

# Separation of variable

## 1 Diffusion equation

Consider the 1D diffusion equation on a bounded domain

$$u_t - ku_{xx} = 0 \quad \forall x \in ]0, 1[ \quad (1)$$

with initial condition

$$u(x, 0) = \phi(x) \quad \forall x \in ]0, 1[ \quad (2)$$

and with some boundary conditions at  $x = 0$  and  $x = 1$ . (These boundary conditions will be specified in sub-questions).

**(a)** Using separation of variables  $u(x, t) = w(t)v(x)$ , find all the separable solutions of Eq.(1).

**(b)** Find the solution to Eq.(1) for the homogeneous Dirichlet boundary conditions  $u(0, t) = u(1, t) = 0 \forall t \geq 0$  and for  $\phi(x) = \sin \pi x$ .

**(c)** Find the solution to Eq.(1) for the homogeneous Dirichlet boundary conditions  $u(0, t) = u(1, t) = 0 \forall t \geq 0$  and for

$$\phi(x) = \begin{cases} x & \forall x \in ]0, \frac{1}{2}], \\ 1 - x & \forall x \in [\frac{1}{2}, 1[. \end{cases} \quad (3)$$

**(d)** Find the solution to Eq.(1) for the homogeneous Dirichlet boundary conditions  $u(0, t) = u(1, t) = 0 \forall t \geq 0$  and for  $\phi(x) = 1$ .

**(e)** Find the solution to Eq.(1) for the homogeneous Neumann boundary conditions  $u_x(0, t) = u_x(1, t) = 0 \forall t \geq 0$  and for  $\phi(x) = \cos \pi x$ .

**(f)** Find the solution to Eq.(1) for the non-homogeneous Dirichlet boundary conditions  $u_x(0, t) = 0$  and  $u_x(1, t) = 1, \forall t \geq 0$  and for  $\phi(x) = x + \sin \pi x$ .

## 2 Wave equation

Consider the 1D wave equation on a bounded domain

$$u_{tt} - c^2 u_{xx} = 0 \quad \forall x \in ]0, 1[ \quad (4)$$

with initial condition

$$u(x, 0) = \phi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x) \quad \forall x \in ]0, 1[ \quad (5)$$

and with some boundary conditions at  $x = 0$  and  $x = 1$ . (These conditions will be specified in sub-questions).

(a) Using separation of variable  $u(x, t) = w(t)v(x)$ , find all the separable solution of Eq.(31).

(b) Find the solution to Eq.(1) for the homogeneous Dirichlet boundary conditions  $u(0, t) = u(1, t) = 0 \forall t \geq 0$ , for  $\phi(x) = \sin \pi x$  and for  $\psi(x) = 0$ .

(c) Find the solution to Eq.(1) for the homogeneous Dirichlet boundary conditions  $u(0, t) = u(1, t) = 0 \forall t \geq 0$ , for

$$\phi(x) = \begin{cases} x & \forall x \in ]0, \frac{1}{2}] \\ 1 - x & \forall x \in [\frac{1}{2}, 1[ \end{cases} \quad (6)$$

and for  $\psi(x) = 0$ .

(d) Find the solution to Eq.(1) for the homogeneous Dirichlet boundary conditions  $u(0, t) = u(1, t) = 0 \forall t \geq 0$  for  $\phi(x) = \sin \pi x$  and for  $\psi(x) = c \sin \pi x$ .