Applied Mathematics - MATH-0504
Exercise statements and solutions

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Contents

1 Characteristic lines

1.1 Exercise 1 [Strauss 1.2, Ex.3]

Consider the following equation: $(1 + x^2)u_x + u_y = 0$.

(a) Find the general solution of this equation.

(b) Draw some of the characteristic lines.

Solution

(a) The directional derivative of u in the direction of $(1 + x^2, 1)$ must always be 0. Therefore, the characteristic lines, along which u is constant, satisfy the relation

$$\frac{dy}{dx} = \frac{1}{1 + x^2}, \tag{1}$$

which can be rewritten as

$$dy = \frac{1}{1 + x^2}\, dx. \tag{2}$$

Direct integration of both sides yields, with the integration constant $C \in \mathbb{R}$,

$$\begin{aligned} y &= \arctan(x) + C \\ \Leftrightarrow C &= y - \arctan(x). \end{aligned} \tag{3}$$

Knowing that u is constant on these characteristic lines, we have the following general solution:

$$\boxed{u(x, y) = f\big(y - \arctan(x)\big),} \tag{4}$$

where f is an arbitrary function.

(b) Some characteristic lines are represented in Fig. 1.

1.2 Exercise 2 [Strauss 1.2, Ex.7]

Consider the following problem: $yu_x + xu_y = 0$ with $u(0, y) = \exp(-y^2)$.

(a) Solve this problem.

(b) Where is the solution uniquely determined in the xy-plane?

Solution

(a) The directional derivative of u along the vector (y, x) must be 0. The equation of the characteristic lines is therefore

$$\frac{dy}{dx} = \frac{x}{y}. \tag{5}$$

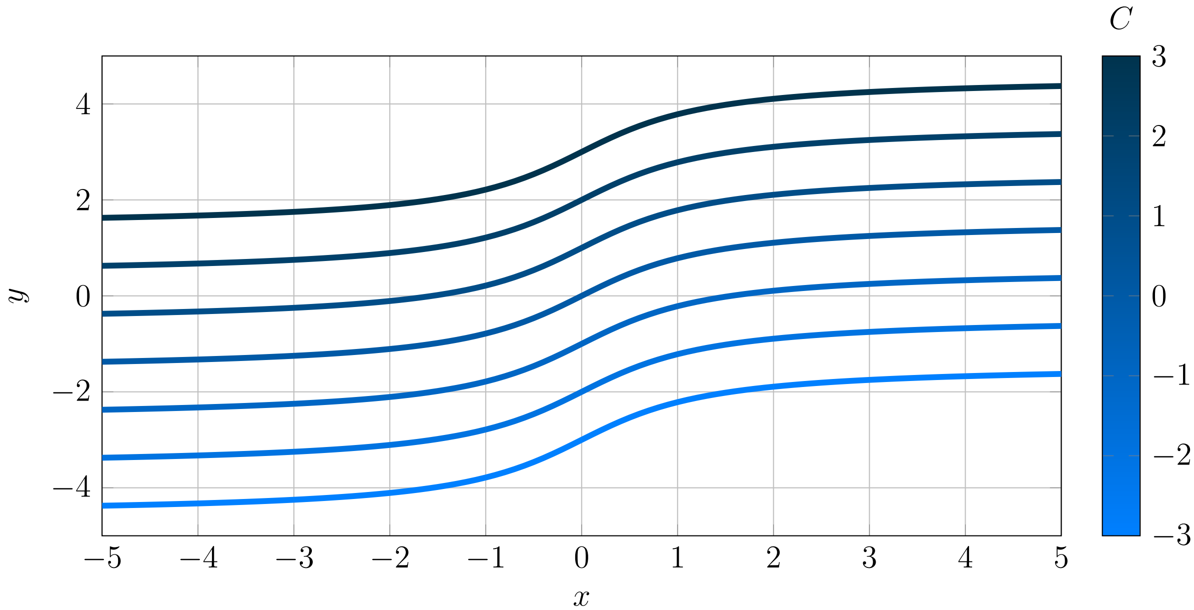

Figure 1: Characteristic curves $y - \arctan(x) = C$.

The characteristic lines satisfy

$$y^2 = x^2 + C \tag{6}$$

such that

$$C = y^2 - x^2. \tag{7}$$

Along these lines u is constant. Thus u is only a function of the parameter C. Then using Eq.(7) yields

$$u(x, y) = f(y^2 - x^2). \tag{8}$$

Thanks to the additional condition, it is possible to determine the shape of the function f. Indeed, at $x = 0$

$$f(y^2) = \exp\{-y^2\}. \tag{9}$$

Thus one can write

$$f(C) = \exp\{-C\} \tag{10}$$

and finally, from Eq.(7), one has

$$u(x, y) = \exp\{x^2 - y^2\}. \tag{11}$$

(b) The additional condition is valid for $x = 0$. The characteristic lines satisfying $C < 0$ never go through $x = 0$, therefore the condition does not apply in that case. The two following cases must be distinguished in the final answer

$$\begin{cases} \text{if } y^2 - x^2 < 0 \text{ then } u(x, y) = f(y^2 - x^2) \\ \text{if } y^2 - x^2 \geq 0 \text{ then } u(x, y) = \exp\{x^2 - y^2\}. \end{cases} \tag{12}$$

1.3 Exercise 3 [Strauss 1.5, Ex. 6]

Find the general solution of the equation $u_x + 2xy^2 u_y = 0$.

Solution

The equation can be rewritten as

$$\boldsymbol{a} \cdot \nabla u = 0 \tag{13}$$

where $\boldsymbol{a} = (1, 2xy^2)$.
The characteristic curves thus satisfy the equation

$$\frac{dy}{dx} = \frac{2xy^2}{1}. \tag{14}$$

Therefore

$$\frac{dy}{y^2} = 2xdx, \tag{15}$$

i.e., upon integration,

$$-\frac{1}{y} + C = x^2 \tag{16}$$

$$\Rightarrow \quad y = \frac{1}{C - x^2}, \tag{17}$$

$$\Rightarrow \quad C = x^2 + \frac{1}{y}. \tag{18}$$

$u(x, y)$ is constant on the characteristic curves

$$y = \frac{1}{C - x^2} \tag{19}$$

. Indeed

$$\frac{du}{dx}\left(x, \frac{1}{C - x^2}\right) = \frac{\partial u}{\partial x} + \frac{2x}{(C - x^2)^2}\frac{\partial u}{\partial y} = u_x + 2xy^2 u_y = 0. \tag{20}$$

Hence, $u(x, y) = f(C)$, *i.e.*

$$u(x, y) = f\left(x^2 + \frac{1}{y}\right), \tag{21}$$

where f is an arbitrary function.

1.4 Exercise 4 [Strauss 1.2, Ex. 8]

Find the general solution of the equation $au_x + bu_y + cu = 0$.

Solution

The equation can be written as

$$\boldsymbol{a} \cdot \nabla u = -cu \tag{22}$$

where $\boldsymbol{a} = (a, b)$. The directional derivative along (a, b) is not 0. Therefore u is not constant along the lines

$$\frac{dy}{dx} = \frac{b}{a} \tag{23}$$

$$\Rightarrow y = \frac{b}{a}x + C. \tag{24}$$

Along these lines however, u satisfies

$$\frac{du}{dx} = -\frac{c}{a}u. \tag{25}$$

Indeed

$$\frac{d}{dx}\left(u\left(x, \frac{b}{a}x + C\right)\right) = u_x + \frac{b}{a}u_y = -\frac{c}{a}u. \tag{26}$$

Solving the Eq.(25) for u yields

$$u = f(C)\exp\{-\frac{c}{a}x\}, \tag{27}$$

where the function $f(C)$ comes from the integration of Eq.(25). From Eq.(24), the final solution is

$$u(x, y) = f(y - \frac{b}{a}x)\exp\{-\frac{c}{a}x\}. \tag{28}$$

2 Wave equation

2.1 Spherical wave equation [Strauss 2.1, Ex. 8]

The *spherical wave equation* reads

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right).$$

(a) Using the substitution $v = ru$, show that the spherical wave equation can be rewritten as $v_{tt} = c^2 v_{rr}$.

(b) Find the general solution of the spherical wave equation.

(c) Give an expression for the solution corresponding to the initial conditions $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$, for given functions ϕ and ψ.

Solution

(a) Using the change of variables $v = ru$ yields

$$v_t = ru_t, \quad v_{tt} = ru_{tt}, \quad v_r = u + ru_r, \quad \text{and} \quad v_{rr} = 2u_r + ru_{rr}. \tag{29}$$

The equation thus becomes

$$\frac{v_{tt}}{r} = c^2 \frac{v_{rr}}{r}, \tag{30}$$

$$\Rightarrow v_{tt} = c^2 v_{rr}. \tag{31}$$

(b)

$$v = f(r+ct) + g(r-ct), \tag{32}$$

$$\Rightarrow u = \frac{v}{r} = \frac{1}{r}\left[f(r+ct) + g(r-ct) \right]. \tag{33}$$

(c) Let us define $\Phi(r) = r\phi(r)$ and $\Psi(r) = r\psi(r)$. Using the general solution of this initial value problem is

$$v(r,t) = \frac{1}{2}\left[\Phi(r+ct) + \Phi(r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} \Psi(s)\, ds. \tag{34}$$

Since $v = ru$, $\Psi(r) = r\psi(r)$ and $\Phi(r) = r\phi(r)$, the solution for $u(r,t)$ is given by

$$u(r,t) = \frac{1}{2r}\left[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)\right] + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\, ds. \tag{35}$$

2.2 Equation factorization [Strauss 2.1, Ex. 10]

Solve the equation $u_{xx} + u_{xt} - 20u_{tt} = 0$ using the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

Solution

The equation can be written

$$(\partial_{xx} + \partial_{xt} - 20\partial_{tt})\, u = 0. \tag{36}$$

Factoring the operator yields

$$(\partial_x + 5\partial_t)\,(\partial_x - 4\partial_t)\, u = 0. \tag{37}$$

Let $v(x,t) = (\partial_x - 4\partial_t)\, u$, the problem can be written as a system first order equations

$$\begin{cases} (\partial_x + 5\partial_t)\, v = 0, & (38)\\ (\partial_x - 4\partial_t)\, u = v, & (39)\\ \quad u(x,0) = \phi(x), & (40)\\ \quad u_t(x,0) = \psi(x). & (41) \end{cases}$$

From Eq.(38), it can be shown that v is constant along the characteristic lines of equations $t - 5x = C$. Therefore

$$v(x,t) = f(t - 5x). \tag{42}$$

Using Eq.(42), Eq.(39) can be written as:

$$(\partial_x - 4\partial_t)\, u = f(t - 5x). \tag{43}$$

It can be verified by differentiation that a particular solution of Eq.(43) is given by $u^p(x,t) = h(t-5x)$, where $h'(s) = -\dfrac{1}{9}f(s)$.

Besides this, using the same reasoning as the one used for v, one can show that the general solution of the homogeneous part of Eq.(43) can be written as $u^h(x,t) = g(t+4x)$.

