Solution to Exercice 2 from TP3

Statement

Let us consider the 2D diffusion equation

$$u_t + a(u_{xx} + u_{yy}) = 0. \tag{1}$$

Using forward differences in time and central differences in space, establish a stability criterion using Von Neumann analysis.
Hint:

$$\epsilon(x, y, t) = \int\int \hat{\epsilon}(k_x, k_y, t) \exp\left(-i(k_x x + k_y y)\right) dxdy$$

such that any mode of the error at any grid point expresses as

$$\epsilon_{rs}^n = \epsilon(x_r, y_s, t_n) = \hat{\epsilon}(k_x, k_y, t_n) \exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right). \tag{2}$$

As in the 1D case, the stability is ensured in Von Neumann sense if the amplitude of each mode is not increasing, *i.e* if

$$\gamma(k_x, k_y) \triangleq \left| \frac{\hat{\epsilon}\left(k_x, k_y, t_{n+1}\right)}{\hat{\epsilon}\left(k_x, k_y, t_n\right)} \right| \leq 1.$$

Solution

First, let's discretize the 2D diffusion equation with spatial step size Δx, Δy and temporal step size Δt.

- Forward in time:

$$u_t = \frac{u_{r,s}^{n+1} - u_{r,s}^n}{\Delta t} \tag{3}$$

- Centered in space:

$$u_{xx} = \frac{u_{r+1,s}^n - 2u_{r,s}^n + u_{r-1,s}^n}{\Delta x^2} \tag{4}$$

$$u_{yy} = \frac{u_{r,s+1}^n - 2u_{r,s}^n + u_{r,s-1}^n}{\Delta y^2} \tag{5}$$

The 2D diffusion equation can thus be expressed as

$$\frac{u_{r,s}^{n+1} - u_{r,s}^n}{\Delta t} + a\left(\frac{u_{r+1,s}^n - 2u_{r,s}^n + u_{r-1,s}^n}{\Delta x^2} + \frac{u_{r,s+1}^n - 2u_{r,s}^n + u_{r,s-1}^n}{\Delta y^2}\right) = 0 \tag{6}$$

or as

$$u_{r,s}^{n+1} = u_{r,s}^{n} - \frac{a\Delta t}{\Delta x^2}\left(u_{r+1,s}^{n} - 2u_{r,s}^{n} + u_{r-1,s}^{n}\right) - \frac{a\Delta t}{\Delta y^2}\left(u_{r,s+1}^{n} - 2u_{r,s}^{n} + u_{r,s-1}^{n}\right). \tag{7}$$

Introducing

$$\beta_x = \frac{a\Delta t}{\Delta x^2} \quad \text{and} \quad \beta_y = \frac{a\Delta t}{\Delta y^2}, \tag{8}$$

the update equation is then

$$u_{r,s}^{n+1} = u_{r,s}^{n} - \beta_x\left(u_{r+1,s}^{n} - 2u_{r,s}^{n} + u_{r-1,s}^{n}\right) - \beta_y\left(u_{r,s+1}^{n} - 2u_{r,s}^{n} + u_{r,s-1}^{n}\right). \tag{9}$$

The stability criterion is obtained by injecting an arbitrary error mode

$$\epsilon_{rs}^{n} = \epsilon(x_r, y_s, t_n) = \hat{\epsilon}\left(k_x, k_y, t_n\right)\exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right) \tag{10}$$

into the discretized equation. For conciseness, $\hat{\epsilon}\left(k_x, k_y, t_n\right)$ is written as $\hat{\epsilon}_n$.
The evolution of the amplitude of any mode is then governed by

