1 Exercice 1

Let us consider the advection equation

$$u_t + a u_x = 0.$$

(a) Using forward finite differences in time and space, *i.e*

$$u_x \approx \frac{u_{j+1}^n - u_j^n}{\Delta x} \text{ and } u_t \approx \frac{u_j^{n+1} - u_j^n}{\Delta t},$$

show that the scheme is explicit by giving the update equation $u_j^{n+1} = f(u_{j+1}^n, u_j^n)$.

(b) Establish a stability criterion using Von Neumann analysis for the preceeding scheme.

(c) (BONUS) Why does this creterion depend on the sign of a? What happens if a downward space differences, *i.e*

$$u_x \approx \frac{u_j^n - u_{j-1}^n}{\Delta x}$$

are used instead?
Hint: The general solution of the transport equation is $f(x - at)$.

(d) Using the so called Lax-Friedrichs scheme, the update equation is then

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right).$$

Establish a stability criterion using Von Neumann analysis.

(e) Using the so called Lax-Wendroff scheme, the update equation is then

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right).$$

Establish a stability criterion using Von Neumann analysis.

2 Exercice 2 (BONUS)

Let us consider the 2D diffusion equation

$$u_t - a(u_{xx} + u_{yy}) = 0. \tag{1}$$

Using forward differences in time and central differences in space, establish a stability criterion using Von Neumann analysis.
Hint:

$$\epsilon(x, y, t) = \int\int \hat{\epsilon}(k_x, k_y, t) \exp\left(-i(k_x x + k_y y)\right) dx dy$$

such that any mode of the error at a grid point expresses as

$$\epsilon_{rs}^n = \epsilon(x_r, y_s, t_n) = \hat{\epsilon}(k_x, k_y, t_n) \exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right). \tag{2}$$

As in the 1D case, the stability is ensured in Von Neumann sense if the amplitude of each mode is decreasing, *i.e* if

$$\gamma(k_x, k_y) \triangleq \left|\frac{\epsilon_{rs}^{n+1}}{\epsilon_{rs}^n}\right| < 1.$$