As Eq.(43) is linear, the solution can be written as the sum of a particular solution and the general solution of the homogeneous PDE *i.e.*

$$u(x,t) = h(t-5x) + g(t+4x). \tag{44}$$

Then using the auxiliary conditions Eq.(40) and Eq.(41) yields

$$\begin{cases} h(-5x) + g(4x) = \phi(x), & (45)\\ h'(-5x) + g'(4x) = \psi(x). & (46) \end{cases}$$

Deriving Eq.(45) with respect to x and using the chain rule, successively gives

$$\begin{cases} -5h'(-5x) + 4g'(4x) = \phi'(x),\\ h'(-5x) + g'(4x) = \psi(x). \end{cases} \tag{47}$$

$$\Leftrightarrow \begin{cases} 9g'(4x) = \phi'(x) + 5\psi(x),\\ h'(-5x) + g'(4x) = \psi(x). \end{cases} \tag{48}$$

$$\Leftrightarrow \begin{cases} g'(4x) = \dfrac{1}{9}\phi'(x) + \dfrac{5}{9}\psi(x),\\ h'(-5x) = -\dfrac{1}{9}\phi'(x) + \dfrac{4}{9}\psi(x). \end{cases} \tag{49}$$

Let solve this system one equation at a time. Integrating both sides of the first equation of Eqs.(49) successively gives

$$\int_0^x g'(4s)ds = \int_0^x \frac{1}{9}\phi'(s)ds + \int_0^x \frac{5}{9}\psi(s)ds, \tag{50}$$

$$\Leftrightarrow \int_0^{4x} \frac{1}{4}g'(\tilde{s})d\tilde{s} = \frac{1}{9}\int_0^x \phi'(s)ds + \frac{5}{9}\int_0^x \psi(s)ds, \tag{51}$$

$$\Leftrightarrow g(4x) = \frac{4}{9}\phi(x) + \frac{20}{9}\int_0^x \psi(s)ds + A, \tag{52}$$

where the change of variable $4s = \tilde{s}$ was used and where A is an integration constant. Posing $4x = y$ in (52) yields

$$g(y) = \frac{4}{9}\phi(\frac{y}{4}) + \frac{20}{9}\int_0^{\frac{y}{4}} \psi(s)ds + A. \tag{53}$$

Then solving the second equation of Eqs.(49) gives

$$\int_0^x h'(-5s)ds = -\int_0^x \frac{1}{9}\phi'(s)ds + \int_0^x \frac{4}{9}\psi(s)ds, \tag{54}$$

$$\Leftrightarrow \int_0^{-5x} \frac{-1}{5}h'(\tilde{s})d\tilde{s} = -\frac{1}{9}\int_0^x \phi'(s)ds + \frac{4}{9}\int_0^x \psi(s)ds, \tag{55}$$

$$\Leftrightarrow h(-5x) = \frac{5}{9}\phi(x) - \frac{20}{9}\int_0^x \psi(s)ds + B, \tag{56}$$

where the change of variable $-5s = \tilde{s}$ was used and where B is an integration constant. Posing $-5x = y$ in (56) yields

$$h(y) = \frac{5}{9}\phi(\frac{-y}{5}) - \frac{20}{9}\int_0^{\frac{-y}{5}} \psi(s)ds + B. \tag{57}$$

From Eq.(44), Eq.(53) and Eq.(57) it follows

$$u(x,t) = \frac{5}{9}\phi\left(\frac{-(t-5x)}{5}\right) + \frac{4}{9}\phi\left(\frac{(t+4x)}{4}\right) - \frac{20}{9}\int_0^{\frac{-(t-5x)}{5}} \psi(s)ds + \frac{20}{9}\int_0^{\frac{(t+4x)}{4}} \psi(s)ds + A + B. \tag{58}$$

From the first equation in Eqs.(45), Eq.(52) and Eq.(56), it can be shown that $A + B = 0$. Therefore, the solution is

$$u(x,t) = \frac{1}{9}\left[5\phi\left(x - \frac{t}{5}\right) + 4\phi\left(x + \frac{t}{4}\right) + 20\int_{x-\frac{t}{5}}^{x+\frac{t}{4}} \psi(s)ds\right]. \tag{59}$$

2.3 Damped string [Strauss 2.2, Ex. 5]

The equation of motion for a damped string is given by

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad r > 0. \tag{60}$$

Show that, in this case, the energy $\mathbb{E} = \mathbb{K} + \mathbb{P}$ decreases.
The kinetic energy is defined here as

$$\mathbb{K} = \frac{1}{2}\int_{\mathbb{R}} u_t^2\, dx \tag{61}$$

while the potential is defined as

$$\mathbb{P} = \frac{1}{2}\int_{\mathbb{R}} c^2\left(u_x^2\right)\, dx. \tag{62}$$

Solution

The energy decreases with time provided that

$$\frac{d\mathbb{E}}{dt} < 0. \tag{63}$$

By definition, this derivative is given by

$$\begin{aligned}
\frac{d\mathbb{E}}{dt} &= \frac{d\mathbb{K}}{dt} + \frac{d\mathbb{P}}{dt} &(64)\\
&= \int_{\mathbb{R}} u_t u_{tt}\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx &(65)\\
&= \int_{\mathbb{R}} u_t c^2 u_{xx}\, dx - \int_{\mathbb{R}} r u_t u_t\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx &(66)\\
&= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx + [u_x u_t]_{-\infty}^{\infty} - \int_{\mathbb{R}} r u_t^2\, dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} c^2 u_x^2\, dx &(67)\\
&= -\int_{\mathbb{R}} r u_t^2\, dx < 0. &(68)
\end{aligned}$$

2.4 Inhomogeneous wave equation [Olver 2.4, Ex. 11]

(a) Solve the initial value problem

$$\partial_{tt} u - c^2 \partial_{xx} u = 0, \quad u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x), \quad -\infty < x < \infty, \quad t \geq 0. \tag{69}$$

(b) Is $u(x,t)$ a periodic function of t?

(c) Solve the forced initial value problem

$$\partial_{tt} u - c^2 \partial_{xx} u = \cos(\omega t), \quad u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x), \quad -\infty < x < \infty, \quad t \geq 0. \tag{70}$$

(d) Does the solution exhibits resonance ?

(e) What would happen if the forcing function is $\sin(\omega t)$ instead of $\cos(\omega t)$?

Solution

(a) Starting from

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u = 0, \tag{71}$$

define the characteristic coordinates

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct. \tag{72}$$

The chain rule gives

$$\partial_x = \partial_\xi + \partial_\eta \quad \text{and} \quad \partial_t = c\partial_\xi - c\partial_\eta \tag{73}$$

such that the equation simplifies as

$$\begin{aligned} & -4c^2\left(\partial_\xi\right)\left(\partial_\eta\right)u = 0 && (74)\\ \Rightarrow\quad & \partial_\xi\partial_\eta u \qquad = 0 && (75)\\ \Rightarrow\quad & u(x,t) \qquad = f(\xi)+g(\eta) = f(x+ct)+g(x-ct). && (76)\end{aligned}$$

The initial conditions yields

$$\begin{cases} u(x,0) = f(x)+g(x) \qquad = \sin(x), & (77)\\ u'(x,0)) = cf'(x) - cg'(x) = \cos(x). & (78)\end{cases}$$

Differentiating the first equation yields

$$f'(x)+g'(x) = \cos(x) \tag{79}$$

Then solving for $f'(x)$ and $g'(x)$ gives

$$\begin{cases} f'(x) = \dfrac{c+1}{2c}\cos(x) & (80)\\ g'(x) = \dfrac{c-1}{2c}\cos(x) & (81)\end{cases}$$

Integrating Eq.(80) and Eq.(81) gives

$$\begin{cases} f(x) = \dfrac{c+1}{2c}\sin(x) + A & (82)\\ g(x) = \dfrac{c-1}{2c}\sin(x) + B & (83)\end{cases}$$

Since $f(x)+g(x) = \sin(x)$, $A+B=0$. Finally, the solution $u(x,t) = f(x+ct)+g(x-ct)$ to the problem is given by

$$u(x,t) = \frac{c+1}{2c}\sin(x+ct) + \frac{c-1}{2c}\sin(x-ct). \tag{84}$$

(b) The function $u(x,t)$ is a periodic function of t if it is possible to find T such that

$$u(x,t+nT) = u(x,t) \tag{85}$$

for any $n\in\mathbb{N}$.
It can be checked that choosing $T=\frac{2\pi}{c}$ this condition is satisfied so that $u(x,t)$ is a periodic function of t.

(c) The equation can be written as

$$\left(\partial_t + c\partial_x\right)\left(\partial_t - c\partial_x\right)u = \cos(\omega t). \tag{86}$$

Define the characteristic coordinates

$$\xi = x+ct \quad \text{and} \quad \eta = x-ct. \tag{87}$$

The chain rule gives

$$\partial_x = \partial_\xi + \partial_\eta \quad \text{and} \quad \partial_t = c\partial_\xi - c\partial_\eta. \tag{88}$$

such that the equation simplifies as

$$-4c^2\left(\partial_\xi\right)\left(\partial_\eta\right)u = \cos\left(\omega\frac{\xi-\eta}{2c}\right) \tag{89}$$

$$\Rightarrow \quad \partial_{\xi\eta}u = -\frac{1}{4c^2}\cos\left(\omega\frac{\xi-\eta}{2c}\right) \tag{90}$$

$$\Rightarrow \quad u(\xi,\eta) = -\frac{1}{\omega^2}\cos\left(\omega\frac{\xi-\eta}{2c}\right) + f(\xi) + g(\eta) \tag{91}$$

$$\Rightarrow \quad u(x,t) = -\frac{1}{\omega^2}\cos(\omega t) + f(x+ct) + g(x-ct). \tag{92}$$

The initial conditions then give

$$\begin{cases} u(x,0) = -\dfrac{1}{\omega^2} + f(x) + g(x) = \sin(x), & (93)\\ u'(x,0)) = cf'(x) - cg'(x) \qquad = \cos(x). & (94)\end{cases}$$

Differentiating the first equation yields

$$f'(x) + g'(x) = \cos(x). \tag{95}$$

Then solving for $f'(x)$ and $g'(x)$ yields

$$\begin{cases} f'(x) = \dfrac{c+1}{2c}\cos(x) & (96)\\ g'(x) = \dfrac{c-1}{2c}\cos(x) & (97)\end{cases}$$

Integrating Eq.(97) gives

$$\begin{cases} f(x) = \dfrac{c+1}{2c}\sin(x) + A & (98)\\ g(x) = \dfrac{c-1}{2c}\sin(x) + B & (99)\end{cases}$$

Since $f(x) + g(x) = \sin(x) + \dfrac{1}{\omega^2}$, $A + B = \dfrac{1}{\omega^2}$ and the final solution is

$$u(x,t) = \frac{1}{\omega^2}(1-\cos(\omega t)) + \frac{c+1}{2c}\sin(x+ct) + \frac{c-1}{2c}\sin(x-ct). \tag{100}$$

(d) The solution Eq.(100) is bounded $\forall\omega > 0$, hence there is no resonance.

(e) In that case, the solutions is

$$u(x,t) = \frac{1}{\omega^2}(\omega t - \sin(\omega t)) + \frac{c+1}{2c}\sin(x+ct) + \frac{c-1}{2c}\sin(x-ct). \tag{101}$$

$u(x,t)$ grows with t which indicates resonance.

3 Von Neumann analysis

3.1 Stability analysis for the advection equation

Consider the advection equation

$$u_t + au_x = 0. \tag{102}$$

(a) Using forward finite differences in time and space, *i.e*

$$u_x \approx \frac{u_{j+1}^n - u_j^n}{\Delta x} \text{ and } u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}, \tag{103}$$

show that the scheme is explicit by giving the update equation $u_j^{n+1} = f(u_{j+1}^n, u_j^n)$.

(b) Establish a stability criterion for this scheme using Von Neumann analysis.

(c) Why does the stability criterion depend on the sign of a? What happens if a downward space difference, *i.e*

$$u_x \approx \frac{u_j^n - u_{j-1}^n}{\Delta x}, \tag{104}$$

is used instead?
Hint: Remember that the general solution of the transport equation is $f(x - at)$.

(d) Consider now the so-called Lax-Friedrichs scheme, whose update equation is

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right). \tag{105}$$

Establish a stability criterion using Von Neumann analysis.

(e) Finally, consider the so-called Lax-Wendroff scheme, whose update equation is

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right). \tag{106}$$

Establish a stability criterion using Von Neumann analysis.

Solution

(a) Using finite differences approximation, the transport equation writes as

$$u_j^{n+1} = (1 + \alpha)\, u_j^n - \alpha u_{j+1}^n \tag{107}$$

where $\alpha = \frac{a\Delta t}{\Delta x}$.

(b) Introducing any error mode

$$\epsilon\,(x, t) = \hat{\epsilon}(k, t) \exp\left(-ikx\right) \tag{108}$$

in the discrete equation gives

$$\hat{\epsilon}(k, t_{n+1}) = (1 + \alpha)\, \hat{\epsilon}(k, t_n) - \alpha\hat{\epsilon}(k, t_n) \exp\left(-ik\Delta x\right). \tag{109}$$

Hence the amplification factor is given by

$$\left|\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)}\right|^2 = \left|1+\alpha\left(1-\exp\left(ik\Delta x\right)\right)\right|^2 \tag{110}$$

$$= 1+2\alpha\left(\alpha+1\right)\left(1-\cos\left(k\Delta x\right)\right) \tag{111}$$

such that the condition on α is

$$1+2\alpha\left(\alpha+1\right)\left(1-\cos\left(k\Delta x\right)\right) \leq 1 \tag{112}$$

$$\Rightarrow 2\alpha\left(\alpha+1\right)\left(1-\cos\left(k\Delta x\right)\right) \leq 0 \tag{113}$$

$$\Rightarrow 2\alpha\left(\alpha+1\right) \leq 0 \tag{114}$$

$$\Rightarrow -1 \leq \alpha \leq 0 \quad . \tag{115}$$

(c) This is because the numerical domain of dependance must not be disjoint from the real domain of dependance.

