Lecture 9 Nonlinear PDEs

Mathématiques appliquées (MATH0504-1)
B. Dewals, Ch. Geuzaine

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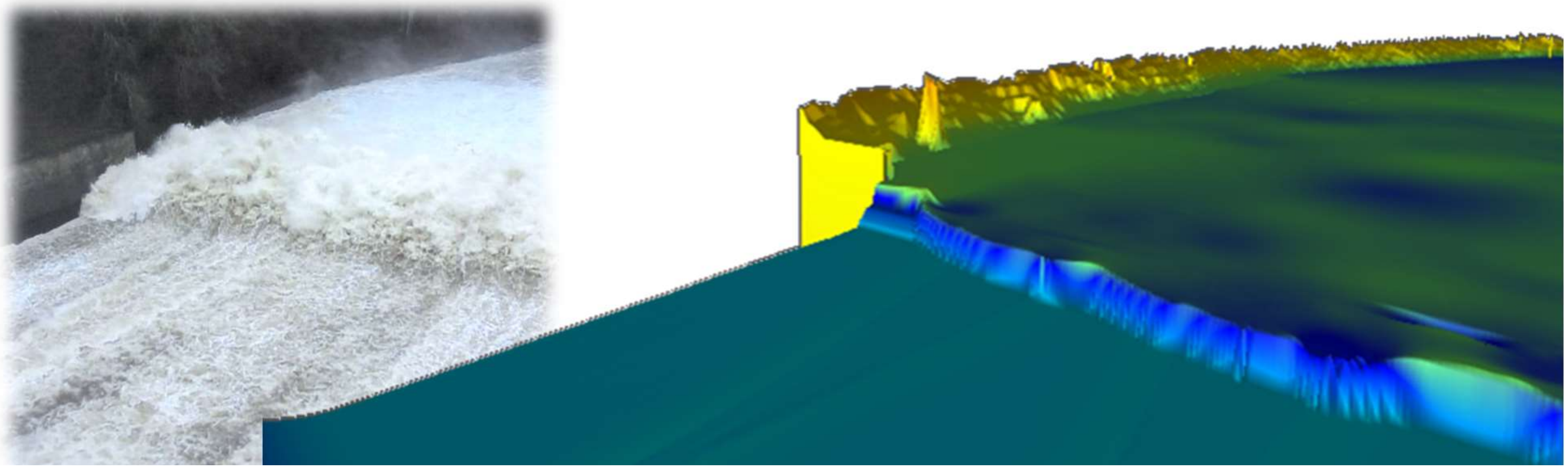
What is new with nonlinear PDEs?

The superposition principle ceases to hold.
Therefore, methods such as those based on summing eigenfunctions may no more be used.

Additionally, new phenomena occur, such as shock waves, solitons …



Site view and computation of a hydraulic jump at Nisramont dam, Belgium

Shock waves are modeled by nonlinear hyperbolic PDEs

The simplest type of nonlinear hyperbolic PDE is the first-order equation

$$u_t + a(u)\, u_x = 0$$

Another example is the system of equations governing fluid motion (Euler equations), written here in 1D:

$$\rho_t + (\rho\, v\,)_x = 0$$

$$v_t + v\, v_x + \rho^{-1}\, [\, f\,(\rho)\,]_x = 0$$

with v the flow velocity and ρ the fluid density.

Learning objectives

Be able to solve nonlinear 1st order PDEs using the characteristics

Become familiar with new concepts such as

- rarefaction waves (or expansion waves),
- shock waves,
- distributions,
- test functions,
- weak solutions …

Learn how to compute shock waves, using Rankine–Hugoniot formula

Outline

1. Reminder: first-order linear equations
2. Typical example of a nonlinear PDE

3. General 1st order nonlinear PDE
   a. Rarefaction waves
   b. Weak solutions
   c. Shock waves
4. Solitons

1 – Reminder: first order linear PDEs

In this section, we briefly review the resolution
of 1st order linear PDEs (Section 14.1 in Strauss, 2008).

Reminder: first-order linear equations

Consider the equation

$$\cancel{b(x,\ t)}\ u_t + a(x,\ t)\ u_x = 0.$$

Note that $b(x,\ t)$ may be taken equal to 1 without any loss of generality.

In **Lecture 1**, we showed that the directional derivative of u is zero along the direction

$$dx/dt = a(x,\ t)$$

i.e. $u(x,\ t)$ is a constant along *characteristic curves* defined as the solutions of the ODE $dx/dt = a(x,\ t)$.

Reminder: first-order linear equations

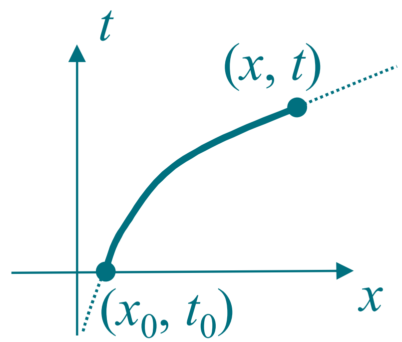
Indeed,

- consider the characteristic curves, which are defined as the solutions of $dx/dt = a(x, t)$;
- every point (x_0, t_0) in the (x, t) plane has a unique characteristic curve passing through it, because the ODE can be uniquely solved with the initial condition $x(t_0) = x_0$;
- along such a curve (parametrized by t) we calculate:

$$0 = u_x a + u_t = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = \frac{d}{dt}[u(\boxed{x(t)}, t)] = \frac{du}{dt}$$

Reminder: first-order linear equations

Thus $u(x(t), t)$ is constant along each characteristic curve: $u(x(t), t) = u(x_0, t_0)$.



Example Let us solve the PDE

$$u_t + e^{x+t}\, u_x = 0.$$

The characteristic equation is $dx/dt = e^{x+t}$.

It can be integrated along the characteristic, between the point (x_0, t_0) and any other point (x, t):

$$\int_{x_0}^{x} e^{-x} dx = \int_{t_0}^{t} e^{t} dt \quad \Rightarrow \quad e^{-x} + e^{t} = e^{-x_0} + e^{t_0} = C$$

with C a constant determined by the values x_0, t_0.

