Lecture 11b: Hand computing a full SVD

MATH0504: Mathématiques appliquées

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Reminder

Let $A \in M^{m\times n}(\mathbb{C})$ be a rectangular matrix.
The singular value decomposition of A is the matrix factorization

$$A = U\Sigma V^*$$

with

$$\begin{aligned} U &\in M^{m\times m}(\mathbb{C}) && \text{such that } UU^* = I, \\ V &\in M^{n\times n}(\mathbb{C}) && \text{such that } VV^* = I, \\ \Sigma &\in M^{m\times n}(\mathbb{R}^+) && \text{such that } D = \operatorname{diag}() . \end{aligned}$$

Hand computing a full SVD

Left singular vectors

From the definition of the SVD

$$\begin{aligned} AA^* &= U\Sigma V^*(U\Sigma V^*)^* \\ &= U\Sigma V^* V\Sigma^* U^* \\ &= U\Sigma\Sigma^* U^* && (VV^* = I) \\ \Rightarrow AA^*U &= U\Sigma\Sigma^* U^* U \\ &= U\Sigma\Sigma^* && (UU^* = I) \end{aligned}$$

Writing explicitly the columns of U gives ($p = \text{rank}\,(A)$)

$$AA^* \left(u_1 \middle| \ldots \middle| u_m \right) = \left(u_1 \middle| \ldots \middle| u_m \right) \text{diag}\left(\sigma_1^2 \ldots \sigma_p^2 \; 0 \ldots 0\right)$$

i.e.,

$$\begin{cases} AA^* u_j = \sigma_j^2 u_j \forall j \in [1, p], \\ AA^* u_j = 0 \qquad \forall j \in [p+1, m]. \end{cases}$$

Right singular vectors

From the definition of the SVD

$$
\begin{aligned}
A^*A &= (U\Sigma V^*)^* U\Sigma V^* \\
&= V\Sigma^* U^* U \Sigma V^* \\
&= V\Sigma\Sigma^* V^* && (UU^* = I) \\
\Rightarrow AA^*V &= V\Sigma\Sigma^* V^* V \\
&= V\Sigma\Sigma^* && (VV^* = I)
\end{aligned}
$$

Writing explicitly the columns of V gives ($p = \text{rank}\,(A)$)

$$
A^*A \begin{pmatrix} v_1 \,\middle|\, \ldots \,\middle|\, v_m \end{pmatrix} = \begin{pmatrix} v_1 \,\middle|\, \ldots \,\middle|\, v_m \end{pmatrix} \text{diag}\left(\sigma_1^2 \ \ldots \ \sigma_p^2 \ 0 \ \ldots \ 0\right)
$$

i.e.,

$$
\begin{cases}
AA^* v_j = \sigma_j^2 v_j \forall j \in [1, p], \\
AA^* v_j = 0 \qquad \forall j \in [p+1, m].
\end{cases}
$$

Recipe for SVD

1. Solve the eigenvalue problem $AA^* u_j = \lambda_j u_j$.
2. From the eigenvalues, deduces the singular values $\sigma_j^2 = \lambda_j$ and form Σ.
3. From the unit eigenvectors u_j, form the matrix U.
4. Find the eigenvectors v_j of $A^* A v_j = \lambda_j v_j \; \forall j \in [1, p]$.
5. From the unit eigenvectors v_j, form the matrix V.

Geometrical interpretation

Deformation of the unit sphere

Consider the set S of unit vectors in the domain space ($\mathbb{C}^n$ or $\mathbb{R}^n$) *i.e*

$$S = \{x \in \mathbb{C}^n | \, \|x\| = 1\} .$$

Properties of the SVD can be viewed from the geometrical shape of the set AS which is defined as

$$AS = \{y = Ax \in \mathbb{C}^m | x \in S\} .$$

In particular, it can be showed that

- the left singular vectors are the principal axis of the ellipse AS in the image space ($\mathbb{C}^m$ or $\mathbb{R}^m$)
- the singular values are the respective lengths of these axis
- the left singular vectors are the preimages of these axis (*i.e* $AV = U\Sigma \Rightarrow Av_j = \sigma_j u_j$).