(d) Introducing any error mode

$$\epsilon\left(x,t\right) = \hat{\epsilon}(k,t)\exp\left(-ikx\right) \tag{116}$$

in the discrete equation gives

$$\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)} = \frac{1}{2}\left[\exp(-ik\Delta x)+\exp(+ik\Delta x)\right] - \frac{\alpha}{2}\left[\exp(-ik\Delta x)-\exp(+ik\Delta x)\right] \tag{117}$$

$$= \cos(k\Delta x)+i\alpha\sin(k\Delta x) \tag{118}$$

thus the norm squared is given by

$$\left|\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)}\right|^2 = \cos^2(k\Delta x)+\alpha^2\sin^2(k\Delta x) \tag{119}$$

$$= 1+\left[\alpha^2-1\right]\sin^2(k\Delta x). \tag{120}$$

Hence, the Lax-Friedrichs scheme is stable if

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 1. \tag{121}$$

(e) Introducing any error mode

$$\epsilon\left(x,t\right) = \hat{\epsilon}(k,t)\exp\left(-ikx\right) \tag{122}$$

in the discrete equation gives

$$\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)} = 1-\frac{\alpha}{2}\left(\exp\left(-ik\Delta x\right)-\exp\left(ik\Delta x\right)\right)+\frac{\alpha^2}{2}\left(\exp\left(-ik\Delta x\right)-2+\exp\left(ik\Delta x\right)\right), \tag{123}$$

$$= 1+i\alpha\sin(k\Delta x)+\alpha^2\left(\cos(k\Delta x)-1\right). \tag{124}$$

thus the norm squared is given by

$$\left|\frac{\hat{\epsilon}(k,t_{n+1})}{\hat{\epsilon}(k,t_n)}\right|^2 = 1+\alpha^4\left(\cos(k\Delta x)-1\right)^2+2\alpha^2\left(\cos(k\Delta x)-1\right)+\alpha^2\sin(k\Delta x)^2, \tag{125}$$

$$= 1+\alpha^4\left(\cos(k\Delta x)-1\right)^2+\alpha^2\left(2\cos(k\Delta x)-2\right)+\alpha^2\left(1-\cos(k\Delta x)^2\right), \tag{126}$$

$$= 1+\alpha^4\left(\cos(k\Delta x)-1\right)^2-\alpha^2\left(\cos(k\Delta x)^2-2\cos(k\Delta x)+1\right), \tag{127}$$

$$= 1+\alpha^4\left(\cos(k\Delta x)-1\right)^2-\alpha^2\left(\cos(k\Delta x)-1\right)^2, \tag{128}$$

$$= 1+\alpha^2\left(\alpha^2-1\right)\left(\cos(k\Delta x)-1\right)^2. \tag{129}$$

Hence, the Lax-Wendroff scheme is stable if

$$-1 \leq \frac{a\Delta t}{\Delta x} \leq 1. \tag{130}$$

3.2 Stability analysis for the 2D diffusion equation

Consider the 2D diffusion equation

$$u_t + a(u_{xx}+u_{yy}) = 0. \tag{131}$$

Using forward differences in time and central differences in space, establish a stability criterion using Von Neumann analysis.
Hint: Remember the following Fourier decomposition

$$\epsilon(x,y,t) = \int\int \hat{\epsilon}(k_x,k_y,t)\exp\left(-i(k_x x+k_y y)\right)dxdy \tag{132}$$

such that any mode of the error at any grid point expresses as

$$\epsilon_{rs}^n = \epsilon(x_r,y_s,t_n) = \hat{\epsilon}(k_x,k_y,t_n)\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right). \tag{133}$$

As in the 1D case, the stability is ensured in Von Neumann sense if the amplitude of each mode is not increasing, *i.e*, if

$$\gamma(k_x,k_y) \triangleq \left|\frac{\hat{\epsilon}\left(k_x,k_y,t_{n+1}\right)}{\hat{\epsilon}\left(k_x,k_y,t_n\right)}\right| \leq 1. \tag{134}$$

Solution

First discretize the 2D diffusion equation with spatial step size Δx, Δy and temporal step size Δt.

- Forward in time:

$$u_t = \frac{u_{r,s}^{n+1}-u_{r,s}^n}{\Delta t} \tag{135}$$

- Centered in space:

$$u_{xx} = \frac{u_{r+1,s}^n-2u_{r,s}^n+u_{r-1,s}^n}{\Delta x^2}, \tag{136}$$

$$u_{yy} = \frac{u_{r,s+1}^n-2u_{r,s}^n+u_{r,s-1}^n}{\Delta y^2}. \tag{137}$$

The 2D diffusion equation can thus be expressed as

$$\frac{u_{r,s}^{n+1}-u_{r,s}^{n}}{\Delta t}+a\left(\frac{u_{r+1,s}^{n}-2u_{r,s}^{n}+u_{r-1,s}^{n}}{\Delta x^{2}}+\frac{u_{r,s+1}^{n}-2u_{r,s}^{n}+u_{r,s-1}^{n}}{\Delta y^{2}}\right)=0 \tag{138}$$

or as

$$u_{r,s}^{n+1}=u_{r,s}^{n}-\frac{a\Delta t}{\Delta x^{2}}\left(u_{r+1,s}^{n}-2u_{r,s}^{n}+u_{r-1,s}^{n}\right)-\frac{a\Delta t}{\Delta y^{2}}\left(u_{r,s+1}^{n}-2u_{r,s}^{n}+u_{r,s-1}^{n}\right). \tag{139}$$

Introducing

$$\beta_x=\frac{a\Delta t}{\Delta x^{2}} \quad \text{and} \quad \beta_y=\frac{a\Delta t}{\Delta y^{2}}, \tag{140}$$

the update equation is then

$$u_{r,s}^{n+1}=u_{r,s}^{n}-\beta_x\left(u_{r+1,s}^{n}-2u_{r,s}^{n}+u_{r-1,s}^{n}\right)-\beta_y\left(u_{r,s+1}^{n}-2u_{r,s}^{n}+u_{r,s-1}^{n}\right). \tag{141}$$

The stability criterion is obtained by injecting an arbitrary error mode

$$\epsilon_{rs}^{n}=\epsilon(x_r,y_s,t_n)=\hat{\epsilon}\left(k_x,k_y,t_n\right)\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right) \tag{142}$$

into the discretized equation. For conciseness, $\hat{\epsilon}\left(k_x,k_y,t_n\right)$ is written as $\hat{\epsilon}_n$.
The evolution of the amplitude of any mode is then governed by

$$\begin{aligned}\hat{\epsilon}_{n+1}\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right)=\hat{\epsilon}_n\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right)\\ -\beta_x\big[\hat{\epsilon}_n\exp\left(-i(k_x(r+1)\Delta x+k_y s\Delta y)\right)-2\hat{\epsilon}_n\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right)\\ +\hat{\epsilon}_n\exp\left(-i(k_x(r-1)\Delta x+k_y s\Delta y)\right)\big]\\ -\beta_y\big[\hat{\epsilon}_n\exp\left(-i(k_x r\Delta x+k_y(s+1)\Delta y)\right)-2\hat{\epsilon}_n\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right)\\ +\hat{\epsilon}_n\exp\left(-i(k_x r\Delta x+k_y(s-1)\Delta y)\right)\big].\end{aligned} \tag{143}$$

Dividing both sides by $\hat{\epsilon}^n\exp\left(-i(k_x r\Delta x+k_y s\Delta y)\right)$ yields

$$\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n}=1-\beta_x\left(e^{-ik_x\Delta x}-2+e^{ik_x\Delta x}\right)-\beta_y\left(e^{-ik_y\Delta y}-2+e^{ik_y\Delta y}\right) \tag{144}$$

$$\Rightarrow \qquad \frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n}=1-2\beta_x\left(\cos k_x\Delta x-1\right)-2\beta_y\left(\cos k_y\Delta y-1\right) \tag{145}$$

$$\Rightarrow \qquad \frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_n}=1+4\beta_x\sin^2\frac{k_x\Delta x}{2}+4\beta_y\sin^2\frac{k_y\Delta y}{2}. \tag{146}$$

Ensuring that all modes are not divergent, *i.e*

$$\left|\frac{\hat{\epsilon}(k_x,k_y,t_{n+1})}{\hat{\epsilon}(k_x,k_y,t_n)}\right|\leq 1 \quad \forall k_x,k_y \tag{147}$$

gives

$$-1\ \leq 1+4\beta_x\sin^2\frac{k_x\Delta x}{2}+4\beta_y\sin^2\frac{k_y\Delta y}{2}\leq 1 \tag{148}$$

$$\Rightarrow -\frac{1}{2}\leq \beta_x\sin^2\frac{k_x\Delta x}{2}+\beta_y\sin^2\frac{k_y\Delta y}{2} \qquad \leq 0. \tag{149}$$

These two inequalities must be verified for any k_x, k_y. The most restrictive cases for the rightmost inequality are obtained by considering $\left\{\sin^2\frac{k_x\Delta x}{2}, \sin^2\frac{k_y\Delta y}{2}\right\} = \{1, 0\}$ and $\left\{\sin^2\frac{k_x\Delta x}{2}, \sin^2\frac{k_y\Delta y}{2}\right\} = \{0, 1\}$ which yields

$$\beta_x \leq 0 \quad \text{and} \quad \beta_y \leq 0. \tag{150}$$

Then β_x and β_y being negative, the most restrictive case for the leftmost inequality is obtained considering $\left\{\sin^2\frac{k_x\Delta x}{2}, \sin^2\frac{k_y\Delta y}{2}\right\} = \{1, 1\}$ which yields

$$-\frac{1}{2} \leq \beta_x + \beta_y = a\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \triangleq \frac{a\Delta t}{\Delta l^2}. \tag{151}$$

The diffusion coefficient a being negative because β_x and β_y are negative, condition Eq.(151) writes

$$\frac{\Delta t}{\Delta l^2} \leq \frac{1}{2(-a)}. \tag{152}$$

Remark The conventional way to write a diffusion equation is

$$u_t - \alpha\left(u_{xx} + u_{yy}\right) = 0 \tag{153}$$

i.e with a minus sign in front of the diffusion coefficient.
With that new convention, the stability condition is then

$$\frac{\Delta t}{\Delta l^2} \leq \frac{1}{2\alpha}. \tag{154}$$

4 Diffusion equation

4.1 Diffusion of a window [Strauss 2.4, Ex. 1]

Solve the following initial value problem using the fundamental solution of the diffusion equation,

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = 1 & \text{for} \quad |x| < l, \\ u(x,0) = 0 & \text{for} \quad |x| \geq l. \end{cases}$$

Write your answer in terms of $\operatorname{erf}(x)$.
Reminder:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta.$$

Solution

The general solution is given by

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{4\pi kt}} \int_{-\ell}^{\ell} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) dy \\ =& \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\ell} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) dy - \frac{1}{\sqrt{4\pi kt}} \int_{0}^{-\ell} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) dy. \end{aligned} \tag{155}$$

Introducing the following change of variables

$$\xi = \frac{x-y}{\sqrt{4kt}} \tag{156}$$

gives

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\frac{x+\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi - \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\frac{x-\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi \tag{157}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\frac{x+\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi + \frac{1}{\sqrt{\pi}} \int_{\frac{x-\ell}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} \exp(-\xi^2) d\xi \tag{158}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-\ell}{\sqrt{4kt}}}^{\frac{x+\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi \tag{159}$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-\ell}{\sqrt{4kt}}} \exp(-\xi^2) d\xi \tag{160}$$

$$= \frac{1}{2} \left[\operatorname{erf}\left(\frac{x+\ell}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-\ell}{\sqrt{4kt}}\right)\right]. \tag{161}$$