Reminder: first-order linear equations

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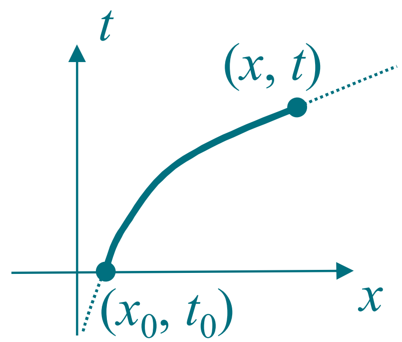

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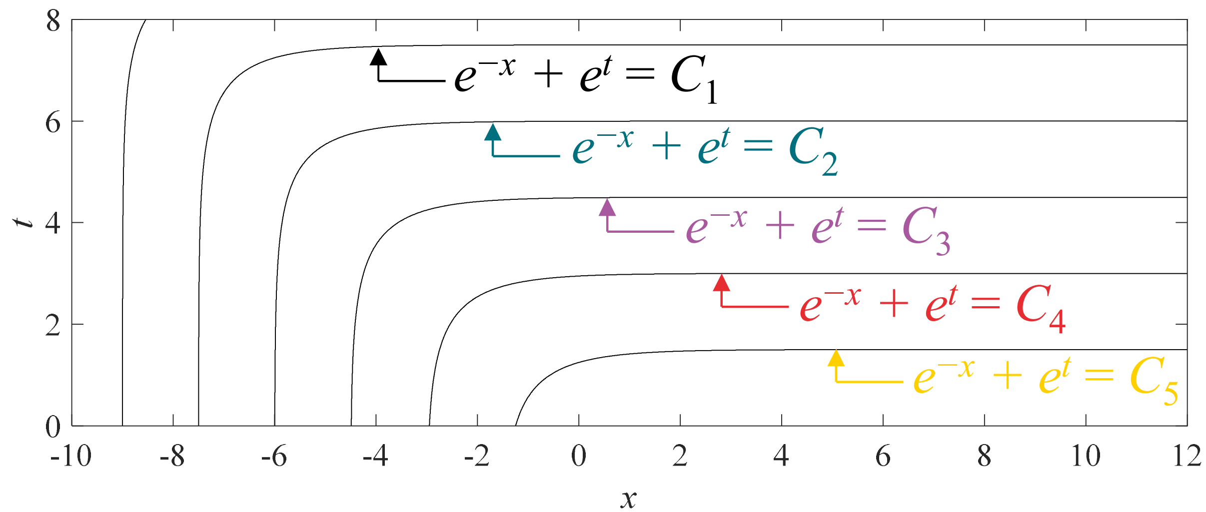


$$e^{-x} + e^t = C$$

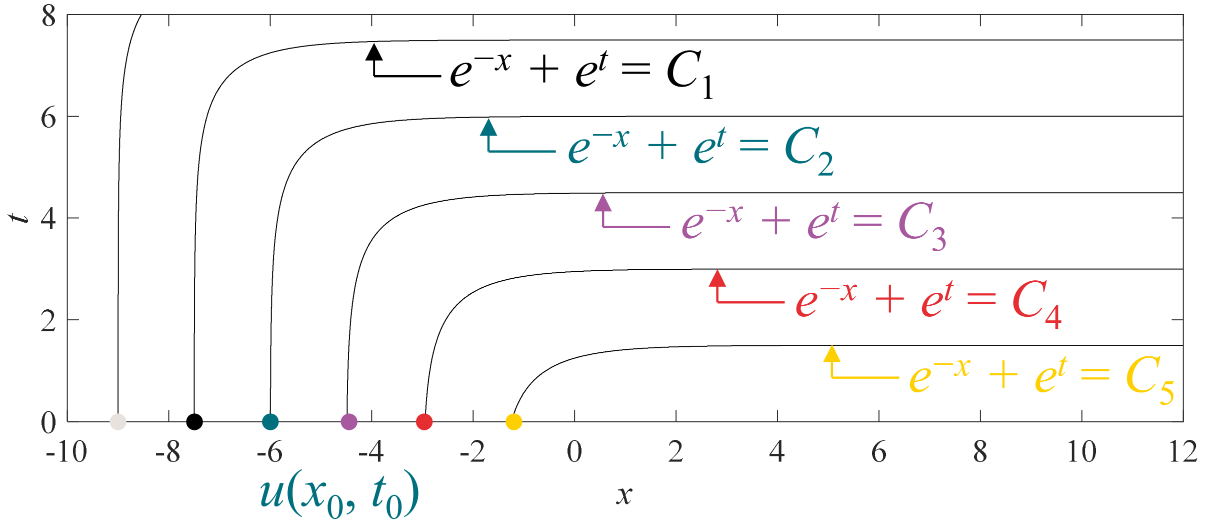
Reminder: first-order linear equations

Since u is a constant along each characteristic, the general solution is

$$u(x,t) = f(C) = f(e^{-x} + e^{t})$$

with f an arbitrary differentiable function.

Next, f can be determined based on an IC $u(x_0, t_0)$.



$$e^{-x} + e^{t} = C$$

2 – Typical example of a nonlinear PDE

In this section, we use a typical example of nonlinear 1^{st} order PDE to highlight to which extent the procedure used for linear 1^{st} order PDEs can still be applied in the nonlinear case; and we point out the possible occurrence of discontinuous solutions which require a more general approach (Section 14.1 in Strauss, 2008).

Typical example of nonlinear equation

The nonlinear equation

$$u_t + u\, u_x = 0.$$

is similar in nature to the basic equation of fluids.

We use the geometric method: the characteristic curves are given by solutions of the ODE

$$\frac{dx}{dt} = u(x,t)$$

Since the PDE is nonlinear,
the characteristic equation depends now
on the unknown function $u(x, t)$ itself!

What do we know about the characteristic curves?

Since we do not know yet the solution $u(x, t)$, we cannot determine the characteristic curves;

BUT we know that u remains a constant on each of them, since (using the chain rule):

$$\frac{d}{dt}[u(x(t), t)] = u_t + \frac{dx}{dt}u_x = u_t + uu_x = 0$$

Hence, the slope of the characteristic curves $dx/dt = u(x, t)$ is also a constant for each of them.

These observations have important consequences …

Three important properties of the characteristic curves for $u_t + u\, u_x = 0$

① Each characteristic curve is a straight line:
each solution $u(x, t)$ has a family of straight lines (of various slopes) as its characteristics.

② The solution is constant on each such line.

③ The slope of each such line is equal to the value of $u(x, t)$ on it.

