\frac{2\hbar}{m} \frac{\Delta t}{\Delta x^2} \sin^2 \left( \frac{k\Delta x}{2} \right) \pm \sqrt{-\rho} \right| \quad (5.4.21)
\]
\[
= \left| -\sqrt{1 - \rho} \pm \sqrt{-\rho} \right|. \quad (5.4.22)
\]

The roots $\xi_2$ must verify the following condition to be stable
\[
-1 \leq -\sqrt{1 - \rho} \pm \sqrt{-\rho} \leq 1. \quad (5.4.23)
\]
But, as $\rho < 0$, $\xi_2$ verifies
\[
\xi_2 < -1. \quad (5.4.24)
\]
The modes for which $\rho < 0$ are thus divergent.

Case 2: $\rho \geq 0$

If $\rho$ is positive, the first term of $\xi_2$ is imaginary while the second one is real. Then, the solutions $\xi_1$ and $\xi_2$ have the same modulus which is equal to
\[
|\xi_2| = \sqrt{\frac{4\hbar^2}{m^2} \frac{\Delta t^2}{(\Delta x)^4} \sin^4 \left( \frac{k\Delta x}{2} \right) + \left( 1 - \frac{4\hbar^2}{m^2} \frac{\Delta t^2}{(\Delta x)^4} \sin^4 \left( \frac{k\Delta x}{2} \right) \right)}, \quad (5.4.25)
\]
\[
= 1. \quad (5.4.26)
\]
The conditions on $\xi_1$ and $\xi_2$ in Eq.(5.4.20) are verified and all the modes for which $\rho \geq 0$ are not divergent.

Conclusion  To obtain $\rho \geq 0$, one has to solve
\[
\rho \geq 0 \quad (5.4.27)
\]
\[
\Rightarrow 1 - \frac{4\hbar^2}{m^2} \frac{\Delta t^2}{(\Delta x)^4} \sin^4 \left( \frac{k\Delta x}{2} \right) \geq 0, \quad (5.4.28)
\]
\[
\Rightarrow \frac{2\hbar}{m} \frac{\Delta t}{(\Delta x)^2} \sin^2 \left( \frac{k\Delta x}{2} \right) \leq 1. \quad (5.4.29)
\]
The worst case is obtained when $k\Delta x = \pm \pi/2$. The method is thus stable only when the following relation is respected
\[
\frac{\Delta t}{(\Delta x)^2} \leq \frac{m}{2\hbar}. \quad (5.4.30)
\]
6 Green’s functions

A given Green’s function is associated to a given partial differential problem (it can be an inhomogeneous problem, an initial value problem, a boundary value problem, a combination of those, . . .). Below, we develop the situation for an inhomogeneous problem.

Consider a linear partial differential operator \( L_x \) (e.g. \( \Delta_x \), \( \partial_t - k\Delta_x \) or \( \partial_{tt} - c^2 \Delta_x \)). A Green’s function \( g(x, s) \) for \( L_x \) can be defined to be the generalized (i.e. it can be a distribution) solution of

\[
L_x(g(x, s)) = \delta(x - s)
\]

with suitable linear homogeneous boundary and/or initial conditions.

The motivation for defining such a function is widespread but one of the most useful properties is that the solution of

\[
L_x(u(x)) = f(x)
\]

(with the same linear homogeneous boundary and/or initial conditions) is

\[
u(x) = \int g(x, s)f(s)\,ds.
\]

Indeed multiplying Eq. (\( \diamond \)) by \( f(s) \) and integrating over \( s \) gives

\[
\int f(s)L_x(g(x, s))\,ds = \int f(s)\delta(x - s)\,ds
\]

and then because \( L_x \) is a linear operator

\[
\int L_x(f(s)g(x, s))\,ds = \int f(s)\delta(x - s)\,ds
\]

\[
L_x\left(\int f(s)g(x, s)\,ds\right) = f(x).
\]

This property is sometimes called the superposition principle.

As mentioned above, a Green’s function can also be defined when the impulsion (i.e when the Dirac distribution \( \delta \)) appears in the boundary condition or in the initial condition. Similar superposition principles follow. Examples of Green’s functions with the associated problems are given below. (Note that in the considered problems \( g(x, s) = S(x - s) \), but this is not always the case).

**Diffusion** The Green’s function of the diffusion equation, on the whole space of dimension \( n \), is the solution to the following initial-value problem

\[
\begin{cases}
S_t - k\Delta S = 0 & \text{for } (x, t) \in \mathbb{R}^n \times ]0, +\infty[,
S(x, 0) = \delta(x) & \text{for } x \in \mathbb{R}^n.
\end{cases}
\]

It can be easily obtained via a Fourier transform of the above equation, and writes

\[
S(x, t) = \frac{1}{(4\pi kt)^{n/2}} \exp\left(-\frac{|x|^2}{4kt}\right), \quad (x, t) \in \mathbb{R}^n \times ]0, +\infty[.
\]

NB: This Green’s function is also the solution to the inhomogeneous problem, with a source \( \delta(x)\delta(t) \) and a zero initial condition. This is Duhamel’s principle.
Wave  The Green’s function of the wave equation, on the whole space of dimension $n$, is the solution to the following initial-value problem

$$
\begin{cases}
  S_{tt} - c^2 \Delta S = 0 & \text{for } (x, t) \in \mathbb{R}^n \times ]0, +\infty[, \\
  S(x, 0) = 0 & \text{for } x \in \mathbb{R}^n, \\
  S_t(x, 0) = \delta(x) & \text{for } x \in \mathbb{R}^n.
\end{cases}
$$

The solution can again be obtained via a Fourier transform of the above equation. In this case, the form of the solution depends on the dimension $n$. Solutions write, for $(x, t) \in \mathbb{R}^n \times ]0, +\infty[_{2}$,

$$
\begin{align*}
  n = 1 & : S(x, t) = \frac{1}{2c} H(ct - |x|), \\
  n = 2 & : S(x, t) = \frac{1}{2\pi c} H(ct - \|x\|) (c^2 t^2 - \|x\|^2)^{-1/2}, \\
  n = 3 & : S(x, t) = \frac{1}{4\pi c} \delta(ct - \|x\|).
\end{align*}
$$

Laplace  The Green’s function for the Laplace equation, on the whole space of dimension $n$, is the solution to the following inhomogeneous problem

$$
\Delta S = \delta(x) \quad \text{for } (x, t) \in \mathbb{R}^n \times ]0, +\infty[.
$$

The solution cannot be obtained via a Fourier transform and the proof is a bit more technical. The form of the solution depends on the dimension $n$. For $x \in \mathbb{R}^n$, solutions write

$$
\begin{align*}
  n = 1 & : S(x) = \frac{1}{2} |x|, \\
  n = 2 & : S(x) = \frac{1}{2\pi} \ln \|x\|, \\
  n = 3 & : S(x) = -\frac{1}{4\pi} \frac{1}{\|x\|}.
\end{align*}
$$

6.1  Diffusion of a window [Strauss 2.4, Ex. 1]

Solve the following initial value problem using the fundamental solution of the diffusion equation,

$$
\begin{cases}
  u_t - ku_{xx} = 0 & \text{for } (x, t) \in \mathbb{R} \times ]0, +\infty[, \\
  u(x, 0) = 1 & \text{for } |x| < l, \\
  u(x, 0) = 0 & \text{for } |x| \geq l.
\end{cases}
$$

Write your answer in terms of $\text{erf}(x)$.

Reminder:

$$
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta.
$$
Solution

The general solution is given by

\[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\ell}^{\ell} \exp \left( -\frac{(x-y)^2}{4kt} \right) dy. \]  

(6.1.1)

Introducing the following change of variables

\[ \xi = \frac{x-y}{\sqrt{4kt}} \]  

(6.1.2)

gives

\[ u(x, t) = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi \]  

(6.1.3)

\[ = \frac{1}{2} \left[ \text{erf} \left( \frac{x+\ell}{\sqrt{4kt}} \right) - \text{erf} \left( \frac{x-\ell}{\sqrt{4kt}} \right) \right]. \]  

(6.1.4)

6.2 Diffusion of an exponential [Strauss 2.4, Ex. 3]

Solve the following initial value problem using the fundamental solution of the diffusion equation,

\[ \begin{align*}
    u_t - ku_{xx} &= 0 \quad \text{for} \quad (x, t) \in \mathbb{R} \times ]0, +\infty[, \\
    u(x, 0) &= \exp(3x) \quad \text{for} \quad (x, t) \in \mathbb{R} \times \{0\}.
\end{align*} \]

Reminder:

\[ \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \quad \text{(Gaussian integral)}. \]

Solution

Replacing the given initial conditions in the general formula successively gives

\[ u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \exp \left( -\frac{(x-y)^2}{4kt} \right) \exp(3y) dy \]

\[ = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \exp \left( -\frac{y^2 + 2xy - x^2}{4kt} + 3y \right) dy \]  

(6.2.1)

\[ = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \exp \left( -\frac{y^2 + 2xy - x^2 + 12kty}{4kt} \right) dy. \]
Completing the square appearing in the exponential, the solution writes
\[
u(x,t) = \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp \left( -\frac{y^2 + 2xy - x^2 + 12kt^2 + 36k^2t^2 + 12ktx - 12ktx}{4kt} \right) dy
\]
(6.2.2)
\[
= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp \left( -\left[ \frac{x - y + 6kt}{\sqrt{4kt}} \right]^2 + \frac{36k^2t^2 + 12ktx}{4kt} \right) dy
\]
(6.2.3)
\[
= \exp(9kt + 3x) \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp \left( -\left[ \frac{x - y + 6kt}{\sqrt{4kt}} \right]^2 \right) dy.
\]
(6.2.4)

Finally using the change of variable
\[
\xi = \frac{x - y + 6kt}{\sqrt{4kt}},
\]
(6.2.5)
the solution can be written as follows
\[
u(x,t) = \exp(3(x + 3kt)).
\]
(6.2.6)

### 6.3 Diffusion with constant dissipation [Strauss 2.4, Ex. 16]

Solve the following diffusion problem with constant dissipation
\[
\begin{cases}
\frac{\partial u}{\partial t} - ku_{xx} + bu = 0 & \text{for } (x,t) \in \mathbb{R} \times ]0, +\infty[, \\
u(x,0) = \delta(x) & \text{for } (x,t) \in \mathbb{R} \times \{0\}.
\end{cases}
\]
where \( b \in \mathbb{R}_0^+ \) is a constant parameter.

Reminder: The Dirac distribution \( \delta \) has the following property
\[
\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0).
\]

Hint: use the substitution \( u(x,t) = e^{-bt}v(x,t) \).

**Solution**

Using the substitution \( u(x,t) = e^{-bt}v(x,t) \), the terms in the equation become
\[
\begin{align*}
\frac{\partial u}{\partial t} &= -b \exp(-bt) v + \exp(-bt) v_t, \\
u_{xx} &= \exp(-bt) v_{xx}.
\end{align*}
\]
(6.3.1)
(6.3.2)

Thus the equation takes the form
\[
\exp(-bt) v_t - k \exp(-bt) v_{xx} = 0,
\]
(6.3.3)
which can be simplified into an equation of the known form:
\[
v_t - kv_{xx} = 0.
\]
(6.3.4)
The initial condition writes \( v(x, 0) = \delta(x) \). The solution of this equation is then written as

\[
v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \exp \left( - \left[ \frac{x - y}{\sqrt{4kt}} \right]^2 \right) \delta(y) \, dy \quad (6.3.5)
\]

\[
= \frac{1}{\sqrt{4\pi kt}} \exp \left( - \frac{x^2}{4kt} \right) \quad (6.3.6)
\]

and finally

\[
u(x, t) = \frac{1}{\sqrt{4\pi kt}} \exp \left( - \frac{x^2}{4kt} \right) \exp(-bt). \quad (6.3.7)
\]

### 6.4 Heat equation with convection [Strauss 2.4, Ex. 18]

Consider the following diffusion problem with convection

\[
\begin{aligned}
\frac{\partial u}{\partial t} - ku_{xx} + Vu_x &= 0 \\
u(x, 0) &= \phi(x)
\end{aligned}
\quad \text{for} \quad (x, t) \in \mathbb{R} \times [0, +\infty[,
\]

where \( V \) is a constant.

(a) Solve this problem for a general \( \phi \).

Hint: Consider a moving reference frame by using the change of variables \( y = x - Vt \) and \( z = t \).

(b) Compute the solution for the initial condition \( \phi(x) = 1 \) for \( |x| \leq l \). Write your answer in terms of \( \text{erf}(x) \).

**Solution**

(a) Performing the proposed change of variable, the differential operators write

\[
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial y}{\partial x} \frac{\partial}{\partial y} + \frac{\partial z}{\partial x} \frac{\partial}{\partial z} = \frac{\partial}{\partial y}, \\
\frac{\partial}{\partial t} &= \frac{\partial y}{\partial t} \frac{\partial}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial}{\partial z} = -V \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.
\end{aligned}
\quad (6.4.1)
\quad (6.4.2)
\]

The equation becomes

\[
-V \frac{\partial u}{\partial y} + u_z - ku_{yy} + Vu_y = 0, \quad (6.4.3)
\]

which simplifies as

\[
u_z - ku_{yy} = 0. \quad (6.4.4)
\]

The general solution of Eq.(6.4.4) is

\[
u(y, z) = \frac{1}{\sqrt{4\pi k z}} \int_{-\infty}^{+\infty} \exp \left( - \left[ \frac{y - \zeta}{\sqrt{4kz}} \right]^2 \right) \phi(\zeta) \, d\zeta, \quad (6.4.5)
\]

\[
= \frac{1}{\sqrt{4\pi k z}} \int_{-\ell}^{+\ell} \exp \left( - \left[ \frac{y - \zeta}{\sqrt{4kz}} \right]^2 \right) \, d\zeta. \quad (6.4.6)
\]
(b) Therefore, the solution writes
\[ u(y, z) = \frac{1}{2} \left[ \text{erf} \left( \frac{y + l}{\sqrt{4kz}} \right) - \text{erf} \left( \frac{y - l}{\sqrt{4kz}} \right) \right], \tag{6.4.7} \]
\[ = \frac{1}{2} \left[ \text{erf} \left( \frac{x - Vt + l}{\sqrt{4kt}} \right) - \text{erf} \left( \frac{x - Vt - l}{\sqrt{4kt}} \right) \right]. \tag{6.4.8} \]

6.5 Electric potential generated by a charged sphere

In the whole space of dimension 3, a sphere of radius \( a > 0 \) has a uniform surface charge \( \sigma \) on its surface, with no volume charge. Using the fundamental solution of the Laplace equation, give the expression of the electrostatic potential and electric field in \( \mathbb{R} \). Laws of electrostatics are, for the electric field \( e \):

\[ \nabla \times e = 0, \quad \nabla \cdot e = \frac{\rho}{\varepsilon_0}. \]

(a) From the first equation (Faraday’s law), one can define the electrostatic potential \( v \) such that \( e = -\nabla v \). Express the second equation (Gauss’s law) in terms of this electrostatic potential.

(b) Express the volume charge \( \rho \) in terms of the surface charge \( \sigma \). Give the expression of the total charge \( q \) contained in the sphere (including its boundary).

(c) Solve the electrostatic problem with the condition \( v(\|x\| \to +\infty) = 0 \), inside and outside the charged sphere: give the expression of \( v \) and \( e \), expressed in terms of \( q \).

(d) Show that outside the sphere, the solution is identical to that for a point charge of the same value \( q \).

Solution

(a) By substitution of \( e \) by \(-\nabla v\), we get the Poisson equation
\[ \Delta v = -\frac{\rho}{\varepsilon_0}. \tag{6.5.1} \]

(b) Due to the spherical symmetry of the problem, we consider a spherical coordinate system, centered at the center of the sphere. The charge is concentrated on the surface of the sphere, therefore
\[ \rho(r) = \sigma \delta(r - a). \tag{6.5.2} \]

The total charge \( q \) is obtained by integrating \( \sigma \), which is constant, on the surface of the sphere. This gives \( q = 4\pi a^2 \sigma \).

(c) The potential can be obtained by the convolution the Green’s function and the source function. In spherical coordinates, with \((r, \theta, \phi)\) the coordinates at which we want the value of the potential and \((r', \theta', \phi')\) the coordinates of integration for the convolution,
\[ v(r, \theta, \phi) = \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \int_0^{+\infty} \frac{\sigma \delta(r' - a)r'^2 \sin \theta'}{4\pi \varepsilon_0 \|x - x'\|} dr'. \tag{6.5.3} \]
By symmetry, \( v(r, \theta, \phi) = v(r) \). For simplicity, we can choose \( \theta = 0 \) (on the z-axis). Consequently, \( \|x - x'\| \) becomes \( \sqrt{(r - r' \cos \theta')^2 + r'^2 \sin^2 \theta'} = \sqrt{r^2 - 2r'r \cos \theta' + r'^2} \). After integration on \( \phi' \) and using the change of variable for \( \theta' \)

\[
\begin{align*}
  r^2 - 2r'r \cos \theta' + r'^2 &= t, \\
  2r'r \sin \theta'd\theta' &= dt,
\end{align*}
\]  

we get

\[
v(r) = 2\pi \frac{\sigma}{4\pi \varepsilon_0} \int_0^{+\infty} dr' \int_{(r-r')^2}^{(r+r')^2} \frac{\delta(r' - a)r'}{2r} \frac{1}{\sqrt{t}} dt,
\]

\[
= \frac{\sigma}{2\varepsilon_0} \int_0^{+\infty} \frac{\delta(r' - a)r'}{2r} 2(\sqrt{(r + r')^2} - \sqrt{(r - r')^2}) dr',
\]

\[
= \frac{\sigma}{2r\varepsilon_0} \int_0^{+\infty} \delta(r' - a)r'(|r + r'| - |r - r'|) dr'.
\]

From the definition of the Dirac distribution, the integral is direct and gives

\[
v(r) = \frac{\sigma a (|r + a| - |r - a|)}{2r \varepsilon_0},
\]

\[
= \frac{q}{4\pi \varepsilon_0} \frac{|r + a| - |r - a|}{2ra},
\]

\[
= \begin{cases} 
  \frac{q}{4\pi \varepsilon_0 r} & \text{if } r \geq a, \\
  \frac{q}{4\pi \varepsilon_0 a} & \text{if } r < a.
\end{cases}
\]

The potential is constant inside the sphere, and decreases as \( 1/r \) to zero outside the sphere. The electric field \( e \) is given by (gradient in spherical coordinates, using the symmetry of the potential \( v \))

\[
e = -\nabla v = -\frac{\partial v}{\partial r} = \begin{cases} 
  \frac{q}{4\pi \varepsilon_0 r^2} \hat{r} & \text{if } r \geq a, \\
  0 & \text{if } r < a.
\end{cases}
\]

(d) If the source is a point charge \( \rho = q \delta(x) \), the solution is the Green’s function multiplied by \(-q/\varepsilon_0\), by definition (and linearity of the problem):

\[
v(r) = \frac{q}{4\pi \varepsilon_0 r}, \quad r > 0 \quad \text{and} \quad e = \frac{q}{4\pi \varepsilon_0 r^2} \hat{r}, \quad r > 0.
\]

This is the same solution than for the uniformly charged surface sphere.