4.2 Diffusion of an exponential [Strauss 2.4, Ex. 3]

Solve the following initial value problem using the fundamental solution of the diffusion equation,

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \exp(3x) & \text{for} \quad (x,t) \in \mathbb{R} \times \{0\}. \end{cases}$$

Reminder:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \qquad \text{(Gaussian integral).}$$

Solution

Replacing the given initial conditions in the general formula successively gives

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) \exp(3y) dy \\ &= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2+2xy-x^2}{4kt} + 3y\right) dy \\ &= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2+2xy-x^2+12kty}{4kt}\right) dy. \end{aligned} \tag{162}$$

Completing the square appearing in the exponential, the solution writes

$$u(x,t) = \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-y^2+2xy-x^2+12kty-36k^2t^2+36k^2t^2+12ktx-12ktx}{4kt}\right) dy \tag{163}$$

$$= \frac{1}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y+6kt}{\sqrt{4kt}}\right]^2 + \frac{36k^2t^2+12ktx}{4kt}\right) dy \tag{164}$$

$$= \frac{\exp(9kt+3x)}{\sqrt{4kt\pi}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y+6kt}{\sqrt{4kt}}\right]^2\right) dy. \tag{165}$$

Finally using the change of variable

$$\xi = \frac{x-y+6kt}{\sqrt{4kt}}, \tag{166}$$

the solution can be written as follows

$$u(x,t) = \frac{\exp(9kt+3x)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left(-\xi^2\right) d\xi \tag{167}$$

$$= \exp\left(3(x+3kt)\right). \tag{168}$$

4.3 Diffusion with constant dissipation [Strauss 2.4, Ex. 16]

Solve the following diffusion problem with constant dissipation

$$\begin{cases} u_t - ku_{xx} + bu = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \delta(x) & \text{for} \quad (x,t) \in \mathbb{R} \times \{0\}. \end{cases}$$

where $b \in \mathbb{R}_0^+$ is a constant parameter.
Reminder: The Dirac distribution δ has the following property

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0).$$

Hint: use the substitution $u(x,t) = e^{-bt}v(x,t)$.

Solution

Using the substitution $u(x,t) = e^{-bt}v(x,t)$, the terms in the equation become

$$u_t = -b\exp(-bt)\, v + \exp(-bt)\, v_t, \tag{169}$$
$$u_{xx} = \exp(-bt)\, v_{xx}. \tag{170}$$

Thus the equation takes the form

$$\exp(-bt)\, v_t - k\exp(-bt)\, v_{xx} = 0, \tag{171}$$

which can be simplified into an equation of the known form:

$$v_t - kv_{xx} = 0. \tag{172}$$

The initial condition writes $v(x,0) = \delta(x)$. The solution of this equation is then written as

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{x-y}{\sqrt{4kt}}\right]^2\right) \delta(y)\mathrm{d}y \tag{173}$$
$$= \frac{1}{\sqrt{4\pi kt}} \exp\left(\frac{-x^2}{4kt}\right) \tag{174}$$

and finally

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(\frac{-x^2}{4kt}\right) \exp(-bt). \tag{175}$$

4.4 Heat equation with convection [Strauss 2.4, Ex. 18]

Consider the following diffusion problem with convection

$$\begin{cases} u_t - ku_{xx} + Vu_x = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \phi(x) & \text{for} \quad (x,t) \in \mathbb{R} \times \{0\}, \end{cases}$$

where V is a constant.

(a) Solve this problem for a general ϕ.
Hint: Consider a moving reference frame by using the change of variables $y = x - Vt$ and $z = t$.

(b) Compute the solution for the initial condition $\phi(x) = 1$ for $|x| \leq l$. Write your answer in terms of $\mathrm{erf}(x)$.

Solution

(a) Performing the proposed change of variable, the differential operators write

$$\partial_x = \frac{\partial y}{\partial x}\,\partial_y + \frac{\partial z}{\partial x}\,\partial_z = \partial_y, \tag{176}$$
$$\partial_t = \frac{\partial y}{\partial t}\,\partial_y + \frac{\partial z}{\partial t}\,\partial_z = -V\,\partial_y + \partial_z. \tag{177}$$

The equation becomes

$$-Vu_y + u_z - ku_{yy} + Vu_y = 0, \tag{178}$$

which simplifies as

$$u_z - ku_{yy} = 0. \tag{179}$$

The general solution of Eq.(179) is

$$u(y,z) = \frac{1}{\sqrt{4\pi kz}} \int_{-\infty}^{+\infty} \exp\left(-\left[\frac{y-\zeta}{\sqrt{4kz}}\right]^2\right)\phi(\zeta)\mathrm{d}\zeta, \tag{180}$$
$$= \frac{1}{\sqrt{4\pi kz}} \int_{-\ell}^{+\ell} \exp\left(-\left[\frac{y-\zeta}{\sqrt{4kz}}\right]^2\right)\mathrm{d}\zeta. \tag{181}$$

(b) Therefore, the solution writes

$$u(y,z) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{y+l}{\sqrt{4kz}}\right) - \operatorname{erf}\left(\frac{y-l}{\sqrt{4kz}}\right)\right], \tag{182}$$
$$= \frac{1}{2}\left[\operatorname{erf}\left(\frac{x-Vt+l}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-Vt-l}{\sqrt{4kt}}\right)\right]. \tag{183}$$

4.5 Viscous Burgers' equation and Cole-Hopf transformation

Consider the momentum equation for an incompressible fluid (Navier-Stokes equations), that is

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{f} + \nu\nabla^2\mathbf{u}.$$

Assume that there are no body forces ($\mathbf{f} = \mathbf{0}$), and that the pressure gradient term can be neglected ($\nabla p = \mathbf{0}$), *i.e.*

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\nabla^2\mathbf{u}.$$

Also assume that the problem is in one-dimension by posing $\mathbf{u} = u(x,t)\mathbf{e_x}$ and writing the momentum equation along the x-axis which finally gives

$$u_t + uu_x = \nu u_{xx}, \quad x\in\mathbb{R}, \quad t\geq 0,$$

the so-called viscous Burgers' equation. This non-linear partial differential equation can be rewritten in the form of the diffusion equation

$$\phi_t = \nu\phi_{xx}$$

by using appropriate methods (*e.g.* Cole-Hopf transformation).

(a) First, let us consider the substitution

$$U_x = u.$$

Show that the Burgers' equation can be rewritten as

$$U_t = \nu U_{xx} - \frac{1}{2}(U_x)^2. \qquad (\star)$$

Reminder: $uu_x = \frac{1}{2}(u^2)_x$.

(b) Use the Cole-Hopf relation

$$U(x,t) = -2\nu \ln(\phi(x,t))$$

to simplify Eq.($\star$) into the diffusion equation

$$\phi_t = \nu \phi_{xx}.$$

(c) Find the solution $\phi(x,t)$ to the diffusion problem and show that the solution of Burgers' equation can be written as

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right] dy}.$$

Use the fact that

$$\partial_x \int_{-\infty}^{+\infty} f(x,y)dy = \int_{-\infty}^{+\infty} \partial_x f(x,y)dy,$$

that $U_x(x,0) = u(x,0)$, *i.e.*,

$$U(x,0) = \int_0^x u(z,0)dz,$$

and that

$$\begin{aligned}\phi(x,0) &= \exp\left(-\frac{U(x,0)}{2\nu}\right), \\ &= \exp\left(-\frac{1}{2\nu}\int_0^x u(z,0)dz\right).\end{aligned}$$

Solution

(a) First write Burgers' equation in a simpler form

$$u_t + uu_x = \nu u_{xx} \qquad (184)$$

$$\Rightarrow u_t + \frac{1}{2}(u^2)_x = \nu u_{xx}. \qquad (185)$$

Now use the substitution suggested in the statement which gives

$$U_{tx} + \frac{1}{2}((U_x)^2)_x = \nu U_{xxx}, \quad (U_{xt} = U_{tx}) \qquad (186)$$

i.e., upon integration

$$U_t + \frac{1}{2}(U_x)^2 = \nu U_{xx}, \qquad (187)$$

$$\Rightarrow \quad U_t = \nu U_{xx} - \frac{1}{2}(U_x)^2. \qquad (188)$$

(b) Consider the Cole-Hopf relation

$$U(x,t) = -2\nu \ln(\phi(x,t)). \tag{189}$$

Writing each derivates of U that appears in the equation above yields

$$\begin{cases} U_t = -2\nu\partial_t \ln(\phi(x,t)) = -2\nu \dfrac{\phi_t}{\phi} & (190) \\ U_x = -2\nu \dfrac{\phi_x}{\phi} & (191) \\ U_{xx} = -2\nu \left(\dfrac{\phi_{xx}}{\phi} - \dfrac{\phi_x \phi_x}{\phi^2}\right). & (192) \end{cases}$$

Plugging these results into Eq.(188) yields

$$-2\nu \frac{\phi_t}{\phi} = -2\nu^2 \frac{\phi_{xx}}{\phi} + 2\nu^2 \frac{(\phi_x)^2}{\phi^2} - \frac{1}{2}\left(4\nu^2 \frac{(\phi_x)^2}{\phi^2}\right). \tag{193}$$

Simplifying each terms indeed yields the diffusion equation

$$\phi_t = \nu \phi_{xx}.$$

(c) The fundamental solution of the diffusion equation is given by

$$\begin{aligned} \phi(x,t) &= \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4\nu t}\right) \phi(y,0) dy & (194) \\ &= \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right) dy, & (195) \end{aligned}$$

using the definition of $\phi(x,0)$ given in the statement.
Then

$$U(x,t) = -2\nu \ln\left[\frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right) dy\right] \tag{196}$$

and finally the solution $u(x,t)$ is

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right] dy}.$$

5 Separation of variables

5.1 Diffusion equation I

Consider the one-dimensional diffusion equation on a bounded domain

$$u_t - k u_{xx} = 0 \quad \forall x \in]0,1[\tag{‡}$$

with initial condition

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[$$

and with some boundary conditions at $x = 0$ and $x = 1$ (these boundary conditions will be specified later).

(a) Using separation of variables $u(x,t) = w(t)v(x)$, find all the separable solutions of Eq. (‡).

(b) Find the solution to Eq. (‡) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\, \forall t \geq 0$ and for $\phi(x) = \sin \pi x$.

(c) Find the solution to Eq. (‡) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\, \forall t \geq 0$ and for

$$\phi(x) = \begin{cases} x & \forall x \in \left]0, \dfrac{1}{2}\right], \\ 1 - x & \forall x \in \left[\dfrac{1}{2}, 1\right[. \end{cases}$$

(d) Find the solution to Eq. (‡) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$ $\forall t \geq 0$ and for $\phi(x) = 1$.

(e) Find the solution to Eq. (‡) for the homogeneous Neumann boundary conditions $u_x(0,t) = u_x(1,t) = 0$ $\forall t \geq 0$ and for $\phi(x) = \cos \pi x$.

(f) Find the solution to Eq. (‡) for the non-homogeneous Dirichlet boundary conditions $u_x(0,t) = 0$ and $u_x(1,t) = 1$, $\forall t \geq 0$ and for $\phi(x) = x + \sin \pi x$.

Solution

(a) Using the ansatz $u = wv$, the diffusion equation writes as

$$w'v - kwv'' = 0 \tag{197}$$

$$\Rightarrow \quad \frac{w'}{kw} = \frac{v''}{v}. \tag{198}$$

Since the right hand side depends only on x while the left hand side depends only on t, both sides must be equal to a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w' - k\lambda w = 0. \tag{199}$$

Spatial dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \qquad \Rightarrow v = Ax + B, \tag{200}$$

$$\text{if } \lambda = \omega^2 > 0 \qquad \Rightarrow v = C\exp(-\omega x) + D\exp(\omega x), \tag{201}$$

$$\text{if } \lambda = -\omega^2 < 0 \qquad \Rightarrow v = E\cos(\omega x) + F\sin(\omega x). \tag{202}$$

Time dependence Whatever the sign of λ, the time dependence is given by

$$w = \exp(\lambda k t). \tag{203}$$

The constant in front the exponential is here omitted because the field of interest is $u = vw$.