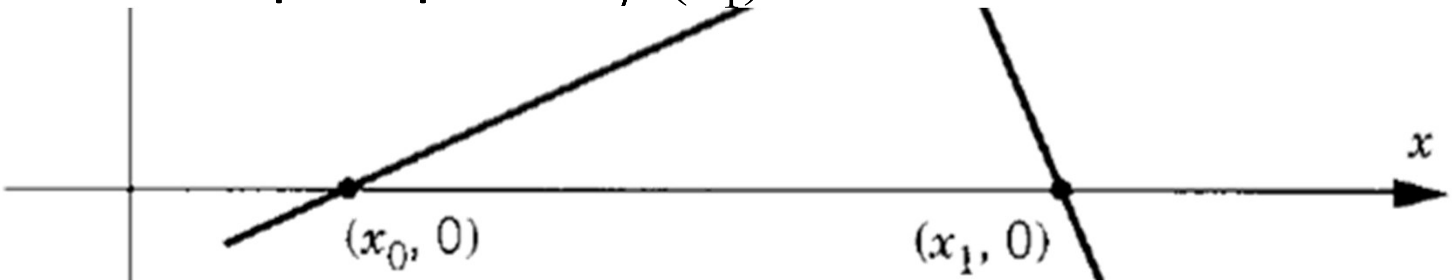
Can two different characteristic lines intersect? What would this mean?

Suppose now that we ask for a solution of the PDE that satisfies the initial condition (on the line $t = 0$):

$$u(x, 0) = \phi(x).$$

Then, by ③,

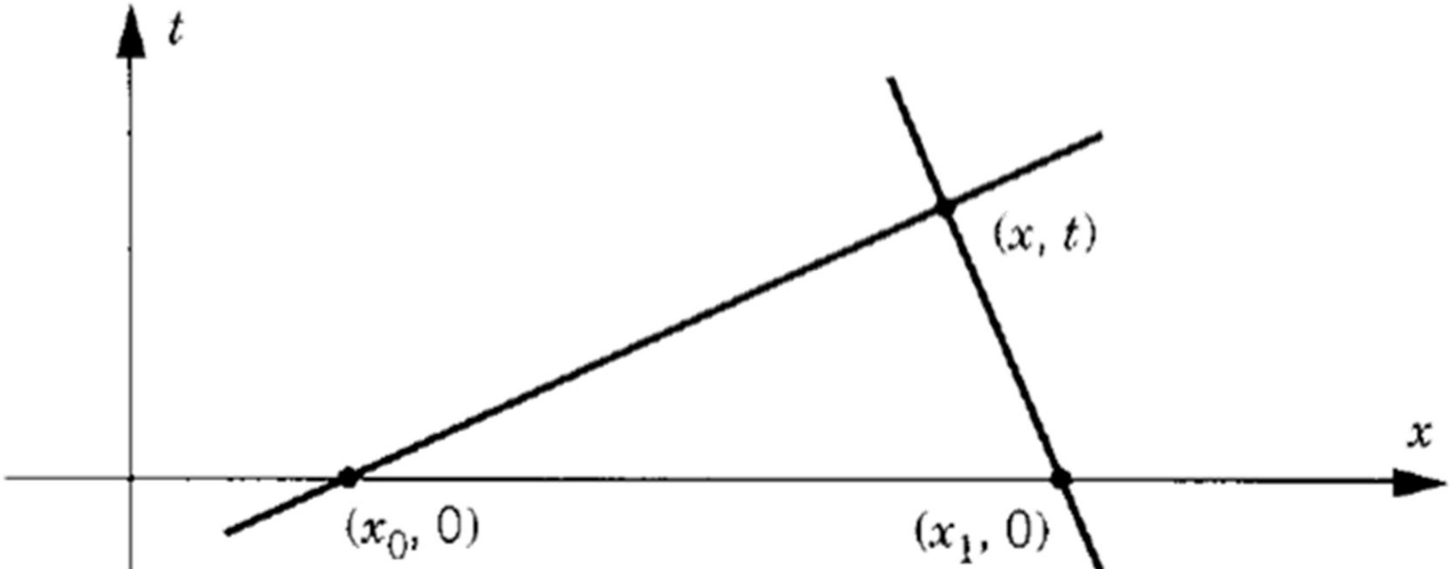
- the characteristic line that passes through $(x_0, 0)$ has a slope equal to $\phi(x_0)$.
- similarly, the characteristic line through $(x_1, 0)$ has a slope equal to $\phi(x_1)$.



Can two different characteristic lines intersect? What would this mean?

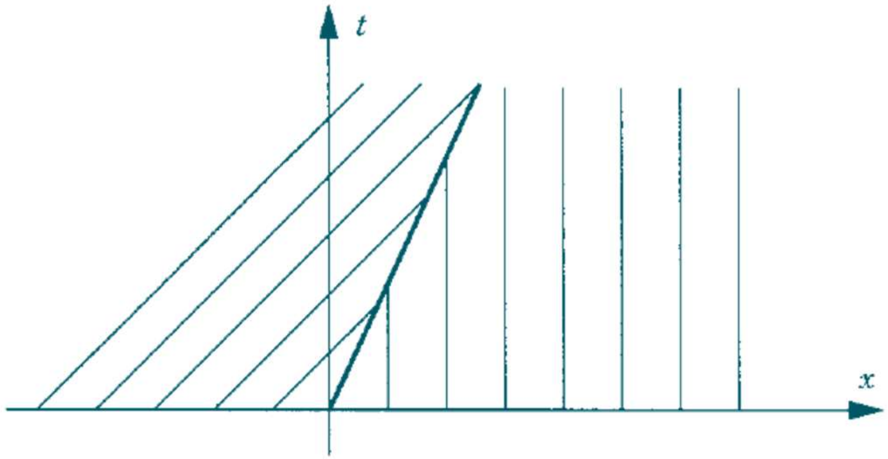
If the two lines intersect, we are in trouble!

- $u = \phi(x_0)$ on one line and $u = \phi(x_1)$ on the other,
- so that $\phi(x_0)$ should be equal to $\phi(x_1)$ (as the value of u should be unique)
- which is impossible since the slopes differ!