### 6.6 Viscous Burgers’ equation and Cole-Hopf transformation [Olver, Sec. 8.4]

Consider the momentum equation for an incompressible fluid (Navier-Stokes equations), that is

\[
\partial_t u + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} f + \nu \nabla^2 u.
\]
Assume that there are no body forces \((f = 0)\), and that the pressure gradient term can be neglected \((\nabla p = 0)\), \textit{i.e.}
\[
\partial_t u + u \cdot \nabla u = \nu \nabla^2 u.
\]
Also assume that the problem is in one-dimension by posing \(u = u(x, t)\) and writing the momentum equation along the \(x\)-axis which finally gives
\[
 u_t + uu_x = \nu u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0,
\]
the so-called viscous Burgers’ equation. This non-linear partial differential equation can be rewritten in the form of the diffusion equation
\[
\phi_t = \nu \phi_{xx}
\]
by using appropriate methods (\textit{e.g.} Cole-Hopf transformation).

(a) First, let us consider the substitution
\[
U_x = u.
\]
Show that the Burgers’ equation can be rewritten as
\[
U_t = \nu U_{xx} - \frac{1}{2}(U_x)^2.
\]
Reminder: \(uu_x = \frac{1}{2}(u^2)_x\).

(b) Use the Cole-Hopf relation
\[
U(x, t) = -2\nu \ln(\phi(x, t))
\]
to simplify Eq.\((\ast)\) into the diffusion equation
\[
\phi_t = \nu \phi_{xx}.
\]

(c) Find the solution \(\phi(x, t)\) to the diffusion problem and show that the solution of Burgers’ equation can be written as
\[
u(\phi, 0) = \exp\left(-\frac{U(\phi, 0)}{2\nu}\right) = \exp\left(-\frac{1}{2\nu} \int_0^x u(z, 0)dz\right).
Solution

(a) First write Burgers’ equation in a simpler form

\[ u_t + uu_x = \nu u_{xx} \]  \hspace{1cm} (6.6.1)

\[ \Rightarrow u_t + \frac{1}{2}(u^2)_x = \nu u_{xx}. \]  \hspace{1cm} (6.6.2)

Now use the substitution suggested in the statement which gives

\[ U_{tx} + \frac{1}{2}((U_x)^2)_x = \nu U_{xxx}, \quad (U_{xt} = U_{tx}) \]  \hspace{1cm} (6.6.3)

i.e., upon integration

\[ U_t + \frac{1}{2}(U_x)^2 = \nu U_{xx}, \]  \hspace{1cm} (6.6.4)

\[ \Rightarrow U_t = \nu U_{xx} - \frac{1}{2}(U_x)^2. \]  \hspace{1cm} (6.6.5)

(b) Consider the Cole-Hopf relation

\[ U(x,t) = -2\nu \ln(\phi(x,t)). \]  \hspace{1cm} (6.6.6)

Writing each derivatives of \( U \) that appears in the equation above yields

\[
\begin{cases}
U_t = -2\nu \frac{\phi_t}{\phi} \\
U_x = -2\nu \frac{\phi_x}{\phi} \\
U_{xx} = -2\nu \left( \frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2} \right).
\end{cases}
\]  \hspace{1cm} (6.6.7-6.6.9)

Plugging these results into Eq.(6.6.5) yields

\[-2\nu \frac{\phi_t}{\phi} = -2\nu^2 \frac{\phi_{xx}}{\phi} + 2\nu^2 \frac{\phi_x^2}{\phi^2} - \frac{1}{2} \left( 4\nu^2 \frac{\phi_x^2}{\phi^2} \right). \]  \hspace{1cm} (6.6.10)

Simplifying each terms indeed yields the diffusion equation

\[ \phi_t = \nu \phi_{xx}. \]

(c) The fundamental solution of the diffusion equation is given by

\[ \phi(x,t) = \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{+\infty} \exp \left( -\frac{(x-y)^2}{4\nu t} \right) \phi(y,0) dy \]  \hspace{1cm} (6.6.11)

\[ = \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{+\infty} \exp \left( -\frac{(x-y)^2}{4\nu t} \right) - \frac{1}{2\nu} \int_0^y u(z,0) dz \right) dy, \]  \hspace{1cm} (6.6.12)

using the definition of \( \phi(x,0) \) given in the statement.

Then

\[ U(x,t) = -2\nu \ln \left[ \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{+\infty} \exp \left( -\frac{(x-y)^2}{4\nu t} \right) - \frac{1}{2\nu} \int_0^y u(z,0) dz \right] dy \]  \hspace{1cm} (6.6.13)
and finally the solution \( u(x, t) \) is

\[
   u(x, t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp \left[ -\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu} \int_0^y u(z, 0) \, dz \right] \, dy}{\int_{-\infty}^{+\infty} \exp \left[ -\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu} \int_0^y u(z, 0) \, dz \right] \, dy}.
\]

### 6.7 Wave generated from an initially activated sphere [Olver, Example 12.16]

Consider the wave equation in the whole three-dimensional space, with the given initial conditions

\[
   \begin{cases}
     u_{tt} - c^2 \Delta u = 0 & \text{for } (x, t) \in \mathbb{R}^3 \times [0, +\infty[,
     \\
     u(x, 0) = 0 & \text{for } x \in \mathbb{R}^3,
     \\
     u_t(x, 0) = g(x) & \text{for } x \in \mathbb{R}^3.
   \end{cases}
\]

with \( g(x) \) the initial condition that will be specified later.

(a) Using the fundamental solution of the wave equation, give the general expression of the solution. Interpret the results.

(b) For \( c = 1 \), solve the problem for the initial condition on \( u_t \) is

\[
   g(x) = \begin{cases} 
     1 & \text{for } \|x\| < 1, \\
     0 & \text{for } \|x\| > 1.
   \end{cases}
\]

(c) Sketch the solution with respect to \( r \) for time instants \( t = 0, t = 0.5, t = 1, \) and \( t = 1.5 \).

**Solution**

(a) The convolution of the Green’s function and the initial condition on \( u_t \) gives

\[
   u(x, t) = \frac{1}{4\pi c} \iiint g(\xi) \frac{\delta(ct - \|x - \xi\|)}{\|x - \xi\|} \, d\xi,
\]

\[
   = \frac{1}{4\pi c^2 t} \iint \frac{g(\xi)}{\|x - \xi\| = ct} \, dS,
\]

\[
   = tM_{ct}^x[g],
\]

where \( M_{ct}^x[g] \) is the average value of \( g \) on the surface of a sphere of center \( x \) and radius \( ct \) (the integral of the value of \( g \) divided by the surface of the sphere \( 4\pi c^2 t^2 \)). In the following, we denote this sphere \( S_{ct}^x \). Therefore, the solution at a given point and a time instant \( t \) only depends on the initial condition \( g \) that was imposed at a distance \( ct \) from it. Local solutions to the wave equation remain local, this is a manifestation of the Huygens’ Principle (valid in 3D).

NB: This principle does not hold in 2D. In a 2D space, an initially local solution leaves a decreasing but non-zero tail after its propagation front. What about in 1D? (Hint: interpret the d’Alembert formula.)

(b) The problem has a spherical symmetry, we use spherical coordinates. Also, \( u(r, \theta, \phi, t) = u(r, t) \). Use Fig. 3.

At a point \( (r, t) \), we have:
\(t\) \(r\) \(O\) \((x,t)\) \(\alpha\) \(\xi\) \(\|\xi - x\| = t\)

Figure 3: Schematics of the two sphere intersection.

- If \(t > r + 1\), or if \(t < r - 1\), then the sphere \(S_t^x\) \((c = 1)\) and the ball \(r < 1\) have no intersection. Thus, \(u(r, t) = 0\).
- If \(t < 1 - r\), which may arise only if \(r < 1\), the sphere \(S_t^x\) is entirely included in the ball \(r < 1\) and the average value of \(g\) is therefore 1. Consequently, \(u(r, t) = t\) (do not forget to multiply by \(t\)).
- In the other situations, the sphere and the ball have a non-trivial intersection. The intersection is a spherical cap. If \(\alpha\) the overture angle of this spherical cap, its area is given by

\[
\text{spherical cap } \alpha = \int_0^{2\pi} d\phi \int_0^\alpha t^2 \sin \theta d\theta = 2\pi t^2 (1 - \cos \alpha). \quad (6.7.4)
\]

What remains to do is to express \(\alpha\) in terms of \(r\) and \(t\). Introducing \(b\) and \(h\) as in Fig. 3, we have
\(b = t \cos \alpha, \ h^2 + b^2 = t^2\) and \((r - b)^2 + h^2 = 1\). We get rid of \(h\) using the last two equations, and then introduce the first equation

\[
1 - (r - b)^2 + b^2 = t^2, \quad (6.7.5)
\]

\[
\Leftrightarrow 1 - r^2 + 2rb = t^2, \quad (6.7.6)
\]

\[
\Leftrightarrow x = \frac{t^2 + r^2 - 1}{2r}, \quad (6.7.7)
\]

\[
\Leftrightarrow \cos \alpha = \frac{t^2 + r^2 - 1}{2rt}. \quad (6.7.8)
\]

The spherical cap area writes

\[
\text{spherical cap } \alpha = 2\pi t^2 (1 - \frac{t^2 + r^2 - 1}{2rt}) = \frac{\pi t}{r} (1 - (t - r)^2), \quad (6.7.9)
\]

and the average value of \(g\) on the whole sphere \(S_t^x\) is \((1 - (t - r)^2) / 4rt\), which, multiplied by \(t\), gives the solution

\[
u(r, t) = \frac{1 - (t - r)^2}{4r} \quad (6.7.10)
\]

To summarize, the solution is

\[
u(x, t) = \begin{cases} 
  t, & 0 \leq t \leq 1 - \|x\|, \\
  \frac{1 - (t - \|x\|)^2}{4\|x\|}, & \|x\| - 1 \leq t \leq \|x\| + 1, \\
  0, & 0 \leq t \leq \|x\| - 1, \ t \geq \|x\| + 1.
\end{cases} \quad (6.7.11)
\]
or

\[
u(x, t) = \begin{cases} 
  t, & 0 \leq \|x\| \leq 1 - t, \\
  \frac{1 - (t - \|x\|)^2}{4\|x\|}, & 1 - t \leq \|x\| \leq 1 + t, \\
  0, & \|x\| \geq 1 + t, \\
  0, & 0 \leq \|x\| \leq t - 1, \\
  \frac{1 - (t - \|x\|)^2}{4\|x\|}, & t - 1 \leq \|x\| \leq t + 1, \\
  0, & \|x\| \geq t + 1, 
\end{cases}
\]

(6.7.12)

(c) See the figure below.

![Graphs showing solution at different time instants](image_url)

Figure 4: Schematics of solution with respect to \(r\) at different time instants.

### 6.8 Wave generated from an initially illuminated sphere [Olver, Example 12.17]

Consider the wave equation in the whole three-dimensional space, with the given initial conditions

\[
\begin{align*}
  v_{tt} - c^2 \Delta v &= 0 & \text{for } (x, t) \in \mathbb{R}^3 \times ]0, +\infty[, \\
  v(x, 0) &= f(x) & \text{for } x \in \mathbb{R}^3, \\
  v_t(x, 0) &= g(x) & \text{for } x \in \mathbb{R}^3.
\end{align*}
\]

with \(f(x)\) and \(g(x)\) the initial conditions that will be specified later.

(a) Show that if a function \(u\) solves the wave equation, then so does its time derivative \(v = u_t\), provided that the function \(u\) is sufficiently smooth. Then, consider an auxiliary homogeneous initial value wave problem for \(u\) with the following initial conditions for \(u\):

\[
\begin{align*}
  u(x, 0) &= \phi(x) & \text{for } x \in \mathbb{R}^3, \\
  u_t(x, 0) &= \psi(x) & \text{for } x \in \mathbb{R}^3.
\end{align*}
\]

Express the initial conditions for \(v = u_t\) \((f\text{ and } g)\) in terms of the initial conditions for \(u\).
(b) In the case where \( c = 1 \), \( g(x) = 0 \) and
\[
f(x) = \begin{cases} 
1 & \text{for } \|x\| < 1, \\
0 & \text{for } \|x\| > 1,
\end{cases}
\]
give the solution to the initial value problem. *Hint:* Use directly the solution of exercise 6.7.

(c) Sketch the solution with respect to \( \|x\| \) for time instants \( t = 0 \), \( t = 0.5 \), \( t = 1 \), and \( t = 1.5 \).

**Solution**

(a) The time derivative of the equation writes \( u_{ttt} - c^2(\Delta u)_t = 0 \). If \( u \) is sufficiently continuous, the order of derivatives can be changed to yield \( (u_t)_{tt} - c^2(\Delta u)_t = 0 \). Therefore, the function \( v = u_t \) is also a solution of the wave equation.

The initial condition of \( v \) is trivial, \( v(x,0) = u_t(x,0) = \psi(x) \). The initial condition on its time derivative is obtained as follows:
\[
v_t(x,0) = u_{tt}(x,0) = c^2\Delta u(x,0) = c^2\Delta \phi(x),
\]
where we used the fact that \( u \) is a solution to the wave equation. The full initial value problem writes:
\[
\begin{align*}
 v_{tt} - c^2\Delta v &= 0 & \text{for } (x,t) \in \mathbb{R}^3 \times ]0, +\infty[ , \\
v(x,0) &= \psi(x) & \text{for } x \in \mathbb{R}^3 , \\
v_t(x,0) &= c^2\Delta \phi(x) & \text{for } x \in \mathbb{R}^3 .
\end{align*}
\]

(b) The problem to solve is
\[
\begin{align*}
 v_{tt} - \Delta v &= 0 & \text{for } (x,t) \in \mathbb{R}^3 \times ]0, +\infty[ , \\
v(x,0) &= f(x) & \text{for } x \in \mathbb{R}^3 , \\
v_t(x,0) &= 0 & \text{for } x \in \mathbb{R}^3 .
\end{align*}
\]

From the previous developments, if a function \( u \) solves the following initial value problem
\[
\begin{align*}
 u_{tt} - c^2\Delta u &= 0 & \text{for } (x,t) \in \mathbb{R}^3 \times ]0, +\infty[ , \\
u(x,0) &= 0 & \text{for } x \in \mathbb{R}^3 , \\
u_t(x,0) &= f(x) & \text{for } x \in \mathbb{R}^3 ,
\end{align*}
\]
then, \( v = u_t \) is also a solution to the wave equation, and verifies the initial conditions of problem (6.8.3). The function \( v = u_t \) is therefore the solution to problem (6.8.3). With the given initial condition \( f \), the solution to problem (6.8.4) has been found in exercise 6.7 and writes
\[
u(x,t) = \begin{cases} 
t, & 0 \leq t \leq 1 - \|x\|, \\
\frac{1 - (t - \|x\|)^2}{4\|x\|}, & \|x\| - 1 \leq t \leq \|x\| + 1, \\
0, & 0 \leq t \leq \|x\| - 1, \quad t \geq \|x\| + 1.
\end{cases}
\]
Therefore, the solution $v$ is given by

$$ v(x, t) = \begin{cases} 
1, & 0 \leq t \leq 1 - \|x\|, \\
\frac{\|x\| - t}{2\|x\|}, & \|x\| - 1 \leq t \leq \|x\| + 1, \\
0, & 0 \leq t \leq \|x\| - 1, \ t \geq \|x\| + 1.
\end{cases} \quad (6.8.6) $$

(c) See the figure below.

![Figure 5: Schematics of solution with respect to $r$ at different time instants.](image_url)

6.9 Method of images for the diffusion equation

Let the evolution of a pollutant concentration $u$ be described by the homogeneous diffusion equation with a unit diffusion constant $k = 1$, in a portion of the three-dimensional space.

(a) Consider the whole three-dimensional space $\mathbb{R}^3$, and the initial condition $u(x, 0) = \delta(x)$. Give the time evolution of the pollutant concentration in $\mathbb{R}^3$. 
(b) Introduce a Cartesian coordinate system \((x, y, z)\), and consider only the half-space \(D = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times ]0, +\infty[\} \). The plane boundary \(z = 0\) is impermeable to the pollutant and there is no pollutant flux across it, \(i.e. \ \hat{z} \cdot \nabla u|_{z=0} = 0\). The initial condition is \(u(x, y, z, 0) = \delta(x)\delta(y)\delta(z - 1)\). By the method of images, give the time evolution of \(u\) in \(D\).

(c) Instead of the Neumann boundary condition, we impose the Dirichlet boundary condition \(u|_{z=0} = 0\). By the method of images, give the time evolution of \(u\) in \(D\).

(d) The domain is now restricted to \(E = \{(x, y, z) \in ]0, +\infty[ \times ]0, +\infty[ \times ]0, +\infty[\}\), with the Neumann boundary condition \(n \cdot \nabla u = 0\) on its boundaries, and with the initial condition \(u(x, y, z, 0) = \delta(x - 1)\delta(y - 2)\delta(z - 3)\). By the method of images, give the time evolution of \(u\) in \(E\). What is the solution for the same problem but with the Dirichlet boundary condition \(u = 0\) on its boundaries?