(b) Among all the eigensolutions found in the previous sub-question (*i.e* for any value of λ), only those that satisfy the homogeneous boundary conditions are kept.

Stationary eigensolutions, *i.e*: $\lambda = 0$ Applying the boundary conditions to $v = Ax + B$ gives

$$\begin{cases} v(0) & = B & = 0 \\ v(1) & = A + B & = 0 \end{cases} \quad \Rightarrow A = B = 0 \tag{204}$$

such that there is no stationary eigensolution satisfying the homogeneous boundary conditions.

Time-growing eigensolutions, *i.e*: $\lambda > 0$ Applying the boundary conditions to $v = C\exp(-\omega x) + D\exp(\omega x)$ gives

$$\begin{cases} v(0) & = C + D & = 0 \\ v(1) & = C\exp(-\omega) + D\exp(\omega) & = 0 \end{cases} \quad \Rightarrow \begin{pmatrix} 1 & 1 \\ \exp(-\omega) & \exp(\omega) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{205}$$

provided that $\omega > 0$, the unique solution to Eqs(205) is

$$C = D = 0 \tag{206}$$

thus there is no time growing eigensolution satisfying the homogeneous boundary condition neither.

Time-decaying eigensolutions, *i.e*: $\lambda < 0$ Applying the boundary conditions to $v = E\cos(\omega x) + F\cos(\omega x)$ gives

$$\begin{cases} v(0) & = E & = 0 \\ v(1) & = E\cos(\omega) + F\sin(\omega) & = 0 \end{cases} \quad \Rightarrow F\sin(\omega) = 0 \quad \Rightarrow \omega \to \omega_n = n\pi, \quad n = 1, 2, 3, \dots \tag{207}$$

One should be carefull that the values $\omega_n = n\pi$ for $n < 0$ are not considered because they yield the same eigenvalue $\lambda_n = -\omega_n^2$.
The only eigensolutions compatible with the boundary conditions are therefore

$$v_n(x) = F_n \sin(\omega_n x), \quad \forall n = 1, 2, 3, \dots \tag{208}$$

and the most general solution compatible with boundary condition is then

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp\left(-\omega_n^2 k t\right). \tag{209}$$

All the constants F_n must still be determined through the initial condition

$$u(x,0) = \phi(x) = \sum_{n=1}^{\infty} F_n \sin n\pi x. \tag{210}$$

Eq.(210) is actually a Fourier sine series which has the following orthogonality property

$$F_m = 2\int_0^1 \phi(x)\sin(m\pi x)\,dx. \tag{211}$$

In the particular case where $\phi(x) = \sin \pi x$, the unknwown coeficient are given by

$$F_m = 2\int_0^1 \sin(\pi x)\sin(m\pi x)\,dx = \delta_{1m} \tag{212}$$

such that the final solution is

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin(\omega_n x)\exp(-\omega_n^2 kt) = \sin(\pi x)\exp(-\pi^2 kt). \tag{213}$$

(c) The boundary conditions are the same than for the previous subquestion such that the same set of eigen-solutions must be conserved, only the coefficient F_n of the Fourier sine series change. The Fourier coefficients are given by

$$\begin{aligned}
F_m &= 2\int_0^1 \phi(x)\sin(m\pi x)\,dx && (214)\\
&= 2\int_0^{1/2} x\sin(m\pi x)\,dx + 2\int_{1/2}^1 (1-x)\sin(m\pi x)\,dx && (215)\\
&= \frac{4\sin\left(\frac{m\pi}{2}\right)}{m^2\pi^2} = \begin{cases} \dfrac{4(-1)^k}{(2k+1)^2\pi^2}, & m = 2k+1.\\ 0, & m = 2k. \end{cases} && (216)
\end{aligned}$$

The final solution is therfore given by

$$u(x,t) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}(-1)^k \frac{\sin\Big((2k+1)\pi x\Big)}{(2k+1)^2}\exp\Big(-(2k+1)^2\pi^2 kt\Big) \tag{217}$$

(f) The problem here is that the boundary conditions are not homogeneous. Therefore, if all modes satisfy the non-homogeneous boundary conditions, their sum will not satisy the boundary condition. The idea is then to first choose a group of eigenfunctions that satisfy the non-homogeneous boundary condition, then to add the eigensolutions of the problem with homogeneous boundary conditions. This is actually the same idea than when solving an initial value problem, the whole solution is the sum of a particular solution and the homogeneous solution.
First, the procedure is therefore to find a set of solutions that satisfy th non homogeneous boundary condition

$$\begin{cases} u(0,t) &= w(t)v(0) &= 0 \qquad \forall t,\\ u(1,t) &= w(t)v(1) &= 1 \qquad \forall t. \end{cases} \tag{218}$$

While obviously the boundary condition at $x = 0$ yields $v(0) = 0$, the boundary condition at $x = 1$ implies that $w(t)$ does not depend on t. The natural eigensolutions to consider to satisfy the non-homogeneous solutions must therefore have a temporal part that is constant, which is typically the case for $\lambda = 0$. Applying the boundary condition to the steady eigensolutions one finds

$$\begin{cases} u(0,t) & = w(t)v(0) & = \exp 0kt\,(B) & = 0 & \forall t \\ u(1,t) & = w(t)v(1) & = \exp 0kt\,(A+B) & = 1 & \forall t \end{cases} \Rightarrow \begin{cases} B & = 0 \\ A & = 1 \end{cases} \tag{219}$$

Now that a unique solution satisfying non-homogeneous boundary conditions is known, all the solutions satisfying the homogeneous boundary condition must be added. Referring to sub-questions, the solution becomes

$$u(x,t) = x \exp 0kt + \sum_{n=1}^{\infty} F_n \sin(\omega_n x) \exp(-\omega_n^2 kt), \tag{220}$$

then to initial condition yields

$$u(x,0) = x + \sum_{n=1}^{\infty} F_n \sin(\omega_n x) = \phi(x) = x + \sin(\pi x), \tag{221}$$

and, as in sub-question, the solution is

$$u(x,t) = x + \sin(\pi x) \exp(-\pi^2 kt). \tag{222}$$

5.2 Wave equation I

Consider the one-dimensional wave equation on a bounded domain

$$u_{tt} - c^2 u_{xx} = 0 \quad \forall x \in]0,1[\tag{$\diamond$}$$

with initial condition

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x) \quad \forall x \in]0,1[$$

and with some boundary conditions at $x = 0$ and $x = 1$ (these conditions will be specified later).

(a) Using separation of variable $u(x,t) = w(t)v(x)$, find all the separable solution of Eq. ($\diamond$).

(b) Find the solution to Eq. ($\diamond$) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\,\forall t \geq 0$, for $\phi(x) = \sin \pi x$ and for $\psi(x) = 0$.

(c) Find the solution to Eq. ($\diamond$) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\,\forall t \geq 0$, for

$$\phi(x) = \begin{cases} x & \forall x \in]0, \frac{1}{2}] \\ 1 - x & \forall x \in [\frac{1}{2}, 1[, \end{cases}$$

and for $\psi(x) = 0$.

(d) Find the solution to Eq. ($\diamond$) for the homogeneous Dirichlet boundary conditions $u(0,t) = u(1,t) = 0\,\forall t \geq 0$ for $\phi(x) = \sin \pi x$ and for $\psi(x) = c \sin \pi x$.

Solution

(a) Using the ansatz $u = wv$, the wave equation writes as

$$w''v - c^2wv'' = 0 \tag{223}$$
$$\Rightarrow \frac{w''}{c^2w} = \frac{v''}{v}. \tag{224}$$

Since the right hand side depends only on x while the left hand side depends only on t, both sides must be equal to a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w'' - c^2\lambda w = 0. \tag{225}$$

Spatial dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \qquad \Rightarrow v = Ax + B, \tag{226}$$
$$\text{if } \lambda = \omega^2 > 0 \qquad \Rightarrow v = C\exp(-\omega x) + D\exp(\omega x), \tag{227}$$
$$\text{if } \lambda = -\omega^2 < 0 \qquad \Rightarrow v = E\cos(\omega x) + F\sin(\omega x). \tag{228}$$

Time dependence Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \qquad \Rightarrow v = Gt + H, \tag{229}$$
$$\text{if } \lambda = \omega^2 > 0 \qquad \Rightarrow v = I\exp(-c\omega t) + J\exp(c\omega t), \tag{230}$$
$$\text{if } \lambda = -\omega^2 < 0 \qquad \Rightarrow v = K\cos(c\omega t) + L\sin(c\omega t). \tag{231}$$

(b) The eigenvalues are given by $\lambda_n = -\omega_n^2 = -n^2\pi^2$ with $n = 1, 2, 3, ...$ such that the only compatible eigensolutions are

$$v_n(x) = F_n \sin(w_n x), \ \ \forall n = 1, 2, 3, ... \tag{232}$$

The most general way to write the solution is

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(nc\pi t) + B_n \sin(nc\pi t)] \sin(w_n x). \tag{233}$$

It remains to determine the constants A_n et B_n using initial conditions. Since the time dependence of the wave equation governed by a second order differential equation there are two initial conditions. One will help finding the A_n and the other will help determining the B_n.
At $t = 0$ one has

$$u(x,0) = \sin(\pi x) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \tag{234}$$
$$u_t(x,0) = 0 = \sum_{n=1}^{+\infty} B_n cn\pi \sin(n\pi x) \tag{235}$$

The B_n are immediately identified as 0. It can be shown that $A_n = \delta_{1n}$ using the orthogonality of Fourier sine series. The final solution is then

$$u(x,t) = \cos(c\pi t)\sin(\pi x). \tag{236}$$

(c) Similarly to the preceding sub-questions, starting from Eq.(233), the constants A_n and B_n are determined with the initial conditions. First,

$$u_t(x,0) = 0 = \sum_{n+1}^{+\infty} nc\pi B_n \sin(w_n x). \tag{237}$$

As the functions $\sin(w_n x)$ form a basis of $]0,1[$, the only possible combination of B_n satisfying the equality is $B_n = 0$. The second initial condition is

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(w_n x) = \phi(x). \tag{238}$$

Using then the orthogonality property of the sin functions,

$$A_n = 2\int_0^1 \phi(x)\sin(n\pi x)dx \tag{239}$$

$$= 2\int_0^{1/2} x\sin(n\pi x)dx + 2\int_{1/2}^1 (1-x)\sin(n\pi x)dx. \tag{240}$$

Integrating by parts gives

$$A_n = \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} = \begin{cases} \frac{4(-1)^k}{(2k+1)^2\pi^2}, & n = 2k+1, \\ 0, & n = 2k. \end{cases} \tag{241}$$

Therefore the solution is

$$u(x,t) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}(-1)^k \frac{\cos\Big((2k+1)c\pi t\Big)\sin\Big((2k+1)\pi x\Big)}{(2k+1)^2}. \tag{242}$$

(d) Similar to preceeding sub-questions, the A_n are given by

$$A_n = \delta_{1n} \tag{243}$$

while the B_n are derived starting from

$$u_t(x,0) = c\sin(\pi x) = \sum_{n=1}^{+\infty} B_n cn\pi \sin(n\pi x). \tag{244}$$

Using the same procedure as for the determination of the A_n one can show that

$$B_n = \frac{\delta_{1n}}{n\pi}. \tag{245}$$

$$u(x,t) = \left[\cos(c\pi t) + \frac{\sin(c\pi t)}{\pi}\right]\sin(\pi x). \tag{246}$$

5.3 Laplace equation I

Consider the two-dimensional Laplace equation on a square

$$u_{xx} + u_{yy} = 0 \quad \{x, y\} \in]0,1[\times]0,1[.$$

(a) Using separation of variables $u(x, y) = v(x)w(y)$, find all the separable solutions of this equation.