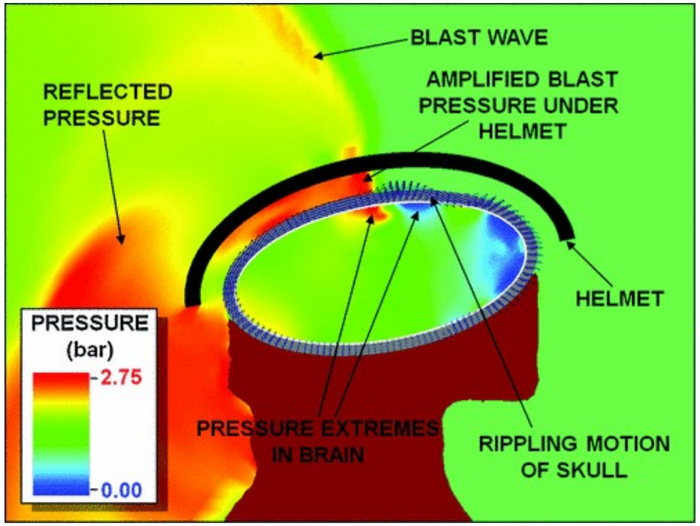


Which is the way out?

A general and practical option is to **extend the very notion of solution** to allow discontinuities.



This leads to the theory of *shock waves*, which occur in blasts, traffic flow, water waves, supersonic flights ...



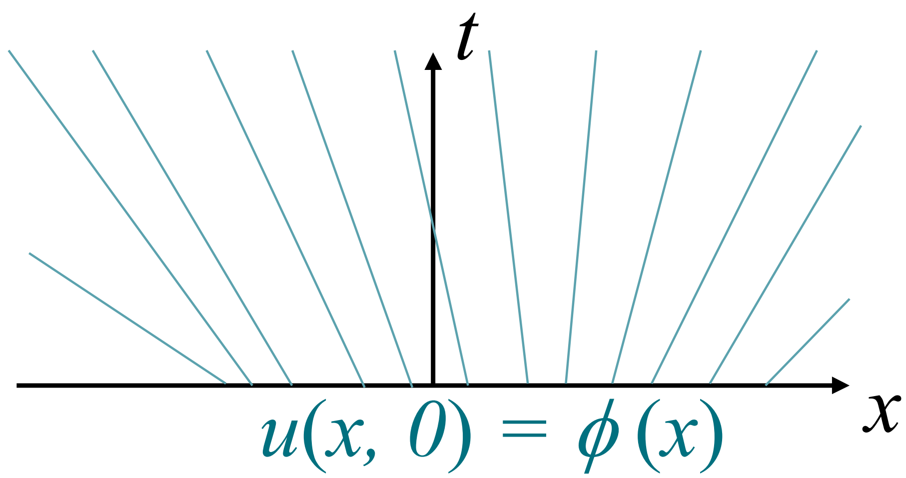
Blast pressure around a helmet

Moss et al. (2009)

Which is the way out?

An alternate option is to avoid any such intersection of characteristic lines.

This will be ensured for $t \geq 0$ provided that the function $\phi(x)$ is increasing for all x.

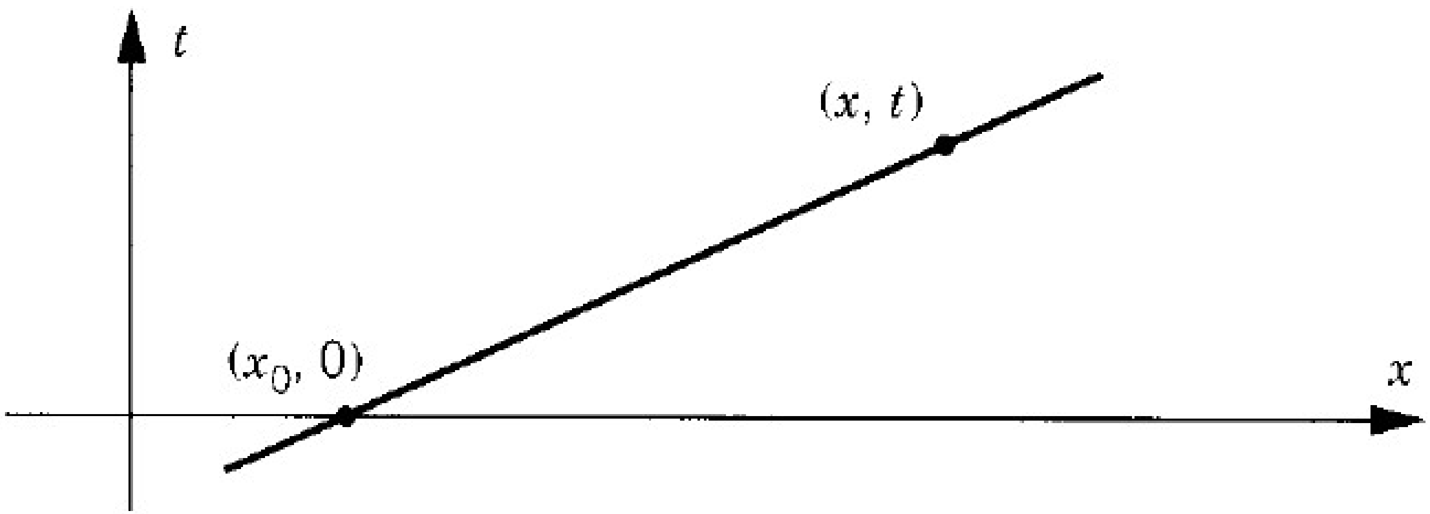


Formula for the solution of $u_t + u\,u_x = 0$, where it exists and in the absence of shock

Consider the characteristic line passing through $(x_0, 0)$ and (x, t).

Its slope is

$$\frac{x - x_0}{t - 0} = \frac{dx}{dt} = u(x, t) = u(x_0, 0) = \phi(x_0)$$



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Its slope is

$$\frac{x - x_0}{t - 0} = \frac{dx}{dt} = u(x, t) = u(x_0, 0) = \phi(x_0)$$

So that

$$\boxed{x - x_0 = t\phi(x_0)}$$

which gives x_0 implicitly as a function of (x, t).

Then the solution writes: $\boxed{u(x, t) = \phi(\, x_0(x, t)\,)}$

Example: closed form solution for a particular initial condition

Let the initial function be $\phi(x) = x^2$ (for $x > 0$).

The slope of the characteristics is given by

$$x - x_0 = t\,\phi(x_0) = t\,(x_0)^2$$

or

$$t\,(x_0)^2 + x_0 - x = 0.$$

We solve this equation explicitly for x_0:

$$x_0 = \frac{-1 \pm \sqrt{1 + 4tx}}{2t}$$

for $t \neq 0$.