$$\begin{aligned}
\hat{\epsilon}_{n+1}\exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right) &= \hat{\epsilon}_{n}\exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right) \\
&- \beta_x\big[\hat{\epsilon}_{n}\exp\left(-i(k_x(r+1)\Delta x + k_y s\Delta y)\right) - 2\hat{\epsilon}_{n}\exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right) \\
&\qquad + \hat{\epsilon}_{n}\exp\left(-i(k_x(r-1)\Delta x + k_y s\Delta y)\right)\big] \\
&- \beta_y\big[\hat{\epsilon}_{n}\exp\left(-i(k_x r\Delta x + k_y(s+1)\Delta y)\right) - 2\hat{\epsilon}_{n}\exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right) \\
&\qquad + \hat{\epsilon}_{n}\exp\left(-i(k_x r\Delta x + k_y(s-1)\Delta y)\right)\big].
\end{aligned} \tag{11}$$

Dividing both sides by $\hat{\epsilon}^n\exp\left(-i(k_x r\Delta x + k_y s\Delta y)\right)$ yields

$$\begin{aligned}
&\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_{n}} = 1 - \beta_x\left(e^{-ik_x\Delta x} - 2 + e^{ik_x\Delta x}\right) - \beta_y\left(e^{-ik_y\Delta y} - 2 + e^{ik_y\Delta y}\right) && (12)\\
\Leftrightarrow\quad &\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_{n}} = 1 - 2\beta_x\left(\cos k_x\Delta x - 1\right) - 2\beta_y\left(\cos k_y\Delta y - 1\right) && (13)\\
\Leftrightarrow\quad &\frac{\hat{\epsilon}_{n+1}}{\hat{\epsilon}_{n}} = 1 + 4\beta_x\sin^2\frac{k_x\Delta x}{2} + 4\beta_y\sin^2\frac{k_y\Delta y}{2}. && (14)
\end{aligned}$$

Ensuring that all modes are not divergent, *i.e*

$$\left|\frac{\hat{\epsilon}(k_x, k_y, t_{n+1})}{\hat{\epsilon}(k_x, k_y, t_n)}\right| \leq 1 \quad \forall k_x, k_y \tag{15}$$

gives

$$\begin{aligned}
&-1 \leq 1 + 4\beta_x\sin^2\frac{k_x\Delta x}{2} + 4\beta_y\sin^2\frac{k_y\Delta y}{2} \leq 1 && (16)\\
\Leftrightarrow\quad &-\frac{1}{2} \leq \beta_x\sin^2\frac{k_x\Delta x}{2} + \beta_y\sin^2\frac{k_y\Delta y}{2} \leq 0. && (17)
\end{aligned}$$

These two inequalities must be verified for any k_x, k_y. The most restrictive cases for the rightmost inequality are obtained by considering $\left\{\sin^2\frac{k_x\Delta x}{2}, \sin^2\frac{k_y\Delta y}{2}\right\} = \{1, 0\}$ and $\left\{\sin^2\frac{k_x\Delta x}{2}, \sin^2\frac{k_y\Delta y}{2}\right\} = \{0, 1\}$ which yields

$$\beta_x \leq 0 \quad \text{and} \quad \beta_y \leq 0. \tag{18}$$

Then β_x and β_y being negative, the most restrictive case for the leftmost inequality is obtained considering $\left\{\sin^2 \frac{k_x \Delta x}{2}, \sin^2 \frac{k_y \Delta y}{2}\right\} = \{1, 1\}$ which yields

$$-\frac{1}{2} \leq \beta_x + \beta_y = a\Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \triangleq \frac{a\Delta t}{\Delta l^2}. \tag{19}$$

The diffusion coefficient a being negative because β_x and β_y are negative, condition Eq.(19) writes

$$\frac{\Delta t}{\Delta l^2} \leq \frac{1}{2(-a)}. \tag{20}$$

Remark The conventional way to write a diffusion equation is

$$u_t - \alpha \left(u_{xx} + u_{yy} \right) = 0 \tag{21}$$

i.e with a minus sign in front of the diffusion coefficient.
With that new convention, the stability condition is then

$$\frac{\Delta t}{\Delta l^2} \leq \frac{1}{2\alpha}. \tag{22}$$