1 Diffusion equation 1

Solve the diffusion equation using the general solution formula:

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = 1 & \text{for} \quad |x| < l, \\ u(x,0) = 0 & \text{for} \quad |x| \geq l. \end{cases} \tag{1}$$

Write your answer in terms of $\text{erf}(x)$.
Reminder:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} \mathrm{d}\zeta.$$

2 Diffusion equation 2

Solve the diffusion equation using the general solution formula:

$$\begin{cases} u_t - ku_{xx} = 0 & \text{for} \quad (x,t) \in \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \exp(3x) & \quad (x,t) \in \mathbb{R} \times 0. \end{cases} \tag{2}$$

Reminder: The gaussian integral

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$$

.

3 Diffusion with constant dissipation(Strauss)

Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \text{ for } -\infty < x < \infty \text{ with } u(x,0) = \delta(x), \tag{3}$$

where $b > 0$ is a constant.
Reminder: The Dirac δ function is defined such that

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\mathrm{d}x = f(0).$$

Hint: make the change of variables $u(x,t) = e^{-bt}v(x,t)$.

4 Heat equation with convection

Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \text{ for } -\infty < x < \infty \text{ with } u(x,0) = \phi(x), \tag{4}$$

where V is a constant.
Hint: Go to a moving reference frame by substituting $y = x - Vt$ and $z = t$.
Compute the solution for the initial condition

$$\phi(x) = 1 \text{ for } |x| \leq l$$

.
Write your answer in terms of $\text{erf}(x)$.

5 Additional exercise: Viscous Burgers' equation and Cole-Hopf transformation

Let us consider the momentum equation for an incompressible fluid (Navier-Stokes equations), that is

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{f} + \nu \nabla^2 \mathbf{u}.$$

Now, let us assume that there are no body forces ($\mathbf{f} = \mathbf{0}$), and that the pressure gradient term can be neglected ($\nabla p = \mathbf{0}$), *i.e.*

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u}.$$

Finally, let us also assume that the problem is in one-dimension by posing $\mathbf{u} = u(x,t)\mathbf{e_x}$ and writing the momentum equation along the x-axis. We finally obtain

$$u_t + u u_x = \nu u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0,$$

which is the celebrated viscous Burgers' equation.

This non-linear partial differential equation can be rewritten in the form of the diffusion equation

$$\phi_t = \nu \phi_{xx}$$

by using appropriate methods (*e.g.* Cole-Hopf transformation).

(a) First, let us consider the change of variables

$$U_x = u.$$

Show that the Burgers' equation can be rewritten as

$$U_t = \nu U_{xx} - \frac{1}{2}(U_x)^2. \tag{†}$$

Reminder: $u u_x = \frac{1}{2}(u^2)_x$.

(b) Then, use the Cole-Hopf relation

$$U(x,t) = -2\nu \ln(\phi(x,t))$$

to simplify Eq. † into the diffusion equation

$$\phi_t = \nu \phi_{xx}.$$

(c) Finally, find the solution $\phi(x,t)$ to the diffusion problem and show that the solution of Burgers' equation can be written as

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y u(z,0)dz\right] dy}.$$

NB:

1. You may use the fact that

$$\partial_x \int_{-\infty}^{+\infty} f(x,y)dy = \int_{-\infty}^{+\infty} \partial_x f(x,y)dy;$$

2. We have that $U_x(x,0) = u(x,0)$, i.e.

$$U(x,0) = \int_0^x u(z,0)dz,$$

and

$$\begin{aligned}\phi(x,0) &= \exp\left(-\frac{U(x,0)}{2\nu}\right) \\ &= \exp\left(-\frac{1}{2\nu}\int_0^x u(z,0)dz\right).\end{aligned}$$