Example: closed form solution for a particular initial condition

Hence, the solution of the PDE writes (for $t \neq 0$):

$$u(x,t) = \phi(x_0) = \left(\frac{-1 \pm \sqrt{1+4tx}}{2t}\right)^2$$

$$= \frac{1 \mp 2\sqrt{1+4tx} + (1+4tx)}{4t^2} = \frac{1+2tx \mp \sqrt{1+4tx}}{2t^2}$$

This formula is supposed to verify $u(x, 0) = x^2$,
but it is not defined along the line $t = 0$.

So we require: $x^2 = u(x,0) = \lim_{t \to 0} \frac{1+2tx \mp \sqrt{1+4tx}}{2t^2}$

Example: closed form solution for a particular initial condition

With the plus sign, this expression tends to infinity when t tends to zero, so it cannot be a solution.

With the minus sign, we use L'Hôpital's rule twice (with x constant) to calculate the limit as

$$\lim_{t \to 0} \frac{2x - 2x(1+4tx)^{-1/2}}{4t} = \lim_{t \to 0} \frac{4x^2(1+4tx)^{-3/2}}{4} = x^2$$

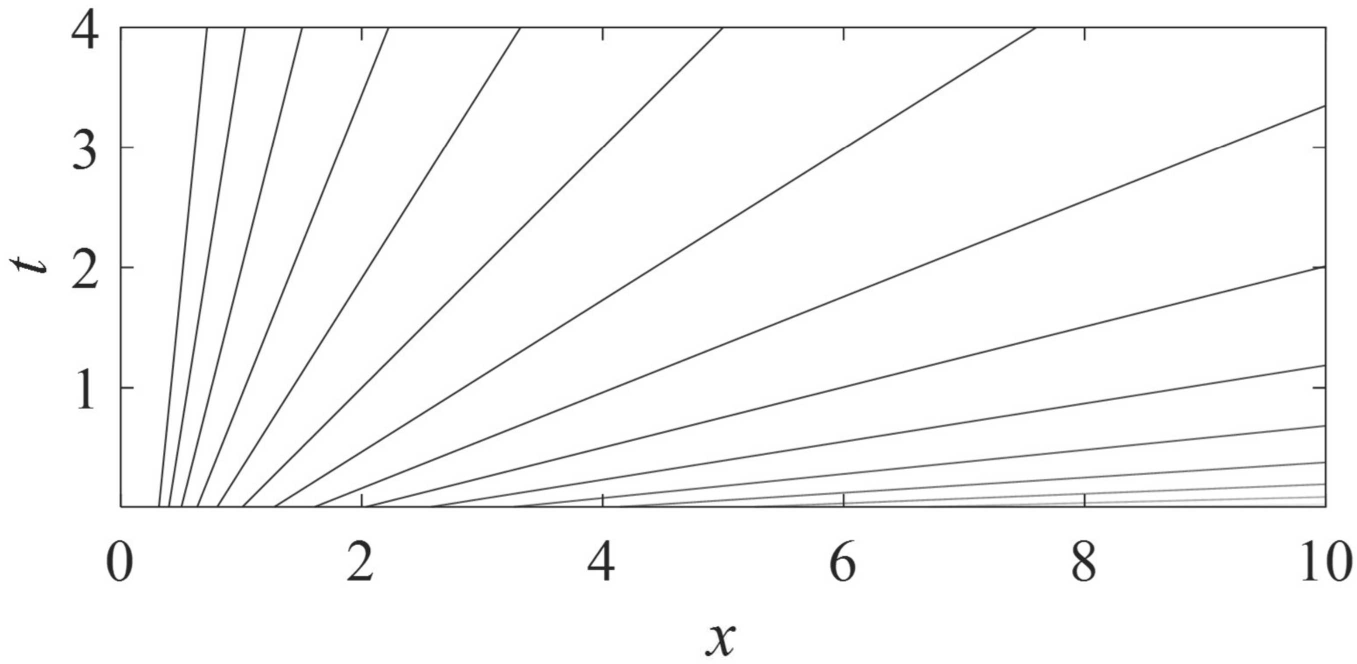
as it should be!

Therefore, the unique solution is (for $1 + 4\,t\,x \geq 0$)

$$u(x,t) = \frac{1 + 2tx - \sqrt{1+4tx}}{2t^2} \qquad \text{for } t \neq 0.$$

Example: closed form solution for a particular initial condition

And the characteristics look like this:



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$$u(x,t) = \frac{1 + 2tx - \sqrt{1 + 4tx}}{2t^2} \qquad \text{for } t \neq 0.$$

3 – General 1st order nonlinear PDE

In this section, we consider a general 1st order nonlinear PDE to introduce key concepts such as *rarefaction* (or *expansion*) *waves*, *weak solutions* and *shock waves*. We also detail the computation of the jump discontinuity at shock waves using Rankine-Hugoniot formula (Section 14.1 in Strauss, 2008). We resort to the notion of *distribution* (Section 12.1 in Strauss, 2008).

Properties of the characteristic lines for the general first-order nonlinear PDE

Let's return now to the general equation

$$u_t + a(u)\, u_x = 0.$$

The characteristic curves are the solutions of

$$\frac{dx}{dt} = a(u(x,t))$$

Calling such a curve $x = x(t)$, we observe that

$$\frac{d}{dt}u(x(t),t) = u_x\frac{dx}{dt} + u_t = u_x a(u) - a(u)u_x = 0$$

Therefore, here also, the characteristics are straight lines and the solution is constant along them.