(b) Find the set of separable solutions that verify the following homogeneous boundary conditions

$$\begin{cases} u(0, y) = 0 & \forall y \in [0, 1], \\ u(1, y) = 0 & \forall y \in [0, 1], \\ u(x, 1) = 0 & \forall x \in [0, 1]. \end{cases}$$

(c) Show that the Fourier sine series expansion of

$$\phi(x) = \begin{cases} x & \forall x \in]0, \frac{1}{2}], \\ 1 - x & \forall x \in [\frac{1}{2}, 1[\end{cases}$$

can be expressed as

$$\phi(x) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin\left((2j + 1)\pi x\right)}{(2j + 1)^2}.$$

(d) With the boundary conditions given in (b) and

$$u(x, 0) = \phi(x), \quad \forall x \in [0, 1]$$

for the last edge, show that the solution to the boundary value problem is then

$$u(x, y) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin\left((2j + 1)\pi x\right) \sinh\left((2j + 1)\pi(1 - y)\right)}{(2j + 1)^2 \sinh\left((2j + 1)\pi\right)}.$$

Solution

(a) Using separation of variables, Laplace equation can be written

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda. \tag{247}$$

Therefore, both spatial part of the solution depends on the sign of the same constant λ. The following table presents all the separable solutions as a function of λ.

	Part $X(x)$	Part $Y(y)$
$\lambda = 0$	$X(x) = Ax + B$	$Y(y) = Cy + D$
$\lambda = \omega^2 > 0$	$X(x) = E\cosh(\omega x) + F\sinh(\omega x)$	$Y(y) = G\cos(\omega y) + H\sin(\omega y)$
$\lambda = -\omega^2 < 0$	$X(x) = I\cos(\omega x) + J\sin(\omega x)$	$Y(y) = K\cosh(\omega y) + L\sinh(\omega y)$

(b) Applying the boundary condition, depending on the sign of λ yields

(1) If $\lambda = 0$

$$\begin{cases} B = 0, \\ A + B = 0, \\ C + D = 0, \end{cases} \tag{248}$$

which admits the only trivial solution $A = B = 0$.

(2) If $\lambda = \omega^2 > 0$

$$\begin{cases} E = 0, \\ E\cosh(\omega) + F\sinh(\omega) = 0, \\ G\cos(\omega) + H\sin(\omega) = 0, \end{cases} \tag{249}$$

which admits the only trivial solution $E = F = 0$.

(3) If $\lambda = -\omega^2 < 0$

$$\begin{cases} I = 0, \\ I\cos(\omega) + J\sin(\omega) = 0, \\ K\cosh(\omega) + L\sinh(\omega) = 0, \end{cases} \tag{250}$$

which admits non trivial solution for $\omega_n = n\pi$ and $L = -\dfrac{K}{\tanh\omega}$.

Therefore, the solution writes

$$u(x, y) = \sum_{n=1}^{\infty} K_n \sin(n\pi x)\left[\cosh(n\pi y) - \frac{\sinh(n\pi y)}{\tanh(n\pi)}\right]. \tag{251}$$

Alternatively, to simplify the notation, one could also write the mode $Y(y)$ when $\lambda = -\omega^2 < 0$ under the form

$$Y(y) = K\cosh(\omega(1 - y)) + L\sinh(\omega(1 - y)), \tag{252}$$

such that the application of the boundary conditions simply gives

$$K = 0. \tag{253}$$

With these notations, the solution writes

$$u(x,y) = \sum_{n=1}^{\infty} L_n \sin(n\pi x)\sinh(n\pi(1-y)). \tag{254}$$

It is this expression that will be used in the following because of its simplicity.

(c) The coefficient of the Fourrier expansion of the function $\phi(x)$ can be obtained by calculating the integrals

$$2\int_0^1 \sin(n\pi x)\phi(x)\mathrm{d}x, \tag{255}$$

which can be integrated by part.

(d) Applying the last boundary condition gives

$$\begin{aligned} & u(x,0) = \phi(x), \\ \Rightarrow & \sum_{n=1}^{\infty} L_n \sin(n\pi x)\sinh(n\pi) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(-1)^j \frac{\sin((2j+1)\pi x)}{(2j+1)^2}, \end{aligned} \tag{256}$$

Identifying the terms in those series, one can find the expression for L_n:

$$L_n = \begin{cases} 0 & \text{if } n \text{ pair}, \\ \dfrac{-4}{\pi^2 n^2 \sinh(n\pi)} & \text{if } n \text{ odd and } n \mod 4 = 1, \\ \dfrac{4}{\pi^2 n^2 \sinh(n\pi)} & \text{if } n \text{ odd and } n \mod 4 = 3. \end{cases} \tag{257}$$

Replacing this in Eq. (254) and redefining the summation index, finally gives

$$u(x,y) = \frac{4}{\pi^2}\sum_{j=0}^{\infty}(-1)^j \frac{\sin((2j+1)\pi x)\sinh((2j+1)\pi(1-y))}{(2j+1)^2\sinh((2j+1)\pi)}. \tag{258}$$

5.4 Laplace equation II

Consider the following equation on a square

$$u_{xx} + \frac{1}{4}u_{yy} = 0 \quad \{x,y\} \in]0,1[\times]0,1[$$

with the boundary conditions

$$\begin{cases} u_x(0,y) = 0 & \forall y \in [0,1], \\ u_x(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = 0 & \forall x \in [0,1], \\ u(x,0) = 2\cos 2\pi x - 1 & \forall x \in [0,1]. \end{cases}$$

Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem.

Solution

Using separation of variables, the equation can be written:

$$\frac{X''}{X} = -\frac{Y''}{4Y} = \lambda. \tag{259}$$

Therefore, both spatial part of the solution depends on the sign of the same constant λ.
The following table presents all the separable solutions as a function of λ.

	Part $X(x)$	Part $Y(y)$
$\lambda = 0$	$X(x) = Ax + B$	$Y(y) = Cy + D$
$\lambda = \omega^2 > 0$	$X(x) = E\cosh(\omega x) + F\sinh(\omega x)$	$Y(y) = G\cos(2\omega y) + H\sin(2\omega y)$
$\lambda = -\omega^2 < 0$	$X(x) = I\cos(\omega x) + J\sin(\omega x)$	$Y(y) = K\cosh(2\omega y) + L\sinh(2\omega y)$

Applying the homogeneous boundary conditions on each mode, depending on the sign of λ, one has:

(1) If $\lambda = 0$

$$\begin{cases} A = 0, \\ A = 0, \\ C + D = 0, \end{cases} \tag{260}$$

which admits non trivial solutions for $A = 0$ and $D = -C$. Redefining a new constant, the associated mode can be written

$$X_0(x)Y_0(y) = C_0(y-1). \tag{261}$$

(2) If $\lambda = \omega^2 > 0$

$$\begin{cases} \omega F = 0, \\ \omega\left[E\sinh(\omega) + F\cosh(\omega)\right] = 0, \\ G\cos(2\omega) + H\sin(2\omega) = 0, \end{cases} \tag{262}$$

which admits the only trivial solution $E = F = 0$.

(3) If $\lambda = -\omega^2 < 0$

$$\begin{cases} J = 0, \\ \omega\left[I\sin(\omega) + J\cos(\omega)\right] = 0, \\ K\cosh(2\omega) + L\sinh(2\omega) = 0, \end{cases} \tag{263}$$

which admits non trivial solution for $\omega_n = n\pi$ and $L = -\dfrac{K}{\tanh(2\omega)}$.

Therefore, the solution writes

$$u(x,y) = C_0(y-1) + \sum_{n=1}^{\infty} K_n \cos(n\pi x)\left[\cosh(2n\pi y) - \frac{\sinh(2n\pi y)}{\tanh(2n\pi)}\right]. \tag{264}$$

Finally, applying the remaining boundary condition

$$\begin{aligned} &u(x,0) = 2\cos 2\pi x - 1, \\ \Rightarrow &-C_0 + \sum_{n=1}^{\infty} K_n \cos(n\pi x) = 2\cos(2\pi x) - 1. \end{aligned} \tag{265}$$

Identifying the coefficient, the final solution is given by

$$u(x,y) = y - 1 + 2\cos(2\pi x)\left[\cosh(4\pi y) - \frac{\sinh(4\pi y)}{\tanh(4\pi)}\right]. \tag{266}$$

5.5 Laplace-like equation

Consider the following equation on a square

$$u_{xx} + 2u_y + u_{yy} = 0 \quad \{x,y\} \in]0,1[\times]0,1[$$

with the boundary conditions, for a given function f,

$$\begin{cases} u(0,y) = 0 & \forall y \in [0,1], \\ u(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = f(x) & \forall x \in [0,1], \\ u(x,0) = 0 & \forall x \in [0,1]. \end{cases}$$

Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem. As the function f is not specified, some constants remain in the final solution. Explain and show how to compute these constants when f is known.

Solution

Using separation of variables, the equation can be written

$$-\frac{X''}{X} = \frac{2Y'}{Y} + \frac{Y''}{Y} = \lambda. \tag{267}$$

Therefore, both spatial part of the solution depend on the sign of the same constant λ.
First focus on the equation for $X(x)$

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = 0, \\ X(1) = 0. \end{cases} \tag{268}$$

This problem admits some not trivial solutions only for $\lambda = (n\pi)^2 > 0$, where n is a positive integer. Those solution write

$$X_n(x) = \tilde{A}_n \sin(n\pi x). \tag{269}$$

After this, the equation for Y writes

$$Y'' + 2Y' - (n\pi)^2 Y = 0. \tag{270}$$

This is an ODE with constant coefficients which can thus simply be solved using the exponential polynomial method. The general solution is given by

$$Y_n(y) = \exp(-y)\left[C_n \sinh\left(\sqrt{(n\pi)^2+1}y\right) + D_n \cosh\left(\sqrt{(n\pi)^2+1}y\right)\right]. \tag{271}$$

Applying the last homogeneous boundary condition directly leads to $D_n = 0\ \forall n$. Therefore, defining a new constants A_n, the solution writes

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\exp(-y)\sinh\left(\sqrt{(n\pi)^2+1}y\right). \tag{272}$$

The determination of the coefficients A_n is obtained by applying the non homogeneous boundary condition to Eq.(272). It can easily be shown that the application of this boundary condition will lead to the decomposition of $f(x)$ in a sine Fourier series.

5.6 Helmholtz equation

Consider the two-dimensional Helmholtz equation on a square

$$u_{xx} + u_{yy} - u = 0 \quad \{x,y\} \in]0,1[\times]0,1[$$

with the boundary conditions, for a given function f,

$$\begin{cases} u(0,y) = 0 & \forall y \in [0,1], \\ u(1,y) = 0 & \forall y \in [0,1], \\ u(x,1) = f(x) & \forall x \in [0,1], \\ u(x,0) = 0 & \forall x \in [0,1]. \end{cases}$$

Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem. As the function f is not specified, some constants remain in the final solution. Explain and show how to compute these constants when f is known.
Hint: introduce two eigenvalues k_x and k_y such that $v''/v = \lambda_x$ and $w''/w = \lambda_y$.

Solution

Using the separation of variables $u(x,y) = X(x)Y(y)$, the Helmoltz equation can be written as

$$\frac{X''}{X} + \frac{Y''}{Y} - 1 = 0. \tag{273}$$

Introducing $\lambda_x = \frac{X''}{X}$ and $\lambda_y = \frac{Y''}{Y}$, Eq.(273) becomes

$$\begin{cases} X'' - \lambda_x X = 0, \\ Y'' - \lambda_y Y = 0, \\ \lambda_x + \lambda_y = 1, \end{cases} \tag{274}$$

i.e.

$$\begin{cases} X'' - \lambda_x X = 0, \\ Y'' - (1 - \lambda_x) Y = 0. \end{cases} \tag{275}$$

Each possible values for λ_x should be investigated

(1) $\lambda_x = 0$

(2) $\lambda_x = -\omega_x^2 < 0$

(3) $0 < \lambda_x < 1$

(4) $\lambda_x = 1$

(5) $\lambda_x > 1$

However, due to the homogeneous Dirichlet boundary condition $u(0, y) = u(1, y) = 0$, $X(x) = 0 \quad \forall \lambda_x \geq 0$. Only the value of λ such that $\lambda_x = -\omega^2 < 0$ will lead to non trivial solutions. When $\lambda_x = -\omega^2 < 0$, the solutions for X and Y read

$$\begin{cases} X(x) = A\cos(\omega x) + B\sin(\omega x), \\ Y(y) = C\cosh\left(\sqrt{1+\omega^2} y\right) + D\sinh\left(\sqrt{1+\omega^2} y\right). \end{cases} \tag{276}$$

Using the initial conditions gives $A = C = 0$ and

$$\omega_n = n\pi, \qquad n = 1, 2, \dots \tag{277}$$

Finally, the solution to the problem can be written as

$$u(x, y) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \sinh(\sqrt{1 + n^2\pi^2} y). \tag{278}$$

The constant A_n can be inferred by using the remaining initial condition

$$u(x, 1) = \sum_{n=1}^{+\infty} A_n \sin(n\pi x) \sinh(\sqrt{1 + n^2\pi^2}) = f(x),$$

which is a sine Fourier expansion of $f(x)$.

