Separation of variable II

1 Laplace equation

Consider the 2D Laplace equation on a square

$$u_{xx} + u_{yy} = 0 \quad \{x, y\} \in]0,1[\times]0,1[\tag{1}$$

with some boundary conditions that will be specified in sub-questions.

(a) Using separation of variables $u(x,y) = v(x)w(y)$, find all the separable solutions of Eq.(1).

(b) Found the set of separable solutions that verify homogeneous boundary conditions

$$u(0,y) = 0 \quad \forall y \in [0,1], \tag{2}$$
$$u(1,y) = 0 \quad \forall y \in [0,1], \tag{3}$$
$$u(x,1) = 0 \quad \forall x \in [0,1]. \tag{4}$$

(c) Show that the Fourier sine series expansion of

$$\phi(x) = \begin{cases} x & \forall x \in]0, \frac{1}{2}], \\ 1 - x & \forall x \in [\frac{1}{2}, 1[\end{cases} \tag{5}$$

is

$$\phi(x) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin((2j+1)\pi x)}{(2j+1)^2}. \tag{6}$$

(c) If the boundary condition

$$u(x,0) = \phi(x) \quad \forall x \in [0,1] \tag{7}$$

is given for the last edge, show that the solution $u(x,y)$ is then

$$u(x,y) = \frac{4}{\pi^2} \sum_{j=0}^{\infty} (-1)^j \frac{\sin((2j+1)\pi x)\sinh((2j+1)\pi(1-y))}{(2j+1)^2 \sinh((2j+1)\pi)}. \tag{8}$$

2 Laplace equation II

Consider the 2D Laplace equation on a square

$$u_{xx} + \frac{1}{4} u_{yy} = 0 \quad \{x, y\} \in]0,1[\times]0,1[\tag{9}$$

with the boundary conditions

$$u_x(0,y) = 0 \qquad \forall y \in [0,1], \tag{10}$$
$$u_x(1,y) = 0 \qquad \forall y \in [0,1], \tag{11}$$
$$u(x,1) = 0 \qquad \forall x \in [0,1], \tag{12}$$
$$u(x,0) = 2\cos 2\pi x - 1 \quad \forall x \in [0,1]. \tag{13}$$

(a) Using separation of variables $u(x,y) = v(x)w(y)$, find this boundary value problem.

3 Laplace-like equation

Consider the 2D equation on a square

$$u_{xx} + 2u_y + u_{yy} = 0 \quad \{x,y\} \in]0,1[\times]0,1[\tag{14}$$

with the boundary conditions

$$\begin{aligned} u(0,y) &= 0 && \forall y \in [0,1], && (15)\\ u(1,y) &= 0 && \forall y \in [0,1], && (16)\\ u(x,1) &= f(x) && \forall x \in [0,1], && (17)\\ u(x,0) &= 0 && \forall x \in [0,1]. && (18)\end{aligned}$$

(a) Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem. (As the function $f(x)$ is not specified, some constants remain in the final solution. Explain how to compute these constants when $f(x)$ is known).

4 Helmholtz equation

Consider the 2D Helmholtz equation on a square

$$u_{xx} + u_{yy} - u = 0 \quad \{x,y\} \in]0,1[\times]0,1[\tag{19}$$

with the boundary conditions

$$\begin{aligned} u(0,y) &= 0 && \forall y \in [0,1], && (20)\\ u(1,y) &= 0 && \forall y \in [0,1], && (21)\\ u(x,1) &= f(x) && \forall x \in [0,1], && (22)\\ u(x,0) &= 0 && \forall x \in [0,1]. && (23)\end{aligned}$$

(a) Using separation of variables $u(x,y) = v(x)w(y)$, find the solution to this boundary value problem. (As the function $f(x)$ is not specified, some constants remain in the final solution. Explain how to compute these constants when $f(x)$ is known).

Hint: Introduce two eigenvalues k_x and k_y such that $\frac{v''}{v} = \lambda_x$ and $\frac{w''}{w} = \lambda_y$.

5 Diffusion equation

Consider the 1D diffusion equation on a bounded domain

$$u_t - ku_{xx} = 0 \quad \forall x \in]0,1[\tag{24}$$

with initial condition

$$u(x,0) = \phi(x) \quad \forall x \in]0,1[. \tag{25}$$

(a) Using separation of variables $u(x,t) = w(t)v(x)$, find all the separable solutions of Eq.(24).

(b) Find the solution to Eq.(24) for the homogeneous Robin boundary conditions

$$u_x(1,t) + \beta u(1,t) = 0 \tag{26}$$

and the homogeneous Dirichlet boundary condition

$$u(0,t) = 0. \tag{27}$$

In particular, show that the solution of this problem requires to solve the transcendental equation

$$\omega = -\beta \tan \omega. \tag{28}$$

As the function $\phi(x)$ is not specified, some constants remain in the final solution. Explain why finding these constants is not immidiate in this case.

(c) Find the solution to Eq.(24) for the periodic boundary conditions

$$u(-1,t) = u(1,t) \quad \text{and} \quad u_x(-1,t) = u_x(1,t). \tag{29}$$

(Be careful the the domain as been extended to $]-1,1[$).
(As the function $\phi(x)$ is not specified, some constants remain in the final solution. Explain how to compute these constants when $\phi(x)$ is known).

6 Wave equation

Consider the 2D wave equation inside an open subset D (in this exercise, D will either be a square or a disk)

$$u_{tt} - c^2 \Delta u = 0 \quad \forall \{x,y\} \in D \tag{30}$$

with Dirichlet condition on the boundary ∂D of D

$$u(x,y,t) = 0 \quad \forall \{x,y\} \in \partial D \tag{31}$$

and with initial condition

$$u(x,y,0) = \phi(x,y) \quad \text{and} \quad u_t(x,y,0) = \psi(x,y). \tag{32}$$

The Laplacian operator is denoted by Δ. In cartesian coordinate, $\Delta u = u_{xx} + u_{yy}$ while in polar coordinates $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$.

(a) Using separation of variables $u(x,y,t) = w(t)v(x)q(y)$, find the solution to this boundary value problem if $D =]0,1[\times]0,1[$. (As the function $\phi(x)$ is not specified, some constants remain in the final solution. Explain how to compute these constants when $\phi(x)$ is known). What is the physical interpretation of any term in the final sum taken individually? How do these terms evolves with time?

(b) Using separation of variables $u(x,y,t) = w(t)g(r)h(\theta)$, find the solution to this boundary value problem if $D = \{x, y \,|\, x^2 + y^2 < 1\}$, *i.e* D is a disk. (As the function $\phi(x)$ and $\psi(x)$ are not specified, some constants remain in the final solution. Explain how to compute these constants when $\phi(x)$ and $\psi(x)$ are known). Make the parallel with (a) as far as eigenfunctions and time evolution is concerned.