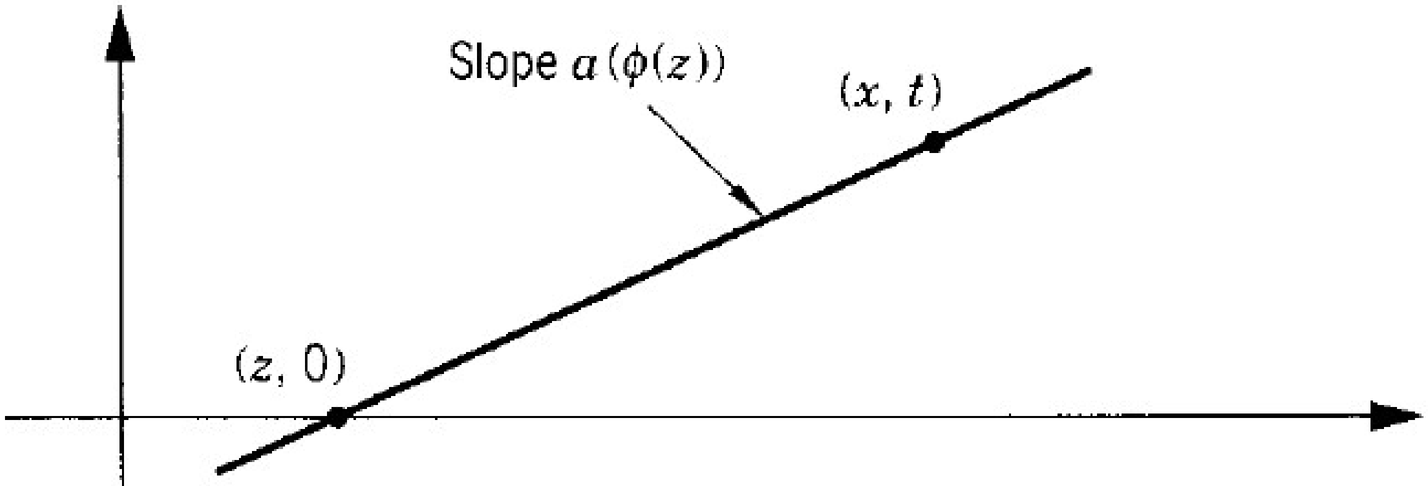
Provided that the characteristics do not intersect, we can solve the initial value problem (IVP)

Consider the initial condition $u(x, 0) = \phi(x)$.

The characteristic line through (x, t) and $(z, 0)$ has the "slope":

$$\frac{x - z}{t - 0} = \frac{dx}{dt} = a(u(x, t)) = a(u(z, 0)) = a(\phi(z))$$

Hence, $$x - z = t\, a(\phi(z)).$$



Provided that the characteristics do not intersect, we can solve the initial value problem (IVP)

Here also, expression $x - z = t\, a(\phi(z))$ gives z implicitly as a function of x and t. We note it $z(x, t)$.

Consequently, the solution of the IVP is:

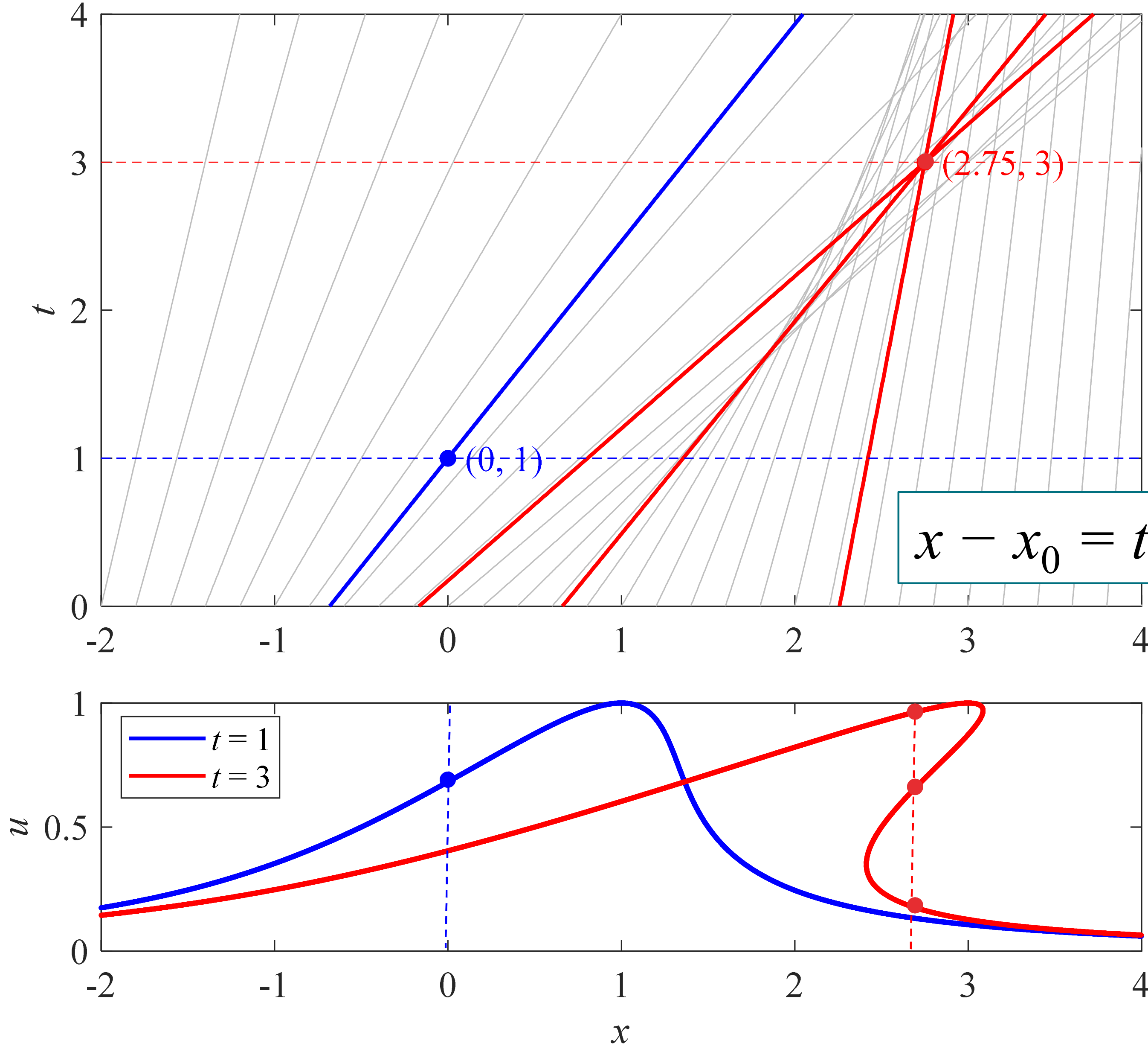
$$u(x, t) = u(z, 0) = \phi(\, z(x, t)\,)$$

No pair of characteristic lines intersect if the slope is increasing as a function of the intercept:

$$a(\, \phi(z)\,) \le a(\phi(w)\,) \qquad \text{for} \qquad z \le w.$$

i.e. the lines spread out for $t > 0$. Such a solution is called an *expansion wave* or *rarefaction wave*.