5.7 Diffusion equation II

Consider the one-dimensional diffusion equation on a bounded domain

$$u_t - ku_{xx} = 0 \quad \forall x \in]0,1[. \tag{$\star$}$$

(a) Using separation of variables $u(x,t) = w(t)v(x)$, find all the separable solutions of Eq. ($\star$).

(b) Find the solution to Eq. ($\star$) for the homogeneous Robin boundary conditions at one end of the domain,

$$u_x(1,t) + \beta u(1,t) = 0,$$

with a constant $\beta \in \mathbb{R}$, the homogeneous Dirichlet boundary condition at the other end,

$$u(0,t) = 0.$$

and an initial condition, for a given function ϕ,

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[.$$

In particular, show that the solution of this problem requires to solve the transcendental equation

$$\omega = -\beta \tan \omega.$$

Draw a schematics and point out (qualitatively) the solutions of this equation. As the function ϕ is not specified, some constants remain in the final solution. Explain why finding these constants is not immediate in this case.

(c) The domain in now extended to the interval $]-1,1[$. Find the solution to Eq. ($\star$) for the periodic boundary conditions

$$u(-1,t) = u(1,t) \quad \text{and} \quad u_x(-1,t) = u_x(1,t),$$

and an initial condition, for a given function ϕ,

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[.$$

As the function ϕ is not specified, some constants remain in the final solution. Explain and show how to compute these constants when ϕ is known.

Solution

(a) Using the ansatz $u = wv$, the diffusion equation writes as

$$w'v - kwv'' = 0 \tag{279}$$

$$\Rightarrow \quad \frac{w'}{kw} = \frac{v''}{v}. \tag{280}$$

Since the right hand side depends only on x while the left hand side depend only on t, both sides must equal a constant (named λ), *i.e*

$$v'' - \lambda v = 0 \quad \text{and} \quad w' - k\lambda w = 0. \tag{281}$$

Spatial dependency Depending on the sign of λ, three solutions arise, *i.e*

$$\text{if } \lambda = 0 \qquad \Rightarrow v = Ax + B, \tag{282}$$
$$\text{if } \lambda = \omega^2 > 0 \qquad \Rightarrow v = C \exp -\omega x + D \exp \omega x, \tag{283}$$
$$\text{if } \lambda = -\omega^2 < 0 \qquad \Rightarrow v = E \cos \omega x + F \sin \omega x. \tag{284}$$

Time dependency Whatever is the sign of λ, the time dependency is given by

$$w = \exp \lambda k t. \tag{285}$$

The constant in front the the exponential is here omitted because the field of interest is $u = vw$.

(b) Among all the eigensolutions found in the previous sub-question (*i.e* for any value of λ), only those that satisfy the homogeneous boundary conditions are kept.

Steady eigenvalues, $\lambda = 0$ The boundary conditions yields

$$\begin{cases} v(0) & = B = 0 \quad (286) \\ v_x(1) + \beta v(1) = 0 & = A(1+\beta) \quad (287) \end{cases}$$

which implies

$$A = 0 \quad \text{if } \beta \neq -1 \tag{288}$$

but if $\beta = -1$ then $v(x) = Ax$ satisfy both boundary condition.

Time growing eigenvalues, $\lambda = \omega^2 > 0$ The boundary conditions yields

$$\begin{cases} v(0) & = 0 = C + D \quad (289) \\ v_x(1) + \beta v(1) & = 0 = -\omega C \exp -\omega + \omega D \exp \omega x + \beta \left(C \exp -\omega + D \exp \omega \right) \quad (290) \end{cases}$$

which implies that

$$C = D = 0 \quad \text{if } \left(\beta - \omega\right) \exp -\omega - \left(\beta + \omega\right) \exp \omega \neq 0 \tag{291}$$

but if

$$\left(\beta - \omega\right) \exp -\omega - \left(\beta + \omega\right) \exp \omega = 0 \qquad \Rightarrow \omega = -\beta \tanh \omega \tag{292}$$

then $v(x) = C\left(\exp -\omega x - \exp \omega x\right)$ is a solution. It can be shown that there exists a single ω that verifies that equation only if $\beta \leq -1$.

Time decaying eigenvalues, $\lambda = -\omega^2 < 0$ The boundary conditions once again yields

$$v(0) = E = 0 \tag{293}$$
$$v_x(1) + \beta v(1) = 0 \qquad \Rightarrow F\left(\omega \cos \omega + \beta \sin \omega\right) = 0 \tag{294}$$

Once again, there is a non trivial solution only if

$$\left(\omega \cos \omega + \beta \sin \omega\right) = 0 \Rightarrow \omega = -\beta \tan \omega. \tag{295}$$

It can be shown that these solutions are infinitely many. There are denoted by ω_n.
The solution compatible with boundary conditions is then

$$u(x,t) = Ax + C\left(\exp -\omega_0 x - \exp +\omega_0 x\right)\exp \omega_0^2 kt + \sum_{n=1}^{\infty} F_n \sin \omega_n x \exp -\omega_n^2 kt. \tag{296}$$

Finally the initial condition must be imposed, *i.e*

$$\phi(x) = Ax + C\left(\exp -\omega_0 x - \exp +\omega_0 x\right) + \sum_{n=1}^{\infty} F_n \sin \omega_n x \tag{297}$$

which is not a Fourier series so the identification is tricky and not necessarily possible.

5.8 Wave equation II

Consider the two-dimensional wave equation inside an open subset D of $\mathbb{R}^2$ (in this exercise, D will either be a square or a disk),

$$u_{tt} - c^2 \Delta u = 0 \quad \forall \{x, y\} \in D,$$

with Dirichlet condition on the boundary ∂D of D

$$u(x,y,t) = 0 \quad \forall \{x,y\} \in \partial D$$

and with initial conditions

$$u(x,y,0) = \phi(x,y) \quad \text{and} \quad u_t(x,y,0) = \psi(x,y).$$

The Laplace operator is denoted by Δ. In Cartesian coordinate, $\Delta u = u_{xx} + u_{yy}$, while in polar coordinates $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$.

(a) Using separation of variables $u(x,y,t) = w(t)v(x)q(y)$, find the solution to this boundary value problem if $D =]0,1[\times]0,1[$. As the functions ϕ and ψ are not specified, some constants remain in the final solution. Explain how to compute these constants when ϕ and ψ are known. What is the physical interpretation of any term in the final sum taken individually? How do these terms evolve with time?

(b) Using separation of variables $u(x,y,t) = w(t)g(r)h(\theta)$, find the solution to this boundary value problem if $D = \{x, y \,|\, x^2 + y^2 < 1\}$, *i.e*, if D is a disk of radius 1. As the functions ϕ and ψ are not specified, some constants remain in the final solution. Explain how to compute these constants when ϕ and ψ are known. Make the parallel with (a) as far as eigenfunctions and time evolution is concerned.

Solution

(a) Using the ansatz $u = wvq$, the wave equation writes as

$$\frac{w''}{c^2 w} = \frac{v''}{v} + \frac{q''}{q}. \tag{298}$$

Since the right hand side depends only on x and y while the left hand side depends only on t, each of the three terms in Eq.(298) must be a constant. These constant are respectively called λ, λ_x and λ_y. The system to solve is then

$$\begin{cases} v'' - \lambda_x v &= 0, \quad (299)\\ q'' - \lambda_y q &= 0, \quad (300)\\ w'' - c^2\lambda w = 0 & \quad (301)\end{cases}$$

with

$$\lambda_x + \lambda_y = \lambda. \tag{302}$$

Time dependency (t) Depending on the sign of λ, three solutions arise, ***i.e***

$$\text{if } \lambda = 0 \quad \Rightarrow w = A_t t + B_t, \tag{303}$$
$$\text{if } c^2\lambda = \omega^2 > 0 \quad \Rightarrow w = C_t \exp -\omega t + D_t \exp \omega t, \tag{304}$$
$$\text{if } c^2\lambda = -\omega^2 < 0 \quad \Rightarrow v = E_t \cos \omega t + F_t \sin \omega t. \tag{305}$$

Spatial dependency (x) Depending on the sign of λ_x, three solutions arise, ***i.e***

$$\text{if } \lambda_x = 0 \quad \Rightarrow v = A_x x + B_x, \tag{306}$$
$$\text{if } \lambda_x = k_x^2 > 0 \quad \Rightarrow v = C_x \exp -k_x x + D_x \exp k_x x, \tag{307}$$
$$\text{if } \lambda_x = -k_x^2 < 0 \quad \Rightarrow v = E_x \cos k_x x + F_x \sin k_x x. \tag{308}$$

The same solutions appears for the spatial dependency in y.
Among all the eigensolutions found previously, only those satisfying the homogeneous boundary condition are kept. Using the ansatz, the boundary conditions become

$$v(0) = v(1) = 0 \tag{309}$$
$$q(0) = q(1) = 0. \tag{310}$$

Stationary eigensolutions, *i.e* $\lambda_x = 0$ Applying the boundary conditions gives $A_x = B_x = 0$ and similarly $A_y = B_y = 0$.

Evanescent eigensolutions, *i.e* $\lambda_x > 0$ Applying the boundary conditions gives $C_x = D_x = 0$ and similarly $C_y = D_y = 0$.

Propagating eigensolutions, *i.e* $\lambda_x < 0$ Applying the boundary condition gives

$$v_n(x) = F_x^n \sin k_x^n x \quad \text{with} \quad k_x^n = n\pi \quad n = 1, 2, 3, \ldots \tag{311}$$

and

$$q_m(x) = F_y^m \sin k_y^m y \quad \text{with} \quad k_x^m = m\pi \quad m = 1, 2, 3, \ldots \tag{312}$$

Because

$$\lambda = \lambda_x + \lambda_y, \tag{313}$$

the pulsation are given by

$$\omega_{m,n} = c\sqrt{m^2+n^2}\pi. \tag{314}$$

The most general solution compatible with boundary condition is then

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F^n F^m \sin n\pi x \sin n\pi y \left(A_{m,n}\cos\left(c\sqrt{m^2+n^2}\pi t\right) + B_{m,n}\sin\left(c\sqrt{m^2+n^2}\pi^2 t\right)\right) \tag{315}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A'_{m,n}\sin n\pi x \sin m\pi y \cos\left(c\sqrt{m^2+n^2}\pi t\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{m,n}\sin n\pi x \sin m\pi y \sin\left(c\sqrt{m^2+n^2}\pi t\right). \tag{316}$$

Finally using the initial condition, one finds

$$u(x,y,0) = \phi(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A'_{m,n}\sin n\pi x \sin m\pi y \tag{317}$$

and

$$u_t(x,y,0) = \psi(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B'_{m,n} c\sqrt{m^2+n^2}\pi \sin n\pi x \sin m\pi y \tag{318}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B''_{m,n}\sin n\pi x \sin m\pi y. \tag{319}$$

Then using the result from 2D Fourier sine series, the coefficients are given by

$$A'_{m,n} = 4\int_0^1\int_0^1 \phi(x,y)\sin n\pi x \sin m\pi y \, dx dy \tag{320}$$

and

$$B''_{m,n} = 4\int_0^1\int_0^1 \psi(x,y)\sin n\pi x \sin m\pi y \, dx dy. \tag{321}$$

From Eq.(316), it can be seen that the solution is a superposition of modes of increasing spatial frequencies, $k_x^n = n\pi$ and $k_y^m = m\pi$ and of increasing time pulsation $\omega_{m,n} = c\sqrt{m^2+n^2}\pi$. At the opposite of the heat equation, none of the modes are damped. Consequently, the discontinuity and the high frequency variations of the initial data are conserved and not smoothed as it is the case for the heat equation.