**Solution**

(a) By definition, the evolution of \(u\) is the diffusion Green’s function, with \(k = 1\),

\[
u(x, t) = \frac{1}{(4\pi t)^{3/2}} \exp \left(-\frac{\|x\|^2}{4t}\right), \quad (x, t) \in \mathbb{R}^3 \times ]0, +\infty[. \quad (6.9.1)
\]

(b) Let us build an auxiliary problem, easier to solve. By virtue of the uniqueness of the solution, if the solution of this auxiliary problem satisfies both the diffusion equation and the Neumann boundary condition on the plane \(z = 0\), its solution is the sought solution. If we consider the whole space \(\mathbb{R}^3\), and an additional fictitious symmetric initial condition at \((x, y, z) = (0, 0, -1)\), we have the following problem:

\[
\begin{aligned}
u_t - \nu_{xx} - \nu_{yy} - \nu_{zz} &= 0, \\
u(x, y, z, 0) &= \delta(x)\delta(y)\delta(z - 1) + \delta(x)\delta(y)\delta(z + 1),
\end{aligned} \quad (x, y, z, t) \in \mathbb{R}^3 \times ]0, +\infty[, \quad (x, y, z) \in \mathbb{R}^3. \quad (6.9.2)
\]

The solution to this problem is, by linearity, for \((x, t) \in \mathbb{R}^3 \times ]0, +\infty[\),

\[
u(x, t) = \frac{1}{(4\pi t)^{3/2}} \left(\exp \left(-\frac{x^2 + y^2 + (z - 1)^2}{4t}\right) + \exp \left(-\frac{x^2 + y^2 + (z + 1)^2}{4t}\right)\right). \quad (6.9.3)
\]

This solution satisfies the Neumann boundary condition (show it) of the original problem as well as the diffusion equation. When it is restricted to the original domain \(D\), this is therefore the sought solution.

(c) The symmetric initial condition is chosen as \(-\delta(x)\delta(y)\delta(z + 1)\). The final solution writes, in \(D\)

\[
u(x, t) = \frac{1}{(4\pi t)^{3/2}} \left(\exp \left(-\frac{x^2 + y^2 + (z - 1)^2}{4t}\right) - \exp \left(-\frac{x^2 + y^2 + (z + 1)^2}{4t}\right)\right). \quad (6.9.4)
\]

(d) Seven image sources must be placed symmetrically. They are all positive and in positions \((-1, 2, 3), (-1, -2, 3), (-1, 2, -3), (-1, -2, -3), (1, -2, 3), (1, 2, -3), (1, -2, -3)\). The solution follows easily, restricted to \(E\). If the Dirichlet boundary condition is imposed, the image sources will switch signs at each reflection, they will be negative at positions \((-1, 2, 3), (1, -2, 3), (1, 2, -3)\) and \((-1, -2, -3)\). The solution also follows easily, still restricted to \(E\).
7 Separation of variables

7.1 Diffusion equation I

Consider the one-dimensional diffusion equation on a bounded domain

\[ u_t - ku_{xx} = 0 \quad \forall x \in [0, 1] \] \hspace{1cm} (7.1.1)

with initial condition

\[ u(x, 0) = \phi(x) \quad \forall x \in [0, 1] \]

and with some boundary conditions at \( x = 0 \) and \( x = 1 \) (these boundary conditions will be specified later).

(a) Using separation of variables \( u(x, t) = w(t)v(x) \), find all the separable solutions of Eq. (7.1.1).

(b) Find the solution to Eq. (7.1.1) for the homogeneous Dirichlet boundary conditions \( u(0, t) = u(1, t) = 0 \quad \forall t \geq 0 \) and for \( \phi(x) = \sin \pi x \).

(c) Find the solution to Eq. (7.1.1) for the homogeneous Dirichlet boundary conditions \( u(0, t) = u(1, t) = 0 \quad \forall t \geq 0 \) and for

\[
\phi(x) = \begin{cases} 
  x & \forall x \in \left[0, \frac{1}{2}\right], \\
  1-x & \forall x \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]

(d) Find the solution to Eq. (7.1.1) for the homogeneous Dirichlet boundary conditions \( u(0, t) = u(1, t) = 0 \quad \forall t \geq 0 \) and for \( \phi(x) = 1 \).

(e) Find the solution to Eq. (7.1.1) for the homogeneous Neumann boundary conditions \( u_x(0, t) = u_x(1, t) = 0 \quad \forall t \geq 0 \) and for \( \phi(x) = \cos \pi x \).

(f) Find the solution to Eq. (7.1.1) for the non-homogeneous Dirichlet boundary conditions

\[ u_x(0, t) = 0 \quad \text{and} \quad u_x(1, t) = 1, \quad \forall t \geq 0 \]

and for \( \phi(x) = x + \sin \pi x \).

Solution

(a) Using the ansatz \( u = wv \), the diffusion equation writes as

\[ w'v - kwv'' = 0 \]

\[ \Rightarrow \quad \frac{w'}{kw} = \frac{v''}{v} \] \hspace{1cm} (7.1.2)

Since the right hand side depends only on \( x \) while the left hand side depends only on \( t \), both sides must be equal to a constant (named \( \lambda \)), \( i.e. \)

\[ v'' - \lambda v = 0 \quad \text{and} \quad w' - k\lambda w = 0. \] \hspace{1cm} (7.1.3)

Spatial dependence \hspace{1cm} Depending on the sign of \( \lambda \), three solutions arise, \( i.e. \)

\[
\begin{align*}
\text{if } \lambda &= 0 \quad \Rightarrow v = Ax + B, \\
\text{if } \lambda &= \omega^2 > 0 \quad \Rightarrow v = C \exp(-\omega x) + D \exp(\omega x), \\
\text{if } \lambda &= -\omega^2 < 0 \quad \Rightarrow v = E \cos(\omega x) + F \sin(\omega x).
\end{align*}
\] \hspace{1cm} (7.1.4) (7.1.5) (7.1.6)
Time dependence  Whatever the sign of $\lambda$, the time dependence is given by

$$w = \exp(\lambda kt).$$  (7.1.7)

The constant in front the exponential is here omitted because the field of interest is $u = vw$.

(b) Among all the eigensolutions found in the previous sub-question (i.e for any value of $\lambda$), only those that satisfy the homogeneous boundary conditions are kept.

Stationary eigensolutions, i.e: $\lambda = 0$  Applying the boundary conditions to $v = Ax + B$ gives

$$\begin{align*}
  v(0) &= B = 0 \\
v(1) &= A + B = 0
\end{align*}$$

⇒ $A = B = 0$  (7.1.8)

such that there is no stationary eigensolution satisfying the homogeneous boundary conditions.

Time-growing eigensolutions, i.e: $\lambda > 0$  Applying the boundary conditions to $v = C \exp(-\omega x) + D \exp(\omega x)$ gives

$$\begin{align*}
  v(0) &= C + D = 0 \\
v(1) &= C \exp(-\omega) + D \exp(\omega) = 0
\end{align*}$$

⇒ $\begin{pmatrix} 1 & 1 \\ \exp(-\omega) & \exp(\omega) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  (7.1.9)

provided that $\omega > 0$, the unique solution to Eqs(7.1.9) is

$$C = D = 0$$  (7.1.10)

thus there is no time growing eigensolution satisfying the homogeneous boundary condition neither.

Time-decaying eigensolutions, i.e: $\lambda < 0$  Applying the boundary conditions to $v = E \cos(\omega x) + F \cos(\omega x)$ gives

$$\begin{align*}
  v(0) &= E = 0 \\
v(1) &= E \cos(\omega) + F \sin(\omega) = 0
\end{align*}$$

⇒ $F \sin(\omega) = 0$  ⇒ $\omega \rightarrow \omega_n = n\pi, \quad n = 1, 2, 3, \ldots$  (7.1.11)

One should be carefull that the values $\omega_n = n\pi$ for $n < 0$ are not considered because they yield the same eigenvalue $\lambda_n = -\omega_n^2$.

The only eigensolutions compatible with the boundary conditions are therefore

$$v_n(x) = F_n \sin(\omega_n x), \quad \forall n = 1, 2, 3, \ldots$$  (7.1.12)

and the most general solution compatible with boundary condition is then

$$u(x, t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp \left(-\omega_n^2 kt \right).$$  (7.1.13)
All the constants $F_n$ must still be determined through the initial condition

$$u(x, 0) = \phi(x) = \sum_{n=1}^{\infty} F_n \sin n\pi x.$$  \hspace{1cm} (7.1.14)

Eq.(7.1.14) is actually a Fourier sine series which has the following orthogonality property

$$F_m = 2 \int_0^1 \phi(x) \sin(m\pi x) \, dx.$$  \hspace{1cm} (7.1.15)

In the particular case where $\phi(x) = \sin \pi x$, the unknown coefficient are given by

$$F_m = 2 \int_0^1 \sin(\pi x) \sin(m\pi x) \, dx = \delta_{1m}$$  \hspace{1cm} (7.1.16)

such that the final solution is

$$u(x, t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp(-\omega_n^2 kt) = \sin(\pi x) \exp(-\pi^2 kt).$$  \hspace{1cm} (7.1.17)

(c) The boundary conditions are the same than for the previous subquestion such that the same set of eigen-solutions must be conserved, only the coefficient $F_n$ of the Fourier sine series change. The Fourier coefficients are given by

$$F_m = 2 \int_0^1 \phi(x) \sin(m\pi x) \, dx$$  \hspace{1cm} (7.1.18)

$$= 2 \int_0^{1/2} x \sin(m\pi x) \, dx + 2 \int_{1/2}^1 (1-x) \sin(m\pi x) \, dx$$  \hspace{1cm} (7.1.19)

$$= \frac{4 \sin \left(\frac{m\pi}{2}\right)}{m^2 \pi^2} = \begin{cases} \frac{4(-1)^l}{(2l+1)^2 \pi^2}, & m = 2l + 1. \\ 0, & m = 2l. \end{cases}$$  \hspace{1cm} (7.1.20)

The final solution is therefore given by

$$u(x, t) = \frac{4}{\pi^2} \sum_{l=0}^{\infty} (-1)^l \frac{\sin \left(\frac{(2l+1)\pi x}{2l+1}\right)}{(2l+1)^2} \exp \left(-\frac{(2l+1)^2 \pi^2 kt}{4}\right)$$  \hspace{1cm} (7.1.21)

(f) The problem here is that the boundary conditions are not homogeneous. Therefore, if all modes satisfy the non-homogeneous boundary conditions, their sum will not satisfy the boundary condition. The idea is then to first choose a group of eigenfunctions that satisfy the non-homogeneous boundary condition, then to add the eigensolutions of the problem with homogeneous boundary conditions. This is actually the same idea than when solving an initial value problem, the whole solution is the sum of a particular solution and the homogeneous solution.

First, the procedure is therefore to find a set of solutions that satisfy the non homogeneous boundary condition

$$\begin{cases} u(0, t) = w(t) v(0) = 0 & \forall t, \\ u(1, t) = w(t) v(1) = 1 & \forall t. \end{cases}$$  \hspace{1cm} (7.1.22)
While obviously the boundary condition at \( x = 0 \) yields \( v(0) = 0 \), the boundary condition at \( x = 1 \) implies that \( w(t) \) does not depend on \( t \). The natural eigensolutions to consider to satisfy the non-homogeneous solutions must therefore have a temporal part that is constant, which is typically the case for \( \lambda = 0 \). Applying the boundary condition to the steady eigensolutions one finds

\[
\begin{aligned}
\begin{cases}
u(0) = 0 \\
\end{cases}
\end{aligned}
\]

Now that a unique solution satisfying non-homogeneous boundary conditions is known, all the solutions satisfying the homogeneous boundary condition must be added. Referring to sub-questions, the solution becomes

\[
u(x, t) = x \exp(0kt) + \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp(-\omega_n^2 kt),
\]

then to initial condition yields

\[
u(x, 0) = x + \sum_{n=1}^{\infty} F_n \sin(\omega_n x) = \phi(x) = x + \sin(\pi x),
\]

and, as in sub-question, the solution is

\[
u(x, t) = x + \sin(\pi x) \exp(-\pi^2 kt).
\]

### 7.2 Wave equation I

Consider the one-dimensional wave equation on a bounded domain

\[
u_{tt} - c^2 \nu_{xx} = 0 \quad \forall x \in ]0, 1[
\]

with initial condition

\[
u(x, 0) = \phi(x) \quad \text{and} \quad \nu_t(x, 0) = \psi(x) \quad \forall x \in ]0, 1[
\]

and with some boundary conditions at \( x = 0 \) and \( x = 1 \) (these conditions will be specified later).

(a) Using separation of variable \( \nu(x, t) = w(t)v(x) \), find all the separable solution of Eq. (\( \diamond \)).

(b) Find the solution to Eq. (\( \diamond \)) for the homogeneous Dirichlet boundary conditions \( \nu(0, t) = \nu(1, t) = 0 \quad \forall t \geq 0 \), for \( \phi(x) = \sin \pi x \) and for \( \psi(x) = 0 \).

(c) Find the solution to Eq. (\( \diamond \)) for the homogeneous Dirichlet boundary conditions \( \nu(0, t) = \nu(1, t) = 0 \quad \forall t \geq 0 \), for

\[
\phi(x) = \begin{cases} x & \forall x \in ]0, \frac{1}{2}] \\
1 - x & \forall x \in ]\frac{1}{2}, 1[,
\end{cases}
\]

and for \( \psi(x) = 0 \).

(d) Find the solution to Eq. (\( \diamond \)) for the homogeneous Dirichlet boundary conditions \( \nu(0, t) = \nu(1, t) = 0 \quad \forall t \geq 0 \) for \( \phi(x) = \sin \pi x \) and for \( \psi(x) = c \sin \pi x \).
Solution

(a) Using the ansatz \( u = wv \), the wave equation writes as

\[
\frac{w''}{c^2w} v - c^2 w v'' = 0 \tag{7.2.1}
\]

\[
\Rightarrow \frac{w''}{c^2w} = \frac{v''}{v}. \tag{7.2.2}
\]

Since the right hand side depends only on \( x \) while the left hand side depends only on \( t \), both sides must be equal to a constant (named \( \lambda \)), i.e

\[
v'' - \lambda v = 0 \quad \text{and} \quad w'' - c^2 \lambda = 0. \tag{7.2.3}
\]

**Spatial dependence**

Depending on the sign of \( \lambda \), three solutions arise, i.e

- if \( \lambda = 0 \) \( \Rightarrow v = Ax + B \), \tag{7.2.4}
- if \( \lambda = \omega^2 > 0 \) \( \Rightarrow v = C \exp(-\omega x) + D \exp(\omega x) \), \tag{7.2.5}
- if \( \lambda = -\omega^2 < 0 \) \( \Rightarrow v = E \cos(\omega x) + F \sin(\omega x) \). \tag{7.2.6}

**Time dependence**

Depending on the sign of \( \lambda \), three solutions arise, i.e

- if \( \lambda = 0 \) \( \Rightarrow v = Gt + H \), \tag{7.2.7}
- if \( \lambda = \omega^2 > 0 \) \( \Rightarrow v = I \exp(-c\omega t) + J \exp(c\omega t) \), \tag{7.2.8}
- if \( \lambda = -\omega^2 < 0 \) \( \Rightarrow v = K \cos(c\omega t) + L \sin(c\omega t) \). \tag{7.2.9}

(b) The eigenvalues are given by \( \lambda_n = -\omega_n^2 = -n^2 \pi^2 \) with \( n = 1, 2, 3, ... \) such that the only compatible eigensolutions are

\[
v_n(x) = F_n \sin(\omega_n x), \quad \forall n = 1, 2, 3, ... \tag{7.2.10}
\]

The most general way to write the solution is

\[
u(x, t) = \sum_{n=1}^{\infty} [A_n \cos(nc\pi t) + B_n \sin(nc\pi t)] \sin(\omega_n x). \tag{7.2.11}
\]

It remains to determine the constants \( A_n \) et \( B_n \) using initial conditions. Since the time dependence of the wave equation governed by a second order differential equation there are two initial conditions. One will help finding the \( A_n \) and the other will help determining the \( B_n \).

At \( t = 0 \) one has

\[
u(x, 0) = \sin(\pi x) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \tag{7.2.12}
\]

\[
u_t(x, 0) = 0 = \sum_{n=1}^{+\infty} B_n cn\pi \sin(n\pi x) \tag{7.2.13}
\]

The \( B_n \) are immediately identified as 0. It can be shown that \( A_n = \delta_{1n} \) using the orthogonality of Fourier sine series. The final solution is then

\[
u(x, t) = \cos(c\pi t) \sin(\pi x). \tag{7.2.14}
\]
(c) Similarly to the preceding sub-questions, starting from Eq. (7.2.11), the constants \( A_n \) and \( B_n \) are determined with the initial conditions. First,

\[
u_t(x, 0) = 0 = \sum_{n+1}^{+\infty} nc\pi B_n \sin(w_n x).
\]

(7.2.15)

As the functions \( \sin(w_n x) \) form a basis of \([0, 1] \), the only possible combination of \( B_n \) satisfying the equality is \( B_n = 0 \). The second initial condition is

\[
u(x, 0) = \sum_{n=1}^{\infty} A_n \sin(w_n x) = \phi(x).
\]

(7.2.16)

Using then the orthogonality property of the \( \sin \) functions,

\[
A_n = 2 \int_0^1 \phi(x) \sin(n\pi x) \, dx
\]

(7.2.17)

\[
= 2 \int_0^{1/2} x \sin(n\pi x) \, dx + 2 \int_{1/2}^1 (1 - x) \sin(n\pi x) \, dx.
\]

(7.2.18)

Integrating by parts gives

\[
A_n = \frac{4 \sin \left( \frac{n\pi}{2} \right)}{n^2 \pi^2} = \begin{cases} 
\frac{4(-1)^k}{(2k+1)^2 \pi^2}, & n = 2k + 1, \\
0, & n = 2k.
\end{cases}
\]

(7.2.19)

Therefore the solution is

\[
u(x, t) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} (-1)^k \cos \left( \frac{(2k+1)c\pi t}{2} \right) \sin \left( \frac{(2k+1)\pi x}{2} \right).
\]

(7.2.20)

(d) Similar to preceding sub-questions, the \( A_n \) are given by

\[A_n = \delta_{1n}\]

(7.2.21)

while the \( B_n \) are derived starting from

\[
u_t(x, 0) = c \sin(\pi x) = \sum_{n=1}^{+\infty} B_n c n\pi \sin(n\pi x).
\]

(7.2.22)

Using the same procedure as for the determination of the \( A_n \) one can show that

\[
B_n = \frac{\delta_{1n}}{n\pi}.
\]

(7.2.23)

\[
u(x, t) = \left[ \cos(c\pi t) + \frac{\sin(c\pi t)}{\pi} \right] \sin(\pi x).
\]

(7.2.24)
7.3 Laplace equation I

Consider the two-dimensional Laplace equation on a square

\[ u_{xx} + u_{yy} = 0 \quad \{x, y\} \in ]0, 1[ \times ]0, 1[. \]

(a) Using separation of variables \( u(x, y) = v(x)w(y) \), find all the separable solutions of this equation.