Let's take a close look at a specific example, by considering $a(u) = u$ and $\phi(x) = (1 + x^2)^{-1}$



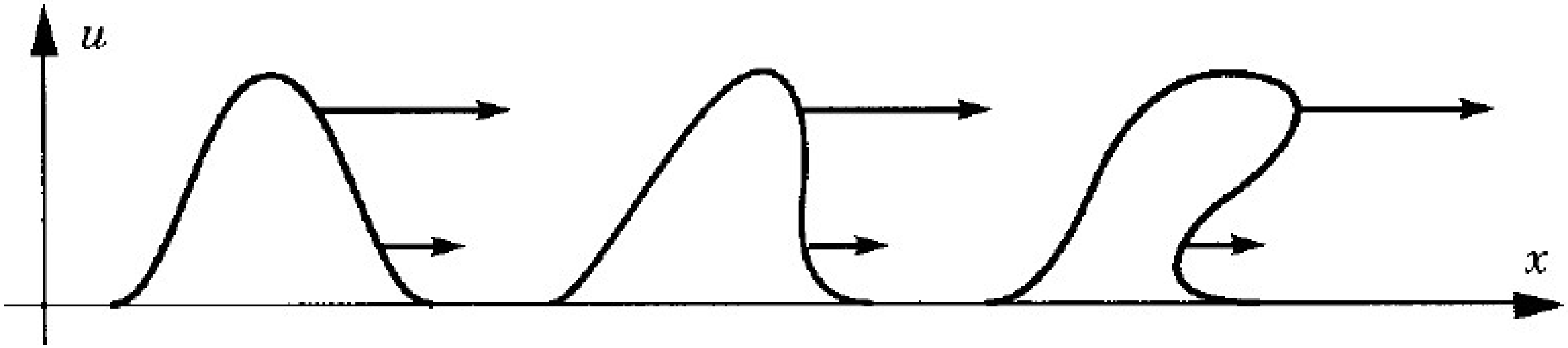
$$x - x_0 = t\,(1 + x_0{}^2)^{-1}$$

How does the solution look like when characteristic lines intersect?

Because the wave speed $a(u)$ depends on u, some parts of the wave move faster than others.

Therefore, it may happen that the "crest" of a wave moves faster and "overtakes" the smaller, slower part of the wave, leading to a 3-value "solution."

E.g. breaking of water waves on the beach …



Mathematically speaking, where a *shock wave* occurs, the solution $u(x, t)$ has a jump discontinuity.

What is the meaning of the PDE if the solution is not even continuous, let alone differentiable?

We need to introduce a "generalized" type of solution of the PDE …

Therefore, to accommodate *discontinuous* solutions of

$$u_t + a(u)\, u_x = 0$$

we will ask the equation to be valid in the sense of *distributions*.

A solution of this type will be called a ***weak solution***.

Concept of distribution

While a function is a rule that assigns numbers to numbers, a distribution is a rule (or transformation or functional) that assigns numbers to functions.

Number → Function → Number

Function → Distribution → Number

Concept of distribution

While a function is a rule that assigns numbers to numbers, a distribution is a rule (or transformation or functional) that assigns numbers to functions.

To come up with a more formal definition, we introduce the notion of *test function*.

A *test function* $\phi(x)$ is a real C^{∞} function
(a function all of whose derivatives exist)
that vanishes outside a finite interval.

Thus $\phi : \mathbb{R} \to \mathbb{R}$ is defined and differentiable for all $-\infty < x < \infty$ and $\phi(x) \equiv 0$ for x large and for x small.

Formal definition of a distribution

Let $\mathbb{D}$ denote the collection of all test functions.

Definition A distribution f is a functional (a rule): $\mathbb{D} \longrightarrow \mathbb{R}$ which is linear and continuous in the sense described below.

If $\phi \in \mathbb{D}$ is a test function, then we note the corresponding real number by $(f\,,\,\phi)$.

By linearity we mean that

$$(f, a\,\phi + b\,\psi) = a(f, \phi) + b(f, \psi)$$

for all constants a, b and all test functions ϕ, ψ.

Formal definition of a distribution

By continuity we mean the following.

If $\{\phi_n\}$ is a sequence of test functions that

- vanish outside a common interval
- and converge uniformly to a test function ϕ,

and if all their derivatives do as well,

then

$$(f, \phi_n) \longrightarrow (f, \phi) \quad \text{as} \quad n \longrightarrow \infty.$$

A distribution f may be noted as: $\phi \longmapsto (f, \phi)$.

Example: the so-called "delta function"

The so-called "delta function", noted δ, is supposed to be infinite at $x = 0$, zero at all $x \neq 0$ and have an integral equal to 1. This is not truly a function …

It is common to use the following notation (which is only a notation!):

$$\int_{-\infty}^{\infty} \delta(x)\, \phi(x)\, dx = \phi(0)$$

Actual definition The "delta function" is the rule that assigns the number $\phi(0)$ to the function $\phi(x)$.

According to this definition, the so-called "delta function" is actually a distribution $\phi \longmapsto \phi(0)$!!!

Example 2: The source function for diffusion

The source function for the diffusion equation on the whole line is

$$S(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \quad \text{for } t > 0.$$

In Lecture 5, we proved that

$$\int_{-\infty}^{+\infty} S(x,t)\,\phi(x)\,dx \to \phi(0) \quad \text{as } t \to 0.$$

Because for each t we may consider the function $S(x, t)$ as a distribution, this means that

$$S(x, t) \to \delta(x) \;\; \text{as } t \to 0.$$

The derivative of a distribution **always exists** and is another distribution

Motivation: Let $f(x)$ be any C^1 function and $\phi(x)$ any test function. Integration by parts shows that

$$\int_{-\infty}^{\infty} f'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx$$

since $\phi(x) = 0$ for large $|x|$.

Definition: For any distribution f, its derivative f' is

$(f', \phi) = -(f, \phi')$ for all test functions $\phi(x)$.

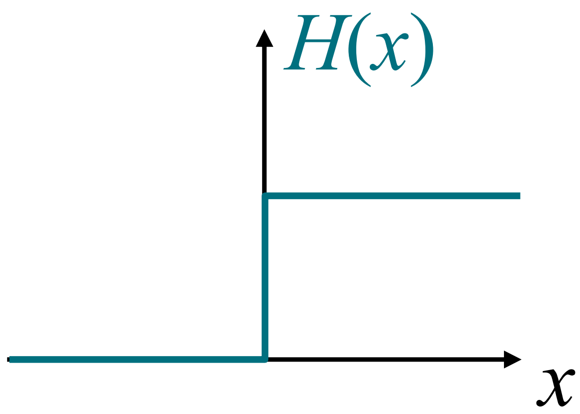
It is possible to show that

- f' satisfies the linearity and continuity properties,
- and that most of the ordinary rules of differentiation are valid for distributions.