(b) If D is a cylinder, the problem written in cylindrical coordinates is

$$u_{tt} - u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \forall\{\theta,r\} \in [0,2\pi]\times[0,1[\tag{322}$$

with the boundary condition

$$u(1,\theta,t) = 0 \quad \forall\theta \in [0,2\pi]. \tag{323}$$

Moreover, to ensure that the solution is single valued,

$$u(r,\theta+2\pi,t) = u(r,\theta,t) \tag{324}$$

must be verified.
As previously, the initial conditions remain

$$u(r,\theta,0) = \phi(r,\theta) \quad \text{and} \quad u_t(r,\theta,0) = \psi(r,\theta) \quad \forall\{r,\theta\} \in D. \tag{325}$$

Using the ansatz $u = wgh$, the wave equation writes as

$$\frac{w''}{c^2 w} = \frac{1}{gh}\left(hg'' + \frac{1}{r}hg' + \frac{1}{r^2}gh''\right). \tag{326}$$

The left hand side only depends on t while the right hand side only depends on r and θ such that both sides must equal the same constant called λ. The wave equation then becomes

$$\begin{cases} r^2hg'' + rhg' + gh'' &= r^2\lambda gh, \qquad (327)\\ r^2\dfrac{g''}{g} + r\dfrac{g'}{g} - r^2\lambda + \dfrac{h''}{h} = 0. & \qquad (328)\end{cases}$$

Once again each term should equal a constant, denoted by $-\mu$ because they only depend on r or θ. The system to solve is then

$$\begin{cases} w'' - c^2\lambda w &= 0, \qquad (329)\\ h'' + \mu h &= 0, \qquad (330)\\ r^2g'' + rg' + (-r^2\lambda - \mu)g = 0. & \qquad (331)\end{cases}$$

Time dependency Depending on λ, this equations yields different time behaviours. However here, only propagating modes are considered, *i.e* $\lambda = -\omega^2 < 0$. To formally reject $\lambda \geq 0$, the same procedure as in the previous sub-question should be done, *i.e* show that the spatial part associated with steady and evanescent modes does not satisfy the boundary conditions.
For propagating modes, the time dependency is

$$w(t) = A_t \cos c\omega t + B_t \sin c\omega t. \tag{332}$$

Azimutal dependency The only solution that ensures the unicity is $\mu = m^2$ where m is an integer, the solution is then

$$h(\theta) = A_\theta \cos m\theta + B_\theta \sin m\theta. \tag{333}$$

Radial dependency The radial equation is now

$$r^2g'' + rg' + \left(r^2\omega^2 - m^2\right) g = 0. \tag{334}$$

Now consider the change of variable $x = \omega r$, the equation becomes

$$x^2g'' + xg' + (x^2 - m^2)g = 0. \tag{335}$$

This equation is known as the *Bessel's equation*. Solving this equations is hard and outside the scope of these exercises.
The solution of this equation can be expressed as the superposition of the Bessel function of the first kind of order m, $J_m(x)$, and of the Bessel function of the second kind of order m, $Y_m(x)$, *i.e*

$$g(x) = A_x J_m(x) + B_x Y_m(x) \tag{336}$$

The boundary condition are $g(1) = 0$ and $|g(0)| < \infty$. Thus $B_x = 0$ because $Y_m(x)$ is unbounded at the origin. Then the Dirichlet boundary condition yields

$$g(r = 1) = 0 = J_m(x(1)) = J_m(\omega) = 0. \tag{337}$$

The nth eigenvalues that satisfy $J_m(\omega) = 0$ is denoted by $\omega_{m,n}$. It can be shown that there are an infinite numbers of such values, the root $\omega_{m,n} = 0$ (*i.e* for $n = 0$) should however be withdrawn because $\omega > 0$. The solution then writes

$$u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [A_t^{m,n} \cos c\omega_{m,n} t + B_t^{m,n} \sin c\omega_{m,n} t] [A_\theta^{m,n} \cos m\theta + B_\theta^{m,n} \sin m\theta] J_m(\omega_{m,n} r) \tag{338}$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[A_\theta'^{m,n} \cos m\theta + B_\theta'^{m,n} \sin m\theta\right] J_m(\omega_{m,n} r) \cos c\omega_{m,n} t \tag{339}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[A_\theta''^{m,n} \cos m\theta + B_\theta''^{m,n} \sin m\theta\right] J_m(\omega_{m,n} r) \sin c\omega_{m,n} t. \tag{340}$$

It is interesting to point out that the values $m = 0$ yields radially symmetric eigenfunctions for all values of n.
As previously, the constant can be withdrawn from the initial conditions

$$u(r, \theta, 0) = \phi(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[A_\theta'^{m,n} \cos m\theta + B_\theta'^{m,n} \sin m\theta\right] J_m(\omega_{m,n} r) \tag{341}$$

$$u_t(r, \theta, 0) = \psi(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \omega_{m,n} \left[A_\theta''^{m,n} \cos m\theta + B_\theta''^{m,n} \sin m\theta\right] J_m(\omega_{m,n} r) \tag{342}$$

Previously, the initial data are decomposed into a Fourier sine series. In this case, the basis used is slightly more complcated and one has to decomposed the initial data in a so called Fourrier-Bessel basis. As in the case of Fourier sine or cosine series, here the coefficient of the Fourier-Bessel series are given by

$$A_\theta'^{m,n} = \frac{2}{\pi \left(J_{m+1}(\omega_{m,n})\right)^2} \int_0^1 \int_{-\pi}^{\pi} \phi(r, \theta) J_m(\omega_{m,n} r) r \cos m\theta \, d\theta dr, \tag{343}$$

$$B_\theta'^{m,n} = \frac{2}{\pi \left(J_{m+1}(\omega_{m,n})\right)^2} \int_0^1 \int_{-\pi}^{\pi} \phi(r, \theta) J_m(\omega_{m,n} r) r \sin m\theta \, d\theta dr \tag{344}$$

and

$$\omega_{m,n} A_\theta''^{m,n} = \frac{2}{\pi \left(J_{m+1}(\omega_{m,n} r)\right)^2} \int_0^1 \int_{-\pi}^{\pi} \psi(r, \theta) J_m(\omega_{m,n} r) r \cos m\theta \, d\theta dr, \tag{345}$$

$$\omega_{m,n} B_\theta''^{m,n} = \frac{2}{\pi \left(J_{m+1}(\omega_{m,n} r)\right)^2} \int_0^1 \int_{-\pi}^{\pi} \psi(r, \theta) J_m(\omega_{m,n} r) r \sin m\theta \, d\theta dr. \tag{346}$$

6 Singular value decomposition

6.1 Exercise 1 [Trefethen Ex.4.1]

Determine the singular value decomposition of the following matrices and give a geometrical interpretation to the singular values and vectors

(a)
$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

(e)
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Solution

(a) A is full rank and $m = n$ such that two non-vanishing singular values are expected.

Left singular vectors

$$AA^* = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}.$$

The eigenvalues are solutions of

$$\begin{aligned} \det AA^* - \lambda I &= 0 \\ \Rightarrow \det \begin{pmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix} &= (9-\lambda)(4-\lambda) = 0 \end{aligned}$$

i.e $\lambda_1 = 9$, $\lambda_2 = 4$ such that $\sigma_1 = 3 > \sigma_2 = 2$. The first eigenvector verifies

$$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$u_1 = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$$

The second eigenvector verifies

$$\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 9 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus

$$u_2 = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$$

Right singular vectors The singular values are known. Only the right eigenvectors then needs to be found
In this very particular exercise, since $AA^* = A^*A$ the left and right eigenvectors are the same. Hence

$$v_1 = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}.$$

However, only some combinations of $\pm$ can be kept. Indeed, from

$$A = U\Sigma V^*$$

It can be seen that $u_1 = v_1$ and $u_2 = -v_1$. This come from the fact that imposing $\|u_j\| = \|v_j\| = 1$ only specifies the vector up to a complex constant of modulus 1 and in real algebra, by a factor ± 1.
A good choice is therefore

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

The geometrical interpretation of this singular value decomposition is given in Figure 2.

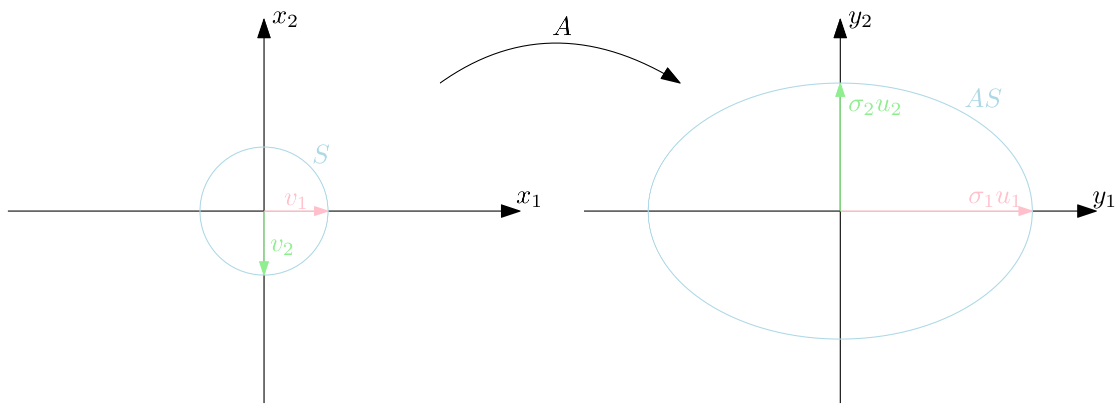

Figure 2: Geometrical interpretation of SVD (a).

(b) A is of rank 1 such that only one non-vanishing singular value is expected.

Left singular eigenvectors

$$AA^* = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are solutions of

$$\det AA^* - \lambda I = 0 \tag{347}$$

$$\Rightarrow \det \begin{pmatrix} 4-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = \lambda^2(4-\lambda) = 0 \tag{348}$$

i.e $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 0$ such that $\sigma_1 = 2$.
The first eigenvector verifies

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 4 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ -4y \\ -4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{349}$$

Thus

$$u_1 = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix} \tag{350}$$

To complete the basis, two orthonormal eigenvectors must be added. For simplicity, they are chosen as

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{351}$$

Right singular vectors The singular value is known. Only the right eigenvectors then needs to be found

$$A^*A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \tag{352}$$

such that the right eigenvector verifies

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{353}$$

Thus normalization gives

$$v_1 = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \tag{354}$$

As previously, the basis is completed by a vector which is chosen as

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{355}$$

However, only some combinations of $\pm$ can be kept. Indeed, from

$$A = U\Sigma V^* \tag{356}$$

It can be seen that $u_1 = v_1$. Once again, this come from the fact that imposing $\|u_j\| = \|v_j\| = 1$ only specifies the vector to a complex constant of modulus 1 and in real algebra, by a factor ± 1.
A good choice is therefore

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{357}$$

The geometrical interpretation of this singular value decomposition is given in Figure 3.

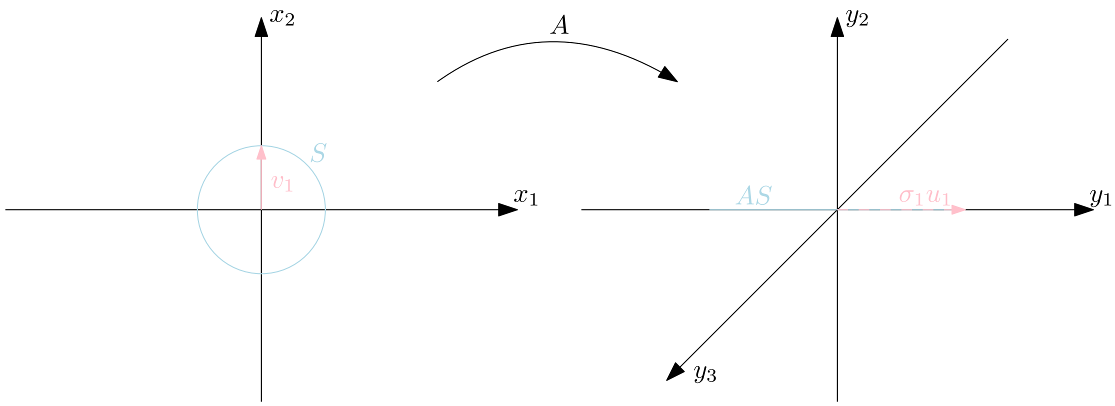

Figure 3: Geometrical interpretation of SVD (b).