(b) Find the set of separable solutions that verify the following homogeneous boundary conditions

\[
\begin{cases}
    u(0, y) = 0 & \forall y \in [0, 1], \\
    u(1, y) = 0 & \forall y \in [0, 1], \\
    u(x, 1) = 0 & \forall x \in [0, 1].
\end{cases}
\]

(c) Show that the Fourier sine series expansion of \( \phi(x) = \begin{cases} 
    x & \forall x \in ]0, \frac{1}{2}[, \\
    1-x & \forall x \in ]\frac{1}{2}, 1[ 
\end{cases} \)

can be expressed as

\[
\phi(x) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin ((2j+1)\pi x)}{(2j+1)^2}.
\]

(d) With the boundary conditions given in (b) and \( u(x, 0) = \phi(x), \quad \forall x \in [0, 1] \)

for the last edge, show that the solution to the boundary value problem is then

\[
u(x, y) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin ((2j+1)\pi x) \sinh ((2j+1)\pi (1-y))}{(2j+1)^2 \sinh ((2j+1)\pi)}.
\]

Solution

(a) Using separation of variables, Laplace equation can be written

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \lambda. \quad (7.3.1)
\]

Therefore, both spatial part of the solution depends on the sign of the same constant \( \lambda \). The following table presents all the separable solutions as a function of \( \lambda \).
Exercise statements and solutions

Part $X(x)$

$X(x) = Ax + B$

Part $Y(y)$

$Y(y) = Cy + D$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$X(x)$</th>
<th>$Y(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 0$</td>
<td>$X(x) = E \cosh(\omega x) + F \sinh(\omega x)$</td>
<td>$Y(y) = G \cos(\omega y) + H \sin(\omega y)$</td>
</tr>
<tr>
<td>$\lambda &gt; 0$</td>
<td>$X(x) = I \cos(\omega x) + J \sin(\omega x)$</td>
<td>$Y(y) = K \cosh(\omega y) + L \sinh(\omega y)$</td>
</tr>
<tr>
<td>$\lambda &lt; 0$</td>
<td>$X(x) = I \cos(\omega x) + J \sin(\omega x)$</td>
<td>$Y(y) = K \cosh(\omega y) + L \sinh(\omega y)$</td>
</tr>
</tbody>
</table>

(b) Applying the boundary condition, depending on the sign of $\lambda$ yields

(1) If $\lambda = 0$

\[
\begin{align*}
B &= 0, \\
A + B &= 0, \\
C + D &= 0,
\end{align*}
\]

which admits the only trivial solution $A = B = 0$.

(2) If $\lambda = \omega^2 > 0$

\[
\begin{align*}
E &= 0, \\
E \cosh(\omega) + F \sinh(\omega) &= 0, \\
G \cos(\omega) + H \sin(\omega) &= 0,
\end{align*}
\]

which admits the only trivial solution $E = F = 0$.

(3) If $\lambda = -\omega^2 < 0$

\[
\begin{align*}
I &= 0, \\
I \cos(\omega) + J \sin(\omega) &= 0, \\
K \cosh(\omega) + L \sinh(\omega) &= 0,
\end{align*}
\]

which admits non trivial solution for $\omega_n = n\pi$ and $L = -\frac{K}{\tanh(\omega)}$.

Therefore, the solution writes

\[
u(x, y) = \sum_{n=1}^{\infty} K_n \sin(n\pi x) \left[ \cosh(n\pi y) - \frac{\sinh(n\pi y)}{\tanh(n\pi)} \right].
\]

Alternatively, to simplify the notation, one could also write the mode $Y(y)$ when $\lambda = -\omega^2 < 0$ under the form

\[
Y(y) = K \cosh(\omega(1 - y)) + L \sinh(\omega(1 - y)),
\]
such that the application of the boundary conditions simply gives

\[ K = 0. \]  (7.3.7)

With these notations, the solution writes

\[ u(x, y) = \sum_{n=1}^{\infty} L_n \sin(n \pi x) \sinh(n \pi (1 - y)). \]  (7.3.8)

It is this expression that will be used in the following because of its simplicity.

(c) The coefficient of the Fourier expansion of the function \( \phi(x) \) can be obtained by calculating the integrals

\[ 2 \int_0^1 \sin(n \pi x) \phi(x) \, dx, \]  (7.3.9)

which can be integrated by part.

(d) Applying the last boundary condition gives

\[ u(x, 0) = \phi(x), \]

\[ \Rightarrow \sum_{n=1}^{\infty} L_n \sin(n \pi x) \sinh(n \pi) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin ((2j + 1) \pi x)}{(2j + 1)^2}, \]  (7.3.10)

Identifying the terms in those series, one can find the expression for \( L_n \):

\[ L_n = \begin{cases} 0 & \text{if } n \text{ pair}, \\ -\frac{4}{\pi^2 n^2 \sinh(n \pi)} & \text{if } n \text{ odd and } n \mod 4 = 1, \\ \frac{4}{\pi^2 n^2 \sinh(n \pi)} & \text{if } n \text{ odd and } n \mod 4 = 3. \end{cases} \]  (7.3.11)

Replacing this in Eq. (7.3.8) and redefining the summation index, finally gives

\[ u(x, y) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin ((2j + 1) \pi x) \sinh ((2j + 1) \pi (1 - y))}{(2j + 1)^2 \sinh ((2j + 1) \pi)}. \]  (7.3.12)

7.4 Laplace equation II

Consider the following equation on a square

\[ u_{xx} + \frac{1}{4} u_{yy} = 0 \quad \{x, y\} \in ]0, 1[ \times ]0, 1[ \]

with the boundary conditions

\[ \begin{cases} u_x(0, y) = 0 & \forall y \in [0, 1], \\ u_x(1, y) = 0 & \forall y \in [0, 1], \\ u(x, 1) = 0 & \forall x \in [0, 1], \\ u(x, 0) = 2 \cos 2 \pi x - 1 & \forall x \in [0, 1]. \end{cases} \]

Using separation of variables \( u(x, y) = v(x)w(y) \), find the solution to this boundary value problem.
Solution

Using separation of variables, the equation can be written:

\[
\frac{X''}{X} = -\frac{Y''}{4Y} = \lambda. \tag{7.4.1}
\]

Therefore, both spatial part of the solution depends on the sign of the same constant \( \lambda \).

The following table presents all the separable solutions as a function of \( \lambda \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Part ( X(x) )</th>
<th>Part ( Y(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( X(x) = Ax + B )</td>
<td>( Y(y) = Cy + D )</td>
</tr>
<tr>
<td>0 ( \omega^2 ) &gt; 0</td>
<td>( X(x) = E \cosh(\omega x) + F \sinh(\omega x) )</td>
<td>( Y(y) = G \cos(2\omega y) + H \sin(2\omega y) )</td>
</tr>
<tr>
<td>0 ( \omega^2 ) &lt; 0</td>
<td>( X(x) = I \cos(\omega x) + J \sin(\omega x) )</td>
<td>( Y(y) = K \cosh(2\omega y) + L \sinh(2\omega y) )</td>
</tr>
</tbody>
</table>

Applying the homogeneous boundary conditions on each mode, depending on the sign of \( \lambda \), one has:

1. If \( \lambda = 0 \)

\[
\begin{cases}
A = 0, \\
A = 0, \\
C + D = 0,
\end{cases} \tag{7.4.2}
\]

which admits non trivial solutions for \( A = 0 \) and \( D = -C \). Redefining a new constant, the associated mode can be written

\[
X_0(x)Y_0(y) = C_0(y - 1). \tag{7.4.3}
\]

2. If \( \lambda = \omega^2 > 0 \)

\[
\begin{cases}
\omega F = 0, \\
\omega [E \sinh(\omega) + F \cosh(\omega)] = 0, \\
G \cos(2\omega) + H \sin(2\omega) = 0,
\end{cases} \tag{7.4.4}
\]

which admits the only trivial solution \( E = F = 0 \).

3. If \( \lambda = -\omega^2 < 0 \)

\[
\begin{cases}
J = 0, \\
\omega [I \sin(\omega) + J \cos(\omega)] = 0, \\
K \cosh(2\omega) + L \sinh(2\omega) = 0,
\end{cases} \tag{7.4.5}
\]
which admits non trivial solution for \( \omega_n = n\pi \) and \( L = -\frac{K}{\tanh(2\omega)} \).

Therefore, the solution writes

\[
\begin{align*}
  u(x, y) &= C_0(y - 1) + \sum_{n=1}^{\infty} K_n \cos(n\pi x) \left[ \cosh(2n\pi y) - \frac{\sinh(2n\pi y)}{\tanh(2n\pi)} \right]. \\
  \text{(7.4.6)}
\end{align*}
\]

Finally, applying the remaining boundary condition

\[
\begin{align*}
  u(x, 0) &= 2 \cos 2\pi x - 1, \\
  \Rightarrow - C_0 + \sum_{n=1}^{\infty} K_n \cos(n\pi x) &= 2 \cos(2\pi x) - 1. \\
  \text{(7.4.7)}
\end{align*}
\]

Identifying the coefficient, the final solution is given by

\[
\begin{align*}
  u(x, y) &= y - 1 + 2 \cos(2\pi x) \left[ \cosh(4\pi y) - \frac{\sinh(4\pi y)}{\tanh(4\pi)} \right]. \\
  \text{(7.4.8)}
\end{align*}
\]

### 7.5 Laplace-like equation

Consider the following equation on a square

\[
\begin{align*}
  u_{xx} + 2u_y + u_{yy} &= 0 \quad \{x, y\} \in [0, 1] \times [0, 1] \\
  \text{with the boundary conditions, for a given function } f,
\end{align*}
\]

\[
\begin{align*}
  \begin{cases}
    u(0, y) = 0 & \forall y \in [0, 1], \\
    u(1, y) = 0 & \forall y \in [0, 1], \\
    u(x, 1) = f(x) & \forall x \in [0, 1], \\
    u(x, 0) = 0 & \forall x \in [0, 1].
  \end{cases}
\end{align*}
\]

Using separation of variables \( u(x, y) = v(x)w(y) \), find the solution to this boundary value problem. As the function \( f \) is not specified, some constants remain in the final solution. Explain and show how to compute these constants when \( f \) is known.

**Solution**

Using separation of variables, the equation can be written

\[
\begin{align*}
  -\frac{X''}{X} = 2\frac{Y''}{Y} + \frac{Y''}{Y} &= \lambda. \\
  \text{(7.5.1)}
\end{align*}
\]

Therefore, both spatial part of the solution depend on the sign of the same constant \( \lambda \).

First focus on the equation for \( X(x) \)

\[
\begin{align*}
  \begin{cases}
    X'' + \lambda X = 0, \\
    X(0) = 0, \\
    X(1) = 0.
  \end{cases} \\
  \text{(7.5.2)}
\end{align*}
\]
This problem admits some not trivial solutions only for $\lambda = (n\pi)^2 > 0$, where $n$ is a positive integer. Those solution write

$$X_n(x) = \tilde{A}_n \sin(n\pi x). \quad (7.5.3)$$

After this, the equation for $Y$ writes

$$Y'' + 2Y' - (n\pi)^2 Y = 0. \quad (7.5.4)$$

This is an ODE with constant coefficients which can thus simply be solved using the exponential polynomial method. The general solution is given by

$$Y_n(y) = \exp(-y) \left[ C_n \sinh\left(\sqrt{(n\pi)^2 + 1} y\right) + D_n \cosh\left(\sqrt{(n\pi)^2 + 1} y\right) \right]. \quad (7.5.5)$$

Applying the last homogeneous boundary condition directly leads to $D_n = 0 \forall n$. Therefore, defining a new constants $A_n$, the solution writes

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \exp(-y) \sinh\left(\sqrt{(n\pi)^2 + 1} y\right). \quad (7.5.6)$$

The determination of the coefficients $A_n$ is obtained by applying the non homogeneous boundary condition to Eq.(7.5.6). It can easily be shown that the application of this boundary condition will lead to the decomposition of $f(x)$ in a sine Fourier series.

### 7.6 Helmholtz equation

Consider the two-dimensional Helmholtz equation on a square

$$u_{xx} + u_{yy} - u = 0 \quad \{x, y\} \in ]0, 1[ \times ]0, 1[$$

with the boundary conditions, for a given function $f$,

$$\begin{cases}
  u(0, y) = 0 & \forall y \in [0, 1], \\
  u(1, y) = 0 & \forall y \in [0, 1], \\
  u(x, 1) = f(x) & \forall x \in [0, 1], \\
  u(x, 0) = 0 & \forall x \in [0, 1].
\end{cases}$$

Using separation of variables $u(x, y) = v(x)w(y)$, find the solution to this boundary value problem. As the function $f$ is not specified, some constants remain in the final solution. Explain and show how to compute these constants when $f$ is known.

**Hint:** introduce two eigenvalues $k_x$ and $k_y$ such that $v''/v = \lambda_x$ and $w''/w = \lambda_y$.

**Solution**

Using the separation of variables $u(x, y) = X(x)Y(y)$, the Helmoltz equation can be written as

$$\frac{X''}{X} + \frac{Y''}{Y} - 1 = 0. \quad (7.6.1)$$
Introducing $\lambda_x = \frac{X''}{X}$ and $\lambda_y = \frac{Y''}{Y}$, Eq.(7.6.1) becomes
\[
\begin{cases}
X'' - \lambda_x X = 0, \\
Y'' - \lambda_y Y = 0, \\
\lambda_x + \lambda_y = 1,
\end{cases}
\]  \hspace{1cm} (7.6.2)
i.e.
\[
\begin{cases}
X'' - \lambda_x X = 0, \\
Y'' - (1 - \lambda_x)Y = 0.
\end{cases}
\]  \hspace{1cm} (7.6.3)
Each possible values for $\lambda_x$ should be investigated
(1) $\lambda_x = 0$
(2) $\lambda_x = -\omega_x^2 < 0$
(3) $0 < \lambda_x < 1$
(4) $\lambda_x = 1$
(5) $\lambda_x > 1$

However, due to the homogeneous Dirichlet boundary condition $u(0, y) = u(1, y) = 0$, $X(x) = 0$ \ $\forall \lambda_x \geq 0$. Only the value of $\lambda$ such that $\lambda_x = -\omega^2 < 0$ will lead to non trivial solutions. When $\lambda_x = -\omega^2 < 0$, the solutions for $X$ and $Y$ read
\[
\begin{cases}
X(x) = A \cos(\omega x) + B \sin(\omega x), \\
Y(y) = C \cosh \left(\sqrt{1 + \omega^2 y}\right) + D \sinh \left(\sqrt{1 + \omega^2 y}\right).
\end{cases}
\]  \hspace{1cm} (7.6.4)

Using the initial conditions gives $A = C = 0$ and
\[
\omega_n = n\pi, \quad n = 1, 2, \ldots
\]  \hspace{1cm} (7.6.5)

Finally, the solution to the problem can be written as
\[
u(x, y) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \sinh\left(\sqrt{1 + n^2 \pi^2 y}\right).
\]  \hspace{1cm} (7.6.6)

The constant $A_n$ can be inferred by using the remaining initial condition
\[
u(x, 1) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \sinh\left(\sqrt{1 + n^2 \pi^2}\right) = f(x),
\]
which is a sine Fourier expansion of $f(x)$. 
7.7 Diffusion equation II

Consider the one-dimensional diffusion equation on a bounded domain

\[ u_t - ku_{xx} = 0 \quad \forall x \in [0, 1]. \]  \hfill (\star)

(a) Using separation of variables \( u(x, t) = w(t)v(x) \), find all the separable solutions of Eq. (\star).

(b) Find the solution to Eq. (\star) for the homogeneous Robin boundary conditions at one end of the domain,

\[ u_x(1, t) + \beta u(1, t) = 0, \]

with a constant \( \beta \in \mathbb{R} \), the homogeneous Dirichlet boundary condition at the other end,

\[ u(0, t) = 0. \]

and an initial condition, for a given function \( \phi \),

\[ u(x, 0) = \phi(x) \quad \forall x \in [0, 1]. \]

In particular, show that the solution of this problem requires to solve the transcendental equation

\[ \omega = -\beta \tan \omega. \]

Draw a schematics and point out (qualitatively) the solutions of this equation. As the function \( \phi \) is not specified, some constants remain in the final solution. Explain why finding these constants is not immediate in this case.

(c) The domain in now extended to the interval \( [-1, 1] \). Find the solution to Eq. (\star) for the periodic boundary conditions

\[ u(-1, t) = u(1, t) \quad \text{and} \quad u_x(-1, t) = u_x(1, t), \]

and an initial condition, for a given function \( \phi \),

\[ u(x, 0) = \phi(x) \quad \forall x \in [0, 1]. \]

As the function \( \phi \) is not specified, some constants remain in the final solution. Explain and show how to compute these constants when \( \phi \) is known.

Solution

(a) Using the ansatz \( u = wv \), the diffusion equation writes as

\[ w'v - kwv'' = 0 \]

\[ \Rightarrow \quad \frac{w'}{kw} = \frac{v''}{v}. \]

Since the right hand side depends only on \( x \) while the left hand side depend only on \( t \), both sides must equal a constant (named \( \lambda \)), i.e

\[ v'' - \lambda v = 0 \quad \text{and} \quad w' - k\lambda w = 0. \]  \hfill (7.7.3)
Spatial dependency  Depending on the sign of \( \lambda \), three solutions arise, i.e

\[
\begin{align*}
\text{if } \lambda = 0 & \quad \Rightarrow v = Ax + B, \\
\text{if } \lambda = \omega^2 > 0 & \quad \Rightarrow v = C \exp(-\omega x) + D \exp(\omega x), \\
\text{if } \lambda = -\omega^2 < 0 & \quad \Rightarrow v = E \cos(\omega x) + F \sin(\omega x).
\end{align*}
\] (7.7.4) (7.7.5) (7.7.6)

Time dependency  Whatever is the sign of \( \lambda \), the time dependency is given by

\[
w = \exp(\lambda kt).
\] (7.7.7)

The constant in front the exponential is here omitted because the field of interest is \( u = vw \).

(b) Among all the eigensolutions found in the previous sub-question (i.e for any value of \( \lambda \)), only those that satisfy the homogeneous boundary conditions are kept.

Steady eigenvalues, \( \lambda = 0 \)  The boundary conditions yields

\[
\begin{align*}
\{ v(0) & = B = 0, \\
v_x(1) + \beta v(1) = 0 & = A(1 + \beta)
\end{align*}
\] (7.7.8) (7.7.9)

which implies

\[
A = 0 \quad \text{if } \beta \neq -1
\] (7.7.10)

but if \( \beta = -1 \) then \( v(x) = Ax \) satisfy both boundary condition.

Time growing eigenvalues, \( \lambda = \omega^2 > 0 \)  The boundary conditions yields

\[
\begin{align*}
\{ v(0) & = 0 = C + D, \\
v_x(1) + \beta v(1) = 0 & = -\omega C \exp(-\omega) + \omega D \exp(\omega x) + \beta (C \exp(-\omega) + D \exp(\omega))
\end{align*}
\] (7.7.11) (7.7.12)

which implies that

\[
C = D = 0 \quad \text{if } (\beta - \omega) \exp(-\omega) - (\beta + \omega) \exp(\omega) \neq 0
\] (7.7.13)

but if

\[
(\beta - \omega) \exp(-\omega) - (\beta + \omega) \exp(\omega) = 0 \quad \Rightarrow \omega = -\beta \tanh \omega
\] (7.7.14)

then \( v(x) = C (\exp(-\omega x) - \exp(\omega x)) \) is a solution. It can be shown that there exists a single \( \omega \) that verifies that equation only if \( \beta < -1 \).

Time decaying eigenvalues, \( \lambda = -\omega^2 < 0 \)  The boundary conditions once again yields

\[
\begin{align*}
v(0) & = E = 0 \\
v_x(1) + \beta v(1) & = 0 \quad \Rightarrow F (\omega \cos(\omega) + \beta \sin(\omega)) = 0
\end{align*}
\] (7.7.15) (7.7.16)

Once again, there is a non trivial solution only if

\[
(\omega \cos(\omega) + \beta \sin(\omega)) = 0 \Rightarrow \omega = -\beta \tan \omega.
\] (7.7.17)
It can be shown that these solutions are infinitely many. There are denoted by $\omega_n$.
The solution compatible with boundary conditions is then

$$u(x, t) = Ax + C (\exp -\omega_0 x - \exp +\omega_0 x) \exp \omega_0^2 kt + \sum_{n=1}^{\infty} F_n \sin \omega_n x \exp -\omega_n^2 kt. \quad (7.7.18)$$

Finally the initial condition must be imposed, i.e

$$\phi(x) = Ax + C (\exp -\omega_0 x - \exp +\omega_0 x) + \sum_{n=1}^{\infty} F_n \sin \omega_n x \quad (7.7.19)$$

which is not a Fourier series so the identification is tricky and not necessarily possible.

### 7.8 Wave equation II

Consider the two-dimensional wave equation inside an open subset $D$ of $\mathbb{R}^2$ (in this exercise, $D$ will either be a square or a disk),

$$u_{tt} - c^2 \Delta u = 0 \quad \forall \{x, y\} \in D,$$

with Dirichlet condition on the boundary $\partial D$ of $D$

$$u(x, y, t) = 0 \quad \forall \{x, y\} \in \partial D$$

and with initial conditions

$$u(x, y, 0) = \phi(x, y) \quad \text{and} \quad u_t(x, y, 0) = \psi(x, y).$$

The Laplace operator is denoted by $\Delta$. In Cartesian coordinate, $\Delta u = u_{xx} + u_{yy}$, while in polar coordinates $\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}$.

(a) Using separation of variables $u(x, y, t) = w(t)v(x)q(y)$, find the solution to this boundary value problem if $D = ]0, 1[ \times ]0, 1[$. As the functions $\phi$ and $\psi$ are not specified, some constants remain in the final solution. Explain how to compute these constants when $\phi$ and $\psi$ are known. What is the physical interpretation of any term in the final sum taken individually? How do these terms evolve with time?

(b) Using separation of variables $u(x, y, t) = w(t)v(r)h(\theta)$, find the solution to this boundary value problem if $D = \{x, y \mid x^2 + y^2 < 1\}$, i.e, if $D$ is a disk of radius 1. As the functions $\phi$ and $\psi$ are not specified, some constants remain in the final solution. Explain how to compute these constants when $\phi$ and $\psi$ are known. Make the parallel with (a) as far as eigenfunctions and time evolution is concerned.

**Solution**

(a) Using the ansatz $u = wvq$, the wave equation writes as

$$\frac{w''}{c^2w} = \frac{v''}{v} + \frac{q''}{q}. \quad (7.8.1)$$
Since the right hand side depends only on $x$ and $y$ while the left hand side depends only on $t$, each of the three terms in Eq.(7.8.1) must be a constant. These constant are respectively called $\lambda$, $\lambda_x$ and $\lambda_y$. The system to solve is then

\[
\begin{align*}
    v'' - \lambda_x v &= 0, \\
    q'' - \lambda_y q &= 0, \\
    w'' - c^2 \lambda w &= 0
\end{align*}
\tag{7.8.2}
\]

with

\[
\lambda_x + \lambda_y = \lambda.
\tag{7.8.5}
\]

**Time dependency ($t$)** Depending on the sign of $\lambda$, three solutions arise, i.e

- If $\lambda = 0$ \quad \Rightarrow w = A_t t + B_t, \tag{7.8.6}
- If $c^2 \lambda = -\omega^2 < 0$ \quad \Rightarrow v = E_t \cos \omega t + F_t \sin \omega t. \tag{7.8.8}
- If $c^2 \lambda = \omega^2 > 0$ \quad \Rightarrow w = C_t \exp -\omega t + D_t \exp \omega t. \tag{7.8.7}

**Spatial dependency ($x$)** Depending on the sign of $\lambda_x$, three solutions arise, i.e

- If $\lambda_x = 0$ \quad \Rightarrow v = A_x x + B_x, \tag{7.8.9}
- If $\lambda_x = k_x^2 > 0$ \quad \Rightarrow v = C_x \exp -k_x x + D_x \exp k_x x, \tag{7.8.10}
- If $\lambda_x = -k_x^2 < 0$ \quad \Rightarrow v = E_x \cos k_x x + F_x \sin k_x x. \tag{7.8.11}

The same solutions appears for the spatial dependency in $y$.

Among all the eigensolutions found previously, only those satisfying the homogeneous boundary condition are kept. Using the ansatz, the boundary conditions become

\[
\begin{align*}
    v(0) &= v(1) = 0 \\
    q(0) &= q(1) = 0
\end{align*}
\tag{7.8.12, 7.8.13}
\]

**Stationary eigensolutions, i.e $\lambda_x = 0$** Applying the boundary conditions gives $A_x = B_x = 0$ and similarly $A_y = B_y = 0$.

**Evanescent eigensolutions, i.e $\lambda_x > 0$** Applying the boundary conditions gives $C_x = D_x = 0$ and similarly $C_y = D_y = 0$.

**Propagating eigensolutions, i.e $\lambda_x < 0$** Applying the boundary condition gives

\[
v_n(x) = F_x^n \sin k_x^n x \quad \text{with} \quad k_x^n = n\pi \quad n = 1, 2, 3, \ldots
\tag{7.8.14}
\]

and

\[
q_m(x) = F_y^m \sin k_y^m y \quad \text{with} \quad k_y^m = m\pi \quad m = 1, 2, 3, \ldots
\tag{7.8.15}
\]

Because

\[
\lambda = \lambda_x + \lambda_y,
\tag{7.8.16}
\]
the pulsation are given by
\[ \omega_{m,n} = c\sqrt{m^2 + n^2 \pi}. \] (7.8.17)

The most general solution compatible with boundary condition is then
\[
\begin{align*}
  u(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F^n F^m \sin n\pi x \sin n\pi y \left( A_{m,n} \cos \left( c\sqrt{m^2 + n^2 \pi t} \right) + B_{m,n} \sin \left( c\sqrt{m^2 + n^2 \pi t} \right) \right) \\
  &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A'_{m,n} \sin n\pi x \sin m\pi y \cos \left( c\sqrt{m^2 + n^2 \pi t} \right) \\
  &\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B'_{m,n} \sin n\pi x \sin m\pi y \sin \left( c\sqrt{m^2 + n^2 \pi t} \right).
\end{align*}
\] (7.8.18)

Finally using the initial condition, one finds
\[
\begin{align*}
  u(x, y, 0) &= \phi(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A'_{m,n} \sin n\pi x \sin m\pi y \\
  u_t(x, y, 0) &= \psi(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B'_{m,n} \sin n\pi x \sin m\pi y \sin \left( c\sqrt{m^2 + n^2 \pi t} \right).
\end{align*}
\] (7.8.20-7.8.21)

Then using the result from 2D Fourier sine series, the coefficients are given by
\[
\begin{align*}
  A'_{m,n} &= 4 \int_{0}^{1} \int_{0}^{1} \phi(x, y) \sin n\pi x \sin m\pi y \, dx \, dy \\
  B'_{m,n} &= 4 \int_{0}^{1} \int_{0}^{1} \psi(x, y) \sin n\pi x \sin m\pi y \, dx \, dy.
\end{align*}
\] (7.8.23-7.8.24)

From Eq.(7.8.19), it can be seen that the solution is a superposition of modes of increasing spatial frequencies, \( k^x_n = n\pi \) and \( k^y_m = m\pi \) and of increasing time pulsation \( \omega_{m,n} = c\sqrt{m^2 + n^2 \pi} \). At the opposite of the heat equation, none of the modes are damped. Consequently, the discontinuity and the high frequency variations of the initial data are conserved and not smoothed as it is the case for the heat equation.

(b) If \( D \) is a cylinder, the problem written in cylindrical coordinates is
\[
\begin{align*}
  u_{tt} - u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0 \quad \forall \{\theta, r\} \in [0, 2\pi] \times [0, 1] \\
  u(1, \theta, t) &= 0 \quad \forall \theta \in [0, 2\pi].
\end{align*}
\] (7.8.25-7.8.26)
Moreover, to ensure that the solution is single valued,

\[ u(r, \theta + 2\pi, t) = u(r, \theta, t) \]  \hspace{1cm} (7.8.27)

must be verified.

As previously, the initial conditions remain

\[ u(r, \theta, 0) = \phi(r, \theta) \quad \text{and} \quad u_t(r, \theta, 0) = \psi(r, \theta) \quad \forall \{r, \theta\} \in D. \]  \hspace{1cm} (7.8.28)

Using the ansatz \( u = wgh \), the wave equation writes as

\[
\frac{w''}{c^2w} = \frac{1}{gh} \left( hg'' + \frac{1}{r} hg' + \frac{1}{r^2} gh'' \right). \]  \hspace{1cm} (7.8.29)

The left hand side only depends on \( t \) while the right hand side only depends on \( r \) and \( \theta \) such that both sides must equal the same constant called \( \lambda \). The wave equation then becomes

\[
\begin{aligned}
)& r^2 hg'' + rhg' + gh'' = r^2 \lambda gh, \\
)& r^2 \frac{g''}{g} + r \frac{g'}{g} - r^2 \lambda + \frac{h''}{h} = 0.
\end{aligned} \]  \hspace{1cm} (7.8.30, 7.8.31)

Once again each term should equal a constant, denoted by \(-\mu\) because they only depend on \( r \) or \( \theta \). The system to solve is then

\[
\begin{aligned}
)& w'' - c^2 \lambda w = 0, \\
)& h'' + \mu h = 0, \\
)& r^2 g'' + rg' + (-r^2 \lambda - \mu) g = 0.
\end{aligned} \]  \hspace{1cm} (7.8.32, 7.8.33, 7.8.34)

**Time dependency** Depending on \( \lambda \), this equations yields different time behaviours. However here, only propagating modes are considered, i.e \( \lambda = -\omega^2 < 0 \). To formally reject \( \lambda \geq 0 \), the same procedure as in the previous sub-question should be done, i.e show that the spatial part associated with steady and evanescent modes does not satisfy the boundary conditions.

For propagating modes, the time dependency is

\[ w(t) = A_t \cos c\omega t + B_t \sin c\omega t. \]  \hspace{1cm} (7.8.35)

**Azimuthal dependency** The only solution that ensures the unicity is \( \mu = m^2 \) where \( m \) is an integer, the solution is then

\[ h(\theta) = A_{\theta} \cos m\theta + B_{\theta} \sin m\theta. \]  \hspace{1cm} (7.8.36)

**Radial dependency** The radial equation is now

\[ r^2 g'' + rg' + (r^2 \omega^2 - m^2) g = 0. \]  \hspace{1cm} (7.8.37)

Now consider the change of variable \( x = \omega r \), the equation becomes

\[ x^2 g'' + xg' + (x^2 - m^2) g = 0. \]  \hspace{1cm} (7.8.38)
This equation is known as the **Bessel’s equation**. Solving this equations is hard and outside the scope of these exercises.

The solution of this equation can be expressed as the superposition of the Bessel function of the first kind of order \( m \), \( J_m(x) \), and of the Bessel function of the second kind of order \( m \), \( Y_m(x) \), i.e.

\[
g(x) = A_x J_m(x) + B_x Y_m(x)
\]

The boundary condition are \( g(1) = 0 \) and \( |g(0)| < \infty \). Thus \( B_x = 0 \) because \( Y_m(x) \) is unbounded at the origin. Then the Dirichlet boundary condition yields

\[
g(r = 1) = 0 = J_m(x(1)) = J_m(\omega) = 0.
\]

The \( n \)th eigenvalues that satisfy \( J_m(\omega) = 0 \) is denoted by \( \omega_{m,n} \). It can be shown that there are an infinite numbers of such values, the root \( \omega_{m,n} = 0 \) (i.e for \( n = 0 \)) should however be withdrawn because \( \omega > 0 \).

The solution then writes

\[
u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [A_{t,m,n}^n \cos c\omega_{m,n}t + B_{t,m,n}^n \sin c\omega_{m,n}t] [A_{m,n}^m \cos m\theta + B_{m,n}^m \sin m\theta] J_m(\omega_{m,n}r)
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [A_{m,n}^m \cos m\theta + B_{m,n}^m \sin m\theta] J_m(\omega_{m,n}r) \cos c\omega_{m,n}t
\]

\[
+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [A_{m,n}^m \cos m\theta + B_{m,n}^m \sin m\theta] J_m(\omega_{m,n}r) \sin c\omega_{m,n}t.
\]

It is interesting to point out that the values \( m = 0 \) yields radially symmetric eigenfunctions for all values of \( n \).

As previously, the constant can be withdrawn from the initial conditions

\[
u(r, \theta, 0) = \phi(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [A_{m,n}^m \cos m\theta + B_{m,n}^m \sin m\theta] J_m(\omega_{m,n}r)
\]

\[
u_t(r, \theta, 0) = \psi(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \omega_{m,n} [A_{m,n}^m \cos m\theta + B_{m,n}^m \sin m\theta] J_m(\omega_{m,n}r)
\]

Previously, the initial data are decomposed into a Fourier sine series. In this case, the basis used is slightly more complicated and one has to decomposed the initial data in a so called Fourier-Bessel basis. As in the case of Fourier sine or cosine series, here the coefficient of the Fourier-Bessel series are given by

\[
A_{m,n}^m = \frac{2}{\pi (J_{m+1}(\omega_{m,n}))^2} \int_{0}^{\pi} \int_{-\pi}^{\pi} \phi(r, \theta) J_m(\omega_{m,n}r) r \cos m\theta d\theta dr,
\]

\[
B_{m,n}^m = \frac{2}{\pi (J_{m+1}(\omega_{m,n}))^2} \int_{0}^{\pi} \int_{-\pi}^{\pi} \phi(r, \theta) J_m(\omega_{m,n}r) r \sin m\theta d\theta dr
\]

and

\[
\omega_{m,n} A_{m,n}^m = \frac{2}{\pi (J_{m+1}(\omega_{m,n}r))^2} \int_{0}^{\pi} \int_{-\pi}^{\pi} \psi(r, \theta) J_m(\omega_{m,n}r) r \cos m\theta d\theta dr,
\]

\[
\omega_{m,n} B_{m,n}^m = \frac{2}{\pi (J_{m+1}(\omega_{m,n}r))^2} \int_{0}^{\pi} \int_{-\pi}^{\pi} \psi(r, \theta) J_m(\omega_{m,n}r) r \sin m\theta d\theta dr.
\]
7.9 Starting flow in circular pipe

Consider an infinite circular pipe of radius $a$, filled with an incompressible and Newtonian fluid of density $\rho$ and dynamic viscosity $\mu$. Consider a cylindrical coordinate system, centered in the pipe, the $z$-direction is parallel to the pipe. The only non-zero velocity component of the fluid $u$ is in the $z$-direction. With the azimuthal symmetry and invariance along the $z$-direction, Navier-Stokes momentum equation for the velocity $u = u(r, t)$ inside the pipe writes

$$\frac{\partial u}{\partial t} - \nu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = -\frac{p_z}{\rho}, \quad \text{with} \quad (r, t) \in ]0, a[, \times ]0, +\infty[,$$

with $\nu = \mu/\rho$ the kinematic viscosity and $p_z$ a constant pressure gradient in the $z$-direction. The boundary conditions are $u(0, t) < \infty$ and $u(a, t) = 0$, $\forall t \in ]0, +\infty[$. The initial condition is $u(r, 0) = 0$, $\forall r \in ]0, a[.$

(a) Show that the steady-state velocity $u_s = u_s(r)$ is given by

$$u_s(r) = \frac{p_z}{4\mu} (r^2 - a^2).$$

(b) Rewrite the initial boundary value problem in terms of $v(r, t) = u(r, t) - u_s(r)$.

(c) Using the separation of variables $v(r, t) = R(r)T(t)$, give the transient solution to the problem.

Help: The equation $x^2 y'' + xy' + (x^2 - m^2)y = 0$ is the Bessel’s equation of order $m \in \mathbb{C}$. For a given $m$, this equation has two linearly independent solutions: $J_m$ the Bessel function of the first kind, which has a finite value at $x = 0$, and $Y_m$ the Bessel function of the second kind, which diverges at $x = 0$. Both functions have an infinite number of roots. See Fig. 6. Some useful properties of the $J_m$ are, with $\alpha_k$ the $k^{th}$ non-zero root of $J_m$,

$$J_{m-1}(x) = \frac{2m}{x} J_m(x) - J_{m+1}(x), \quad x^m J_{m-1}(x) = \frac{d}{dx} \left( x^m J_m(x) \right), \quad J_m(0) = 0, \forall m \in \mathbb{N}_0,$$

$$\int_0^1 x J_m(\alpha_k x) J_m(\alpha_l x) \, dx = \begin{cases} 0, & \text{if } k \neq l, \\ \frac{J_{m+1}^2(\alpha_k)}{2}, & \text{if } k = l. \end{cases}$$

The equation $x^2 y'' + xy' - (x^2 + n^2)y = 0$ is the modified Bessel’s equation of order $n \in \mathbb{C}$. For a given $n$, it has two linearly independent solutions: $I_n$ the modified Bessel function of the first kind, which has a finite value at $x = 0$ and no non zero root, and $K_n$ the modified Bessel function of the second kind, which diverges at $x = 0$ and has no roots. Unlike the ordinary Bessel functions $J_m$ and $Y_m$, the modified Bessel functions are exponentially growing or decaying, they exhibit no oscillations.
Solution (brief solution only)

(a) Impose $\partial_t = 0$ and solve the resulting ODE.

(b) The diffusion equation writes $v_t - \nu (r^{-1} v_r + v_{rr}) = 0$. The boundary conditions write $v(0, t) < \infty$ and $v(a, t) = 0$, $\forall t \in ]0, +\infty[$. And the initial condition is $v(r, 0) = -u_s(r)$, $\forall r \in ]0, a[$.

(c) The separation of variables yields one ODE for $R$ and one ODE for $T$, with a separation constant $\lambda^2 \in \mathbb{R}$. Three cases are discussed for the sign of $\lambda^2$.

Stationary solution: $\lambda = 0$. We get $R(r) = B \ln r + C$, for $B, C \in \mathbb{R}$, which diverges at $r = 0$ so the identification with the initial condition will give $B = 0$. The boundary condition at $r = a$ yields $C = 0$.

Time-growing solutions: $\lambda^2 < 0$. We get the modified Bessel’s equation of order 0 after a change of variables. Because the solutions $I_0$ and $K_0$ have no non zero roots, we cannot satisfy the boundary conditions with them and the associated constants are zero. Time-growing solutions thus do not contribute to the final solution (which makes sense physically).

Time-decaying solutions: $\lambda^2 > 0$. We get the Bessel’s equation of order 0, whose solutions are $J_0$ and $Y_0$. The boundary condition at $r = 0$ discards the solution $Y_0$. The no-slip boundary condition $v(a, t) = 0$, $\forall t \in ]0, +\infty[$, forces $\lambda$ to have discrete values, related to the roots of $J_0$.

Before applying the initial condition, the most general solution expresses

$$v(r, t) = \sum_{k=1}^{\infty} A_k J_0 \left( \frac{\lambda_k r}{\sqrt{\nu}} \right) \exp \left( -\lambda_k^2 \frac{t}{\nu} \right),$$

(7.9.1)

with $\lambda_k = \alpha_k \sqrt{\nu}/a$, $\alpha_k$ the $k$th root of $J_0$, and the $A_k$ constants to be determined with the initial condition:

$$\sum_{k=1}^{\infty} A_k J_0 \left( \frac{\lambda_k r}{\sqrt{\nu}} \right) = -\frac{p_z}{4\mu} (r^2 - a^2).$$

(7.9.2)
Multiplying this condition by \( r J_0(\alpha_l r/a) \), \( l \in \mathbb{N}_0 \), integrating over \([0,a]\), and using the orthogonality property yields

\[
\int_0^a r J_0(\alpha_l r/a) A_l J_0(\alpha_l r/a) dr = - \int_0^a r J_0(\alpha_l r/a) \frac{P_z}{4\mu} (r^2 - a^2) dr. \tag{7.9.3}
\]

The change of variable \( x = r/a \) and the norm property leads to the following integral for computing each \( A_l \):

\[
A_l = -\frac{a^2 p_z}{2\mu} \frac{1}{J_1^2(\alpha_l)} \int_0^1 x(x^2 - 1) J_0(\alpha_l x) dx. \tag{7.9.4}
\]

Using the given recurrence relation, we have \( x^3 J_0(\alpha_l x) = 2x^2 J_1(\alpha_l x)/\alpha_l - x^3 J_2(\alpha_l x) \). The given derivative property and the fact that \( J_m(0) = 0 \), \( \forall m \in \mathbb{N}_0 \), allows to compute directly the integral. We get

\[
A_l = -\frac{a^2 p_z}{2\mu} \frac{1}{J_1^2(\alpha_l)} \left( \frac{2J_2(\alpha_l)}{\alpha_l^2} - \frac{J_3(\alpha_l)}{\alpha_l} - \frac{J_1(\alpha_l)}{\alpha_l} \right), \tag{7.9.5}
\]

and with \( J_1(x) + J_3(x) = 4J_2(x)/x \), then with \( J_2(\alpha_l) = 2J_1(\alpha_l)/\alpha_l \) (because \( J_0(\alpha_l) = 0 \), by definition of \( \alpha_l \) being a root of \( J_0 \)) we finally get

\[
A_l = \frac{a^2 p_z}{\mu} \frac{J_2(\alpha_l)}{\alpha_l^2 J_1(\alpha_l)} = \frac{2a^2 p_z}{\mu \alpha_l^3 J_1(\alpha_l)}, \tag{7.9.6}
\]

such that the transient solution writes (see Fig. 7 for an illustration of the solution)

\[
u \frac{\mu}{P_z a^2} u(r,t) = \frac{P_z}{4\mu} (r^2 - a^2) + \sum_{k=1}^{\infty} \frac{2a^2 p_z}{\mu \alpha_k^3 J_1(\alpha_k)} J_0\left( \frac{\alpha_k r}{a} \right) \exp\left( -\frac{\alpha_k^2 \nu t}{a^2} \right), \text{ with } \alpha_k \text{ the } k^{th} \text{ root of } J_0(x). \tag{7.9.7}
\]

![Figure 7: Velocity profile at different time instants.](image-url)
7.10 Spherical harmonics [Olver, Sec. 12.2]

Consider the three-dimensional Laplace equation inside the unit ball $B = \{ r \leq 1 \}$, in spherical coordinates $(r, \theta, \phi) \in [0, 1] \times [0, \pi] \times [-\pi, \pi]$,

$$
\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin^2 \theta} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0,
$$

with arbitrary boundary conditions of the sphere $r = 1$.

**Warning**: In this course (as in other courses), $\theta$ is the zenith angle (or latitude angle) and $\phi$ is the azimuthal angle (or longitude angle). The associated volume Jacobian is $r^2 \sin \theta$. This convention is usually used in physics. In mathematics, the opposite convention is often used (as in the Olver reference). Be very careful when comparing different formulae.

In order to solve this problem, follow successively the steps below. Explain your developments. (You are not asked to show that the various sets of functions form bases of the associated vector spaces.)

(a) Using the separation of variables $u(r, \theta, \phi) = v(r)p(\theta)q(\phi)$, show that we obtain the following three ordinary differential equations for $v$, $p$ and $q$,

\[
\begin{cases}
  r^2 v'' + 2rv' - \mu v = 0, \\
  \sin^2 \theta \ p'' + \sin \theta \cos \theta \ p' + (\mu \sin^2 \theta - m^2) p = 0, \\
  q'' + m^2 q = 0.
\end{cases}
\]

(b) Solve the azimuthal equation for $q(\phi)$. Remember to impose the $2\pi$-periodicity condition, explain why it is necessary.

(c) Solve the zenith equation for $p(\theta)$, with the following steps:

- Use the change of variable $t = \cos \theta$ to write the equation for $p$ in a more convenient form, with $p(\theta) = P(\cos \theta) = P(t)$. The obtained equation is the Legendre equation of order $m$. Combined with the boundary conditions $|P(-1)| < \infty$ and $|P(1)| < \infty$, this yields the order $m$ Legendre boundary value problem.

- As a lemma for the next questions, show by induction that the function $Q_n(t) = (1 - t^2)^n$, $n = 0, 1, 2 \ldots$, verifies the induction equation, with the notation $Q_n^{(k)}(t) = \frac{d^k Q_n(t)}{dt^k}$,

\[
(1 - t^2)Q_n^{(k+2)} = -2(n - k - 1) t Q_n^{(k+1)} - (k + 1)(2n - k)Q_n^{(k)}, \quad \forall k = 0, 1, 2 \ldots
\]

- Show that, in the particular case $m = 0$, $Q_n^{(n)}(t)$ solve the Legendre boundary value problem for the particular value $\mu = n(n+1)$. Consequently, the functions

\[
P_n(t) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dt^n} (1 - t^2)^n, \quad \forall n \in \mathbb{N},
\]

are also solutions order 0 Legendre boundary value problem. These functions are the Legendre polynomials. The factor is a common convention.
• In the general case \( m \geq 0 \), the solutions are the associated Legendre functions, or Ferrers functions, defined by

\[
P_n(t) = (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_n(t) = (-1)^n \frac{(1 - t^2)^{m/2}}{2^m n!} \frac{d^{n+m}}{dt^{n+m}} (1 - t^2)^n, \quad n = m, m+1, \ldots.
\]

Show that these functions are indeed solutions to the order \( m \) Legendre boundary value problem, by writing the induction equation with \( k = m+n \) and the substitution \( Q_{n}^{(m+n)}(t) = (1 - t^2)^{-m/2} S_n^m(t) \).

One can show that the Ferrers functions provide a complete list of solutions to the order \( m \) Legendre boundary value problem. It has the eigenvalues \( \mu_n = n(n+1) \) for \( n \in \mathbb{N} \) and the associated eigenfunctions \( P_n^m(t) \), with \( m = 0, \ldots, n \) (notice that \( m \) is at most \( n \)). The Ferrers eigenfunctions form an orthogonal basis relative to the \( L^2 \) inner product on \([-1, 1]\).

• Express (directly) the Ferrers functions in terms of the original variable \( \theta \) of the problem to get the zenith function \( p_n^m(\theta) \).

(d) Combine the azimuthal and zenith solution to get the spherical harmonics:

\[
Y_n^m(\theta, \phi) = p_n^m(\theta) \cos m\phi,
\]

\[
\tilde{Y}_n^m(\theta, \phi) = p_n^m(\theta) \sin m\phi,
\]

with \( n = 0, 1, 2, \ldots \) and \( m = 0, 1, \ldots, n \). Following the steps below, show that these functions are orthogonal with respect to the classical inner product \( \langle \cdot, \cdot \rangle = \int\int \cdot \sin \theta d\theta d\phi \) on the sphere:

• Show that the spherical Laplacian \( \Delta_S \), defined by,

\[
\Delta_S(w) = \frac{\partial^2 w}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial w}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 w}{\partial \phi^2},
\]

for a function \( w = w(\theta, \phi) \), is a self-adjoint operator with respect to the inner product \( \langle \cdot, \cdot \rangle \).

Reminder: an operator \( L \) is self-adjoint if \( \langle u, L(v) \rangle = \langle L(u), v \rangle \) for any vector \( u \) and \( v \).

• Show that the spherical harmonics are the eigenvectors of the eigenvalue problem

\[
\Delta_S(w) = -\mu_n w.
\]

• In general, show that eigenfunctions associated with different eigenvalues of a self-adjoint operator are mutually orthogonal.

• Using the results above and a direct calculation for the remaining cases, show that all spherical harmonics are orthogonal.

The norms of these functions are given by (if interested, see the nice proof in Olver, exercise 12.2.16):

\[
\|Y_n^0\|^2 = \frac{4\pi}{2n+1}, \quad \|Y_n^m\|^2 = \|\tilde{Y}_n^m\|^2 = \frac{2\pi(n+m)!}{(2n+1)(n-m)!}, \quad m = 0, 1, \ldots, n.
\]

Finally, one can also show that these functions form a complete orthogonal system of functions on the unit sphere (this is not easy).
(e) Solve the radial equation for \( v(r) \). Discard unbounded solutions at \( r \to 0 \) and combine them with the spherical harmonics to get the harmonic polynomials. Show that they are expressed as

\[
H_n^m = r^n Y_n^m(\theta, \phi),
\]

\[
\tilde{H}_n^m = r^n \tilde{Y}_n^m(\theta, \phi).
\]

Show that, in particular, \( H_0^0, H_0^1, H_1^1, \tilde{H}_1^1 \) and \( H_0^2 \) are indeed polynomials when expressed in Cartesian coordinates. This is actually the case for all of them (try some other ones). It is also possible to show that these functions form a complete system in the unit ball.

(f) Because the harmonic polynomials form a complete system in the unit ball, any (sufficiently smooth) harmonic function \( u(r, \theta, \phi) \) can be expressed as (1/2 factors for convenience)

\[
u(r, \theta, \phi) = \frac{c_{0,0}}{2} + \sum_{n=1}^\infty \left( \frac{c_{0,n}}{2} r^n Y_n^0(\theta, \phi) + \sum_{m=1}^{n} \left( c_{m,n} r^n Y_n^m(\theta, \phi) + \tilde{c}_{m,n} r^n \tilde{Y}_n^m(\theta, \phi) \right) \right),
\]

with coefficients \( c_{m,n} \) and \( \tilde{c}_{m,n} \) to be fixed by the boundary conditions. Explain how to compute them for a Dirichlet boundary condition \( u(1, \theta, \phi) = h(\theta, \phi) \) on the unit sphere.

(g) How is the final solution modified if the mathematical domain is outside the unit ball?

**Solution (brief solution only)**

(a) Algebraic manipulations only.

(b) The periodicity condition is necessary to ensure the matching of the solution at \( \phi = -\pi \) and \( \phi = \pi \). The solution is indeed asked to be continuous. If \( m^2 > 0 \), the solutions are \( q(\phi) = \cos m\phi \) and \( \sin m\phi \), with \( m \) an integer, that can be chosen non-negative without loss of generality. If \( m^2 < 0 \), the solutions are real exponentials and no non-zero solution is periodic. Therefore, eigenfunctions exist only for the eigenvalues \( m \in \mathbb{N} \).

(c)  
- The Legendre equation of order \( m \) writes, for \( p(\theta) = P(\cos \theta) = P(t) \),

\[
(1 - t^2) P''(t) - 2t(1 - t^2) P'(t) + \left( \mu(1 - t^2) - m^2 \right) P(t) = 0, 
\]

with \( t \in [-1, 1] \) (because \( \theta \in [0, \pi] \)).
- Show this is valid for \( k = 0 \); then show that if it is valid for a given \( k \), then this is also valid for \( k + 1 \); conclude (do not forget to conclude!).
- Immediate.
- Follow the prescribed steps.
- Immediate.

(d)  
- This follows by integration by parts. Using

\[
vw_{\theta\theta} = (vw_{\theta})_\theta - v_{\theta}w_\theta = (vw_\theta - v_\theta w)_\theta + v_{\theta\theta}w,
\]

\[
\sin \theta (vw_\theta - v_\theta w)_\theta = (\sin \theta (vw_\theta - v_\theta w))_\theta - \cos \theta (vw_\theta - v_\theta w)
\]
as well as the fact that \( |\sin \theta \, (vw_\theta - v_\theta w)|_0^2 = 0 \) (trivial, because \( v \) and \( w \) are bounded) and

\[
[(vw_\phi - v_\phi w)]_\pi^\pi = 0
\]

(by periodicity), we have

\[
\langle v, \Delta_S(w) \rangle = \int \int \left( \sin \theta \, v \, w_{\theta \theta} + \cos \theta \, v \, w_\theta + \frac{1}{\sin \theta} \, v \, w_{\phi \phi} \right) \, d\theta d\phi,
\]

(7.10.5)

\[
= \int \int \left( \sin \theta \, v_{\theta \theta} + \cos \theta \, v_\theta w + \frac{1}{\sin \theta} \, v_{\phi \phi} \right) \, d\theta d\phi
\]

(7.10.6)

\[
= \langle \Delta_S(v), w \rangle.
\]

(7.10.7)

- Direct from the separation of variables \( u(r, \theta, \phi) = v(r)w(\theta, \phi) \), the resulting angular equation is the eigenvalue problem under consideration whose solutions have been found above, they are the spherical harmonics.

- For a self-adjoint operator \( L \) and eigenfunctions \( v_i \) and \( v_j \) associated with eigenvalues \( \lambda_i \) and \( \lambda_j \), respectively, we have

\[
\langle v_j, L(v_i) \rangle = \langle L(v_j), v_i \rangle,
\]

(7.10.8)

\[
\Leftrightarrow \langle \lambda_j v_j, v_i \rangle = \langle v_j, \lambda_i v_i \rangle,
\]

(7.10.9)

\[
\Leftrightarrow (\lambda_i - \lambda_j) \langle v_j, v_i \rangle = 0,
\]

(7.10.10)

and thus if \( \lambda_i \neq \lambda_j \), the eigenvectors are orthogonal.

- The orthogonality relations are \( \int \int Y_n^m Y_{k}^k \, dS = \delta_{nk} \delta_{mk} \), \( \int \int \bar{Y}_n^m \bar{Y}_{k}^k \, dS = \delta_{nk} \delta_{mk} \) and \( \int \int Y_n^m \bar{Y}_{k}^k \, dS = 0 \), \( \forall n, m, l, k \). If \( n \neq l \), the functions are eigenfunctions associated with different eigenvalues \( (\mu_n \) and \( \mu_l) \) of a self-adjoint operator \( \Delta_S \). Therefore, they are orthogonal (with respect to the same inner product, of course). If \( n = l \) but \( m \neq k \), the azimuthal integration gives zero for the first two orthogonality relations; and if \( n = l \), the azimuthal integration gives zero for all couples \( (m, k) \) for the third orthogonality relation.

(e) Radial solutions: \( r^n \) and \( r^{-n-1} \). Must be bounded for \( r \to 0 \) so that we only keep \( r^n \). We have, by simple change of coordinates and algebraic manipulations, \( H_0^0 = 1 \), \( H_1^0 = z \), \( H_1^1 = x \), \( H_1^2 = y \) and \( H_2^0 = z^2 - 0.5x^2 - 0.5y^2 \).

(f) The boundary condition expresses

\[
u(1, \theta, \phi) = \frac{c_{0,0}}{2} + \sum_{n=1}^{\infty} \left( \frac{c_{0,n}}{2} Y_n^0(\theta, \phi) + \sum_{m=1}^{n} \left( c_{m,n} Y_n^m(\theta, \phi) + \tilde{c}_{m,n} \bar{Y}_n^m(\theta, \phi) \right) \right) = h(\theta, \phi),
\]

(7.10.11)

and the coefficients \( c_{m,n} \) or \( \tilde{c}_{m,n} \) are the projection of \( h \) on the spherical harmonics \( Y_n^m \) and \( \bar{Y}_n^m \), respectively, taking care of the norms, as follows:

\[
c_{0,n} = \frac{2 \langle h, Y_n^0 \rangle}{\|Y_n^0\|^2}, \quad c_{m,n} = \frac{\langle h, Y_n^m \rangle}{\|Y_n^m\|^2}, \quad \tilde{c}_{m,n} = \frac{\langle h, \bar{Y}_n^m \rangle}{\|Y_n^m\|^2},
\]

(7.10.12)

with \( 0 \leq n \) and \( 1 \leq m \leq n \) (notice the squared norm).

(g) We would have kept \( r^{-n-1} \) instead of \( r^n \) for the radial solution, to ensure that the solution is bounded at infinity.
Figure 8: First few spherical harmonics, for information. What is represented is the absolute value of \( Y_n^m(\theta, \phi) \) in the angular direction \((\theta, \phi)\). Blue indicates positive regions and green indicates negative regions. Try to find the \((n, m)\) couple for each graph.
8 Nonlinear transport

The simplest nonlinear partial differential equation to solve is certainly the homogenous transport equation whose most general form is expressed as

\[ u_t + a(u)u_x = 0, \tag{\bigcirc} \]

or equivalently as

\[ \frac{d}{dt}[u(x(t), t)] = 0, \]

or even

\[ u_t + [A(u)]_x = 0, \]

with \( A'(u) = a(u) \).

Such equations can be solved with the characteristic lines technique. In certain cases, the characteristic lines can display an intersection point meaning that two different solutions are possible. One technique to circumvent such problem would be to allow a jump discontinuity in the solution, also called a shock.

When a shock is present, it is very important to notice that the PDE (\( \bigcirc \)) does not hold anymore, as the derivatives \( u_t \) and \( u_x \) stop existing at a discontinuity. Only the integral form of the equation on a compact \( C \) still holds:

\[ \frac{d}{dt} \int_C u \, dx = 0. \tag{\bigstar} \]

Therefore, a solution in the sense of distributions is necessary to solve the problem but its uniqueness would not be guaranteed anymore. It implies the admissibility of the solution should always be discussed.

A shock wave is a function, which must satisfy along its curves of discontinuity the Rankine–Hugoniot formula and the entropy criterion.

The Rankine–Hugoniot formula is given by

\[ \frac{A(u^+) - A(u^-)}{u^+ - u^-} = s(t), \]

with \( s(t) \) the speed of the shock wave.

To determine whether a solution is rejected or not, the entropy criterion must be verified. For a shock wave, the entropy criterion is defined as

\[ a(u^-) > s(t) > a(u^+), \]

It requires that \( a(u^-) \) the wave speed behind the shock is greater than \( a(u^+) \) the wave speeds ahead of it.

8.1 Nonlinear Transport 1 [Strauss 14.1, Ex.3]

Solve the nonlinear equation \( u_t + uu_x = 0 \) with the auxiliary condition \( u(x, 0) = x \). Sketch some of the characteristic lines. Give a physical interpretation of the solution.
Solution

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t), t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x},$$  \hspace{1cm} (8.1.1)

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t), t)] = 0, \hspace{1cm} (8.1.2)$$

provided $u_x$ and $u_t$ are defined on the line $x = x(t)$. Then along any such line the solution is constant, i.e.

$$u(x(t), t) = C_1, \quad C_1 \in \mathbb{R}. \hspace{1cm} (8.1.3)$$

The characteristic curves are given by

$$\frac{dx}{dt} = u(x(t), t), \hspace{1cm} (8.1.4)$$

$$\Rightarrow \frac{dx}{dt} = C_1, \hspace{1cm} (8.1.5)$$

$$\Rightarrow x(t) = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}. \hspace{1cm} (8.1.6)$$

Then using the initial condition gives ($x_0 \triangleq x(0)$)

$$u(x(t), t) = C_1 = u(x(0), 0) = \phi(x_0). \hspace{1cm} (8.1.7)$$

Note that the above equation ceases to hold if $u_x$ and/or $u_t$ are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$. This result can be replaced in the characteristic line equation, i.e.

$$x(t) = \phi(x_0) t + C_2. \hspace{1cm} (8.1.8)$$

At $t = 0$, one has

$$x(0) = x_0 = C_2. \hspace{1cm} (8.1.9)$$

Hence

$$x(t) = \phi(x_0) t + x_0. \hspace{1cm} (8.1.10)$$

Now, the given initial condition $\phi(x_0) = x_0$ is replaced into the characteristic line equation

$$x(t) = x_0 t + x_0, \hspace{1cm} (8.1.11)$$

$$\Rightarrow x(t) = x_0 (1 + t), \hspace{1cm} (8.1.12)$$

$$\Rightarrow x_0 = \frac{x}{1 + t}. \hspace{1cm} (8.1.13)$$

Therefore

$$u(x, t) = \phi(x_0) = \frac{x}{1 + t}. \hspace{1cm} (8.1.14)$$

To draw the characteristic lines, the Eq.(8.1.12) can be used for different $x_0$. It is possible to observe on the graph that all the lines are never crossing each other as long as $t \geq 0$. It means that in a physical situation, no shock will occur with such initial condition. This is a perfect example of a rarefaction wave causing the solution to spread out as time progress.
8.2 Nonlinear Transport 2 [Strauss 14.1, Ex.5]

Solve \( u_t + u^2 u_x = 0 \) with \( u(x, 0) = 2 + x \). Sketch some of the characteristic lines.

**Solution**

From the definition of the total derivative

\[
\frac{d}{dt}[u(x(t), t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x},
\]

the PDE can be written equivalently as

\[
\frac{d}{dt}[u(x(t), t)] = 0,
\]

provided \( u_x \) and \( u_t \) are defined on the line \( x = x(t) \) and provided \( \frac{dx}{dt} = u(x(t), t)^2 \).

Then along any such line the solution is constant, i.e

\[
u(x(t), t) = C_1, \quad C_1 \in \mathbb{R}.
\]

The characteristic curves are given by

\[
\frac{dx}{dt} = u(x(t), t)^2,
\]

\[
\Rightarrow \quad \frac{dx}{dt} = C_1^2,
\]

\[
\Rightarrow \quad x(t) = C_1^2 t + C_2, \quad C_2 \in \mathbb{R}.
\]

Using the initial condition and Eq(8.2.3), one has

\[
u(x(t), t) = C_1 = u(x(0), 0) = \phi(x_0).
\]
Note that the above equation ceases to hold if $u_x$ and/or $u_t$ are not defined at some point on the line between $(x_0, 0)$ and $(x(t), t)$.

Inserting this result in the equation of the characteristic lines gives

$$x(t) = \phi^2(x_0)t + x_0,$$
$$= (2 + x_0)^2t + x_0.$$ (8.2.8)

To determine which curves \textit{i.e.} which $x_0$ pass through the point $(x, t)$, the following polynomial equation has to be solved for $x_0$

$$tx_0^2 + (4t + 1)x_0 + (4t - x) = 0.$$ (8.2.10)

The two roots are

$$x_{01} = \frac{-(4t + 1) \pm \sqrt{1 + 4t(2 + x)}}{2t}$$
$$x_{02} = \frac{-(4t + 1) \pm \sqrt{1 + 4t(2 + x)}}{2t}$$ (8.2.11)

which means that for a given point $(x, t)$, there are two characteristic lines passing through that point. These lines are plotted in Figure 10. No characteristic line penetrates the region

$$1 + 4t(2 + x) < 0$$ (8.2.12)

because Eq.(8.2.10) admits no solution for such $(x, t)$.

![Figure 10: Characteristic lines are depicted in blue while the red region represents the values impossible to take.](image)

Therefore, the two possible solutions are

$$u_1(x, t) = 2 + x_{01},$$
$$= 2 + \frac{1 \pm \sqrt{1 + 4t(2 + x)}}{2t}.$$ (8.2.13)

$$u_2(x, t) = 2 + x_{02},$$
$$= 2 + \frac{1 \pm \sqrt{1 + 4t(2 + x)}}{2t}.$$ (8.2.14)
The solution procedure used here is valid provided \( u_x \) and \( u_t \) are defined at all points on the line between \((x_0, 0)\) and \((x(t), t)\). Using the above possible solutions, the derivative w.r.t \( x \) is given by

\[
\partial_x u_1 = \pm \frac{1}{\sqrt{1 + 4t(2 + x)}} \tag{8.2.15}
\]

which means that \( \partial_x u_1 \) does not exist when \( 1 + 4t(2 + x) = 0 \). As can be seen from Figure 10, the characteristic line corresponding to \( x_0^2 \) is actually tangent to the hyperbola \( 1 + 4t(2 + x) = 0 \) before it arrives at the point \((x, t)\). This characteristic line is thus not valid beyond the tangent point. Consequently, there is only one valid characteristic curve that passes through \((x, t)\) given by

\[
x_0 = \frac{-(4t + 1) + \sqrt{1 + 4t(2 + x)}}{2t} \tag{8.2.16}
\]

and thus

\[
u(x, t) = \frac{-1 + \sqrt{1 + 4t(2 + x)}}{2t}. \tag{8.2.17}
\]

If one is not convinced that the solution \( u_1 \) should not be kept, one can check if the initial conditions are verified.

\[
\lim_{t \to 0} u_1(x, t) = \lim_{t \to 0} \frac{-1 + \sqrt{1 + 4t(2 + x)}}{2t} \tag{8.2.18}
\]

L’Hôpital’s rule can thus be applied to this expression

\[
\lim_{t \to 0} u_1(x, t) \overset{H}{=} \lim_{t \to 0} \frac{2 + x}{\sqrt{1 + 4t(2 + x)}} = 2 + x. \tag{8.2.19}
\]

The initial condition is recovered as expected, and the solution \( u_1(x, t) \) is thus valid.

If the same procedure is now applied to the second solution

\[
\lim_{t \to 0} u_2(x, t) = \lim_{t \to 0} \frac{-1 - \sqrt{1 + 4t(2 + x)}}{2t} = -\infty \neq 2 + x. \tag{8.2.20}
\]
8.3 Shocks and Entropy criterion 1 [Strauss 14.1, Examples 6 and 7] 🎁

Solve \( u_t + uu_x = 0 \) with the following initial conditions

(a) 
\[
   u(x, 0) = \phi(x) = \begin{cases} 
   0 & \forall x > 0, \\
   1 & \forall x < 0.
\end{cases} 
\]  

(b) 
\[
   u(x, 0) = \phi(x) = \begin{cases} 
   1 & \forall x > 0, \\
   0 & \forall x < 0.
\end{cases} 
\]

In both cases, sketch the characteristic lines and determine the speed of the shock waves. Which one is physically correct?

Solution

(a) From the definition of the total derivative

\[
   \frac{d}{dt}[u(x(t), t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x},
\]

the PDE can be written equivalently as

\[
   \frac{d}{dt}[u(x(t), t)] = 0,
\]
provided \( u_x \) and \( u_t \) are defined on the line \( x = x(t) \) and provided \( \frac{dx}{dt} = u(x(t), t) \).
Then along any such line the solution is constant, \( i.e \)
\[
 u(x(t), t) = C_1, \quad C_1 \in \mathbb{R} \quad (8.3.5)
\]

The characteristic curves are given by
\[
 \frac{dx}{dt} = u(x(t), t), \quad (8.3.6)
\]
\[
 \Rightarrow \quad \frac{dx}{dt} = C_1, \quad (8.3.7)
\]
\[
 \Rightarrow \quad x(t) = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}. \quad (8.3.8)
\]

Then using the initial condition gives \( (x_0 \triangleq x(0)) \)
\[
 u(x(t), t) = C_1 = u(x(0), 0) = \phi(x_0). \quad (8.3.9)
\]

Note that the above equation ceases to hold if \( u_x \) and/or \( u_t \) are not defined at some point on the line between \( (x_0, 0) \) and \( (x(t), t) \).
This result can be replaced in the characteristic line equation, \( i.e \)
\[
 x(t) = \phi(x_0) t + C_2. \quad (8.3.10)
\]

At \( t = 0 \), one has
\[
 x(0) = x_0 = C_2. \quad (8.3.11)
\]

Hence
\[
 x(t) = \phi(x_0) t + x_0 \quad (8.3.12)
\]
and the characteristic lines are given by
\[
 \begin{cases}
 x(t) = x_0 & \forall x_0 > 0, \\
 x(t) = t + x_0 & \forall x_0 < 0.
\end{cases} \quad (8.3.13)
\]

The line(s) that passes through a given \( (x, t) \) is(are) therefore
\[
 \begin{cases}
 x_0 = x & 0 < t < x, \\
 x_0 = x - t & x < 0,
\end{cases} \quad (8.3.14)
\]

Thus for \( (x, t) \) such that \( 0 < t < x \) or \( x < 0 \), the solution takes the value of the unique characteristic curve passing through \( (x, t) \), \( i.e \)
\[
 u(x, t) = \begin{cases}
 0 & 0 < t < x, \\
 1 & x < 0
\end{cases} \quad (8.3.15)
\]
but for \( (x, t) \) such that \( 0 < x < t \), the solution is not defined as there is two possible values corresponding to the two possible characteristics.
A possible way to determine the solution in the region $0 < x < t$ is to consider that there is a shock inside that region. The position of the shock determines which characteristic line ($x_0^1$ or $x_0^2$) should be kept inside this region.

The shock wave speed $s(t)$ is given by the Rankine-Hugoniot formula

$$s(t) = \frac{A(u^+) - A(u^-)}{u^+ - u^-} \quad (8.3.16)$$

where $A(u)$ is the flux (defined such that $\partial_x A(u) = a(u)u_x$, i.e. $A(u) = \frac{1}{2}u^2$ here) while $u^+$ and $u^-$ are the value of $u$ just before and just after the shock ($u^+ = 0$ and $u^- = 1$ here).

The speed is then given by

$$s(t) = \frac{1}{2} \quad (8.3.17)$$

and the entropy criterion is respected as

$$a(u^-) \geq s(t) \geq a(u^+), \quad (8.3.18)$$

$$1 \geq \frac{1}{2} \geq 0. \quad (8.3.19)$$

Therefore, this solution is physically acceptable.

The shock wave characteristic line $x_s(t)$ is given by

$$x_s(t) = \frac{1}{2}t, \quad \forall t > 0. \quad (8.3.20)$$

because the first intersection of characteristic occurs at $(x, t) = (0, 0)$.

Finally, a generalized solution is thus

$$u(x, t) = \begin{cases} 
0 & 0 < t < 2x, \\
1 & t > 2x.
\end{cases} \quad (8.3.21)$$

Figure 12: Characteristic lines for $x_0 < 0$ and $x_0 > 0$ are respectively represented in red and blue. The black line between both regions is the shock wave.
(b) Similarly, one finds

\[ u(x, t) = \begin{cases} 
0 & x > t, \\
1 & x < 0 
\end{cases} \]  \hspace{1cm} (8.3.22)

Yet, the traditional technique does not give any direct solution in the region \( 0 < x < t \).

One possible solution would be to consider

\[ u(x, t) = \begin{cases} 
1 & 0 < t < 2x, \\
0 & t > 2x 
\end{cases} \]  \hspace{1cm} (8.3.23)

but now the entropy criterion is violated, indeed

\[ a(u^-) \geq s(t) \geq a(u^+), \]  \hspace{1cm} (8.3.24)

\[ 0 \geq \frac{1}{2} \geq 1. \]  \hspace{1cm} (8.3.25)

An engineer would therefore reject this solution as it is not physically valid :(

Figure 13: Representation of a non-physical solution. Characteristic lines for \( x_0 < 0 \) and \( x_0 > 0 \) are respectively depicted in blue and in red. The black line between both regions is the shock wave.

Another possibility would be to consider the physically valid classical solution

\[ u(x, t) = \begin{cases} 
0 & \forall x > t, \\
x/t & \forall 0 < x < t, \\
1 & \forall x < 0 
\end{cases} \]  \hspace{1cm} (8.3.26)
The characteristics are not crossing each other and no shock appears; this solution is then continuous. **Caution:** the most continuous solution is not always the best!

### 8.4 Shocks and Entropy criterion 2 [Strauss 14.1, Ex.10]

Solve $u_t + uu_x = 0$ with the following initial condition

$$u(x, 0) = \phi(x) = \begin{cases} 
1 & x \leq 0, \\
1 - x & 0 \leq x \leq 1, \\
0 & x \geq 1.
\end{cases} \quad (8.4.1)$$

Find exactly where the shock is and show that it satisfies the entropy condition. Sketch the characteristics.

**Solution**

From the definition of the total derivative

$$\frac{d}{dt}[u(x(t), t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x}, \quad (8.4.2)$$

the PDE can be written equivalently as

$$\frac{d}{dt}[u(x(t), t)] = 0 \quad (8.4.3)$$

provided $u_x$ and $u_t$ are defined on the line $x = x(t)$ and provided $\frac{dx}{dt} = u(x(t), t)$.

Then along any such line the solution is constant, i.e.

$$u(x(t), t) = C_1, \quad C_1 \in \mathbb{R} \quad (8.4.4)$$
The characteristic curves are given by
\[
\frac{dx}{dt} = u(x(t), t),
\] (8.4.5)
\[
\Rightarrow \frac{dx}{dt} = C_1,
\] (8.4.6)
\[
\Rightarrow x(t) = C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}.
\] (8.4.7)

Then using the initial condition gives \((x_0 \triangleq x(0))\)
\[
u(x(t), t) = C_1 = u(x(0), 0) = \phi(x_0).
\] (8.4.8)

Note that the above equation ceases to hold if \(u_x\) and/or \(u_t\) are not defined at some point on the line between \((x_0, 0)\) and \((x(t), t)\).

This result can be replaced in the characteristic line equation, \(i.e\)
\[
x(t) = \phi(x_0) t + C_2.
\] (8.4.9)

At \(t = 0\), one has
\[
x(0) = x_0 = C_2.
\] (8.4.10)

Hence
\[
x(t) = \phi(x_0) t + x_0,
\] (8.4.11)

and the characteristic lines are given by
\[
\begin{cases}
  x(t) = t + x_0 & x_0 \leq 0, \\
  x(t) = (1 - x_0) t + x_0 & 0 \leq x_0 \leq 1, \\
  x(t) = x_0 & x_0 \geq 1.
\end{cases}
\] (8.4.12)

The line(s) that passes through a given \((x, t)\) is(are) therefore
\[
\begin{cases}
  x_0 = x - t & x \leq 1 \text{ and } x \leq t \quad \text{(region 1)}, \\
  x_0 = \frac{x - t}{1 - t} & 0 \leq x \leq 1 \text{ and } x \geq t \quad \text{(region 2)}, \\
  x_0 = x & x \geq 1 \text{ and } x \geq t \quad \text{(region 3)}, \\
  x_0_{-1,2,3} = \begin{cases}
  x - t & x \geq 1 \text{ and } x \leq t \quad \text{(region 4)}.
\end{cases}
\] (8.4.13)

Thus, for \((x, t)\) in regions 1, 2 or 3, the solution takes the value of the unique characteristic curve passing through \((x, t)\), \(i.e\)
\[
u(x, t) = \begin{cases}
  1 & \text{in region 1}, \\
  \frac{1 - x}{1 - t} & \text{in region 2}, \\
  0 & \text{in region 3}.
\end{cases}
\] (8.4.14)
but for \((x, t)\) in region 4, the solution is not defined as there are three possible values corresponding to the three possible characteristics.

![Characteristic Lines](image)

Figure 15: Characteristic lines are depicted in red for \(x_0 \leq 0\), in green for \(0 \leq x_0 \leq 1\) and in blue for \(x_0 \geq 1\).

The solution’s procedure used here is valid provided \(u_x\) and \(u_t\) are defined at all points on the line between \((x_0, 0)\) and \((x(t), t)\). Using the above incomplete solution, it appears that \(u_x\) and \(u_t\) are not defined at \((x, t) = (1, 1)\) (the solution is discontinuous at that point). The characteristic lines passing through that point are therefore not valid anymore beyond that point. Consequently there are only two possibilities left in region 4.

A possible way to determine the solution in region 4 is to consider that there is a shock inside that region. The position of the shock determines which characteristic line \((x_{0-1}\) or \(x_{0-3}\)) should be kept inside that region.

The shock wave speed \(s(t)\) is given by the Rankine-Hugoniot formula

\[
s(t) = \frac{A(u^+) - A(u^-)}{u^+ - u^-} \tag{8.4.15}
\]

where \(A(u)\) is the flux (defined such that \(\partial_x A(u) = a(u)u_x\), i.e. \(A(u) = \frac{1}{2}u^2\) here) while \(u^-\) and \(u^+\) are the two possible values of \(u\) at the shock \((u^- = 1\) and \(u^+ = 0\) here).

The speed is then given by

\[
s(t) = \frac{1}{2} \tag{8.4.16}
\]

and the entropy criterion is respected as

\[
a(u^-) \geq s(t) \geq a(u^+), \tag{8.4.17}
\]

\[
1 \geq \frac{1}{2} \geq 0. \tag{8.4.18}
\]
Therefore, this solution is physically acceptable. The shock wave characteristic line $x_s(t)$ is given by

$$x_s(t) = \frac{1}{2}(t + 1), \forall t \geq 1.$$  \hfill (8.4.19)

because the first intersection of characteristic occurs at $(x, t) = (1, 1)$.

Figure 16: Characteristic lines are depicted in red for $x_0 \leq 0$, in green for $0 \geq x_0 \leq 1$ and in blue for $x_0 \geq 1$. The shock wave is represented in black.

Finally, a generalized solution is thus

$$u(x, t) = \begin{cases} 
1 & 2x \leq 1 + t \text{ and } x \leq t, \\
\frac{1 - x}{1 - t} & 0 \leq x \leq 1 \text{ and } x \geq t, \\
0 & x \geq 1 \text{ and } 2x \geq 1 + t.
\end{cases}$$  \hfill (8.4.20)
Figure 17: Representation of the solutions at different time, where the red line refers to the region 1, the green one to region 2 and the blue one to region 3. At $t = 2$, the black line represents the shock wave. The dashed lines would have been the solutions obtained along characteristics beyond the shock (the rejected solution).

**Note:** The solution of a shock represented in Fig.17 is actually forming a step wave cutting the rejected solutions such as the removed area between the red dashed line and the green dashed line is the same as the removed area between the green dashed line and the blue dashed line. In other words, the shock obtained with the Rankine-Hugoniot formula actually verifies the Eq.($\star$).

### 8.5 Inhomogeneous Nonlinear Transport [Strauss 14.1, Ex.12]

Solve $u_t + uu_x = 1$ with $u(x, 0) = x$. 
Solution

From the definition of the total derivative
\[
\frac{d}{dt}[u(x(t), t)] = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x},
\] (8.5.1)
the PDE can be written equivalently as
\[
\frac{d}{dt}[u(x(t), t)] = 1,
\] (8.5.2)
provided \(u_x\) and \(u_t\) are defined on the line \(x = x(t)\) and provided \(\frac{dx}{dt} = u(x(t), t)\).

Then along any such line the solution grows linearly, i.e.
\[
u(x(t), t) = t + C_1, \quad C_1 \in \mathbb{R}.
\] (8.5.3)

The characteristic curves are given by
\[
\frac{dx}{dt} = u(x(t), t),
\] (8.5.4)
\[
\Rightarrow \frac{dx}{dt} = t + C_1,
\] (8.5.5)
\[
\Rightarrow x(t) = \frac{1}{2} t^2 + C_1 t + C_2, \quad C_1, C_2 \in \mathbb{R}.
\] (8.5.6)

Then using the initial condition gives \((x_0 \triangleq x(0))\)
\[
u(x(0), 0) = C_1 = \phi(x_0).
\] (8.5.7)

Note that the above equation ceases to hold if \(u_x\) and/or \(u_t\) are not defined at some point on the line between \((x_0, 0)\) and \((x(t), t)\).

This result can be replaced in the characteristic line equation, i.e.
\[
x(t) = \frac{1}{2} t^2 + \phi(x_0)t + C_2.
\] (8.5.8)

At \(t = 0\), one has
\[
x(0) = x_0 = C_2.
\] (8.5.9)

Hence
\[
x(t) = \frac{1}{2} t^2 + \phi(x_0)t + x_0.
\] (8.5.10)
Figure 18: Characteristic lines of the problem are depicted in blue.

The line passing through a given \((x, t)\) is therefore

\[
x_0 = \frac{2x - t^2}{2(t + 1)}.
\]  

(8.5.11)

and finally the solution is

\[
u(x, t) = t + x_0
\]

(8.5.12)

\[
u(x, t) = t + \frac{2x - t^2}{2(t + 1)}.
\]  

(8.5.13)

### 8.6 Triangular signal [partly from Olver, Example 2.12]

Solve \(u_t + uu_x = 1\) with the triangular initial data

\[
u(x, 0) = \begin{cases} 
  x, & 0 \leq x \leq 1, \\
  2 - x, & 1 \leq x \leq 2, \\
  0, & \text{otherwise,}
\end{cases}
\]

following the successive steps below. We define \(U(t) = \int_{\mathbb{R}} u(x, t)dx\).

(a) Show that \(u\) is a conserved quantity for this equation. Find the expression of the associated flux. Compute the initial total quantity \(U(0)\).

(b) Determine the characteristic curves. Draw some of them to get a representative figure.

(c) At which time instant \(t_*\) do these lines start to intersect? Give the solution for \(0 < t < t_*\). Check that \(U(t)\) is indeed constant in that time interval.
(d) For \( t > t_\star \), the solution contains a shock. Let \( \sigma(t) \) be the position of the shock with respect to time, and \( \sigma_\tau(t) = s(t) \) be the speed of the shock, use Rankine-Hugoniot formula to derive a first order ordinary differential equation for the shock position \( \sigma(t) \). Solve this equation, with the appropriate initial position \( \sigma(t_\star) \) for the shock.

(e) Give the final solution. Draw it, with the shock. How does the strength of the shock \( u^- - u^+ \) evolve with time? Does this shock satisfies the entropy criterion?

(f) Check that \( U(t) \) is still constant for \( t > t_\star \).

**Solution (brief solution only)**

(a) /

(b) /

(c) We get \( t_\star = 1 \). The solution is, for \( 0 < t < 1 \),

\[
 u(x, t) = \begin{cases} 
 \frac{x}{1+t}, & 0 < \frac{x}{1+t} < 1, \\
 \frac{2-x}{1-t}, & 1 < \frac{x-2t}{1-t} < 2, \\
 0, & \text{otherwise}. 
\end{cases}
\]  
\[ (8.6.1) \]

We indeed have \( U(t) = 1 \) for \( 0 < t < 1 \).

(d) The equation follows from \( u^- = u(\sigma^-), t = \sigma(1+t) \), and \( u^+ = u(\sigma^+, t) = 0 \), and \( A(u) = u^2/2 \). The initial condition is \( \sigma(1) = 2 \). After solving the resulting equation by separation of variables, we get the shock position \( \sigma(t) = \sqrt{2(t+1)} \). (The only difference with simpler exercises is that, here, \( u^- \) depends on the position of the shock. The shock speed is therefore not constant.)

(e) The solution is, for \( t > t_\star \),

\[
 u(x, t) = \begin{cases} 
 \frac{x}{1+t}, & 0 < x < \sigma(t), \\
 0, & \text{otherwise}. 
\end{cases}
\]  
\[ (8.6.2) \]

The strength of the shock is \( \sqrt{2/(1+t)} \). The entropy criterion is satisfied.

(f) We directly get, for \( t > t_\star \),

\[
 U(t) = \int_{-\infty}^{\infty} u(x, t) dx = \int_{0}^{\sigma(t)} \frac{x}{1+t} \, dx = \frac{1}{1+t} \frac{\sigma(t)^2}{2} = 1. 
\]  
\[ (8.6.3) \]
9 Singular value decomposition

Let $A \in \mathbb{C}^{m \times n}$ be a rectangular matrix. A singular value decomposition of $A$ is a matrix factorization

$$A = U \Sigma V^* \quad (\circledast)$$

with

$U \in \mathbb{C}^{m \times m}$ such that $UU^* = I$,
$V \in \mathbb{C}^{n \times n}$ such that $VV^* = I$,
$\Sigma \in \mathbb{R}^{m \times n}$ such that

$$\begin{cases}
\Sigma_{ij} = 0 & \text{for } i \neq j, \\
\Sigma_{ij} = \sigma_i > 0 & \text{for } i = j \leq \text{rank}(A), \\
\Sigma_{ij} = 0 & \text{for } i = j > \text{rank}(A).
\end{cases}$$

The columns of $U$ and $V$ are called the left and right singular vectors respectively. They are denoted by $u_j$ and $v_j$.

**Left singular vector**  Left multiplying definition Eq($\circledast$) by $U^*$ gives

$$U^*A = U^*U \Sigma V^*$$

$$\Rightarrow U^*A = \Sigma V^*$$

$$\Rightarrow AV = V \Sigma^*$$

$$\Rightarrow A^*u_j = \sigma_j v_j \quad \text{for } j \leq \text{rank}(A) \quad \text{(no sum on } j)$$

which means that the image by $A^*$ of a left singular vector $u_j$ is the corresponding right singular vector $v_j$ weighted by the corresponding singular value $\sigma_j$.

From definition ($\circledast$) it appears that the left singular vectors are the eigenvectors of $AA^*$ and that the corresponding eigenvalues are the singular value squared. Indeed

$$AA^* = U \Sigma V^* (U \Sigma V^*)^*$$

$$= U \Sigma V^* V \Sigma^* U^*$$

$$= U \Sigma \Sigma^* U^*$$

$$\Rightarrow AA^*U = U \Sigma \Sigma^* U^*$$

$$= U \Sigma \Sigma^*$$

i.e.

$$\begin{cases}
AA^*u_j = \sigma_j^2 u_j & \text{for } j \leq \text{rank}(A) \\
AA^*u_j = 0 & \text{for } j > \text{rank}(A).
\end{cases}$$

**Right singular vector**  Right multiplying definition Eq($\circledast$) by $V$ gives

$$AV = U \Sigma V^* V$$

$$\Rightarrow AV = U \Sigma$$

$$\Rightarrow Av_j = \sigma_j u_j \quad \text{for } j \leq \text{rank}(A) \quad \text{(no sum on } j)$$
which means that the image by $A$ of a right singular vector $v_j$ is the corresponding left singular vector $u_j$ weighted by the corresponding singular value $\sigma_j$.

From definition (⋄) it appears that the right singular vectors are the eigenvectors of $A^*A$ and that the corresponding eigenvalues are the singular value squared. Indeed

$$A^*A = (U\Sigma V^*)^*U\Sigma V^*$$
$$= V\Sigma^*U^*U\Sigma V^*$$
$$= V\Sigma^*\Sigma V^*$$
$$\Rightarrow A^*AV = V\Sigma\Sigma^*V^*V$$
$$= V\Sigma\Sigma^*$$

i.e

$$\begin{cases}
A^*Av_j = \sigma_j^2 v_j & \text{for } j \leq \text{rank}(A) \\
A^*Av_j = 0 & \text{for } j > \text{rank}(A).
\end{cases}$$

**Computing a singular value decomposition** A singular value decomposition can be obtained through the following two steps

1. Find the eigenvectors and eigenvalues of $A^*A$ (resp. $AA^*$). Deduce the right (resp. left) singular vector and the singular values

2. Find the left (resp. right) singular vector through $u_j = \frac{1}{\sigma_j}Av_j$ (resp. $v_j = \frac{1}{\sigma_j}A^*u_j$).

### 9.1 Exercise 1 [Trefethen Ex.4.1]

Determine the singular value decomposition of the following matrices and give a geometrical interpretation to the singular values and vectors

(a) $$\begin{pmatrix}
3 & 0 \\
0 & -2
\end{pmatrix}$$

(b) $$\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}$$

(c) $$\begin{pmatrix}
0 & 2 \\
0 & 0 \\
0 & 0
\end{pmatrix}$$

(d) $$\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}$$

(e) $$\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}$$
Solution

(a) $A$ is full rank and $m = n$ such that two non-vanishing singular values are expected.

**Left singular vectors**

\[ AA^* = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \]

The eigenvalues are solutions of

\[
\det AA^* - \lambda I = 0
\]

\[ \Rightarrow \det \begin{pmatrix} 9 - \lambda & 0 \\ 0 & 4 - \lambda \end{pmatrix} = (9 - \lambda)(4 - \lambda) = 0 \]

i.e $\lambda_1 = 9$, $\lambda_2 = 4$ such that $\sigma_1 = 3 > \sigma_2 = 2$. The second eigenvector verifies

\[
\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 9 \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 \\ -4y \end{pmatrix} = 0
\]

Thus

\[ u_1 = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \]

The second eigenvector verifies

\[
\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 4 \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 5x \\ 0 \end{pmatrix} = 0
\]

Thus

\[ u_2 = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \]

The + sign are chosen.

**Right singular vectors**

\[ v_1 = \frac{1}{\sigma_1} A^* u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \frac{1}{\sigma_2} A^* u_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \] \hspace{1cm} (9.1.1)

Finally

\[ U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \]

The geometrical interpretation of this singular value decomposition is given in Figure 19.
Figure 19: Geometrical interpretation of SVD.

(b) / 

(c) $A$ is of rank 1 such that only one non-vanishing singular value is expected.

**Left singular eigenvectors**

$$AA^* = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

The eigenvalues are solutions of

$$\det AA^* - \lambda I = 0$$

$$\Rightarrow \det \begin{pmatrix} 4 - \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \lambda^2(4 - \lambda) = 0$$

*i.e $\lambda_1 = 4, \lambda_2 = \lambda_3 = 0$ such that $\sigma_1 = 2$.*

The first eigenvector verifies

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -4y \\ -4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus

$$u_1 = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}.$$  

The $+$ sign is chosen.  

To complete the basis, two orthonormal eigenvectors must be added. For simplicity, they are chosen as

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
Right singular vectors

\[ v_1 = \frac{1}{\sigma_1} A^* u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]  
\[ (9.1.7) \]

As previously, the basis is completed by a vector which is chosen as

\[ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]  
\[ (9.1.8) \]

Finally

\[ U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}. \]  
\[ (9.1.9) \]

The geometrical interpretation of this singular value decomposition is given in Figure 20.

Figure 20: Geometrical interpretation of SVD.