The derivative of a distribution enables a generalization of the concept of derivatives

Example The *Heaviside function* (or step function) is defined by

- $H(x) = 1$ for $x > 0$,
- and $H(x) = 0$ for $x < 0$.



For any test function,

$$(H', \phi) = -(H, \phi') = -\int_{-\infty}^{\infty} H(x)\phi'(x)\, dx$$

$$= -\int_{0}^{\infty} \phi'(x)\, dx = -\phi(x)\Big|_{0}^{\infty} = \phi(0)$$

Thus

$$H' = \delta.$$

What does it mean that a PDE is valid "in the sense of distributions"?

Let us note $A'(u) = a(u)$. Then, equation $u_t + a(u)\, u_x = 0$ can be written as *(conservative form)*

$$u_t + [A(u)]_x = 0.$$

The PDE being valid "in the sense of distributions" means precisely that

$$\int_0^{\infty} \int_{-\infty}^{+\infty} \left[u\, \psi_t + A(u)\, \psi_x \right] dx\, dt = 0$$

for any *test function* $\psi(x, t)$ defined in the half-plane.

A solution u of this type is called a *weak solution*.

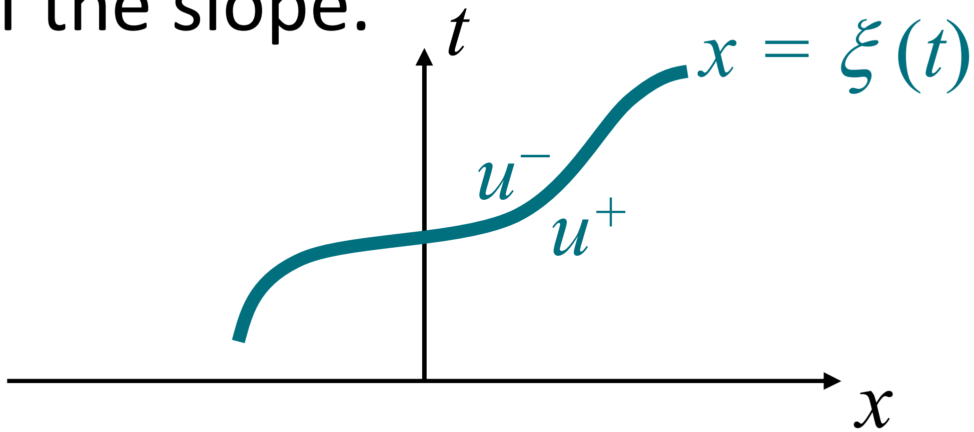
We demonstrate in four steps that the solution at the shock can be computed using *jump relations*

Suppose that the jump discontinuity, called a *shock*, occurs along the curve $x = \xi(t)$.

Because it is a jump, the limits $u^+(t) = u(x^+, t)$ and $u^-(t) = u(x^-, t)$ from the right and the left exist.

We assume that the solution is smooth elsewhere.

The speed of the shock is $s(t) = d\xi/dt$, which is the reciprocal of the slope.



We demonstrate in four steps that the solution at the shock can be computed using *jump relations*

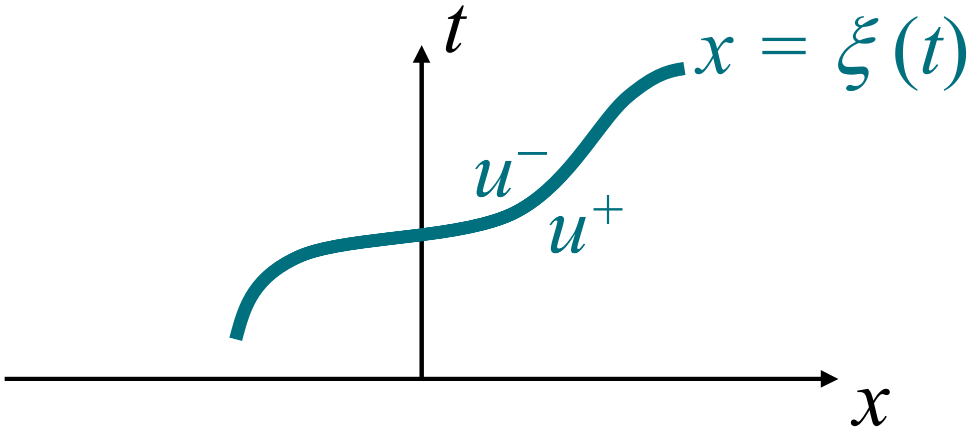
① We split the inner integral in

$$\int_0^{\infty}\int_{-\infty}^{+\infty}\left[u\,\psi_t + A(u)\,\psi_x\right]dx\,dt = 0$$

into two pieces around $\xi(t)$:

$$\int_0^{\infty}\int_{-\infty}^{\xi(t)}\left[u\,\psi_t + A(u)\,\psi_x\right]dx\,dt$$

$$+\int_0^{\infty}\int_{\xi(t)}^{+\infty}\left[u\,\psi_t + A(u)\,\psi_x\right]dx\,dt = 0$$



We demonstrate in four steps that the solution at the shock can be computed using *jump relations*

② On each piece separately (where the function is C^1), we intend to apply the divergence theorem:

$$\int_0^\infty \int_{-\infty}^{\xi(t)} \left[u\,\psi\right]_t + \left[A(u)\,\psi\right]_x \,dx\,dt$$

$$-\int_0^\infty \int_{-\infty}^{\xi(t)} \left[u_t\,\psi + A(u)_x\,\psi\right] dx\,dt$$

$$+\int_0^\infty \int_{\xi(t)}^{+\infty} \left[u\,\psi\right]_t + \left[A(u)\,\psi\right]_x \,dx\,dt$$

$$-\int_0^\infty \int_{\xi(t)}^{+\infty} \left[u_t\,\psi + A(u)_x\,\psi\right] dx\,dt = 0$$

We demonstrate in four steps that the solution at the shock can be computed using *jump relations*

③ On each piece separately (where the function is C^1), the divergence theorem leads to:

$$\int_{x=\xi(t)} u^- \psi n_t + A\left(u^-\right)\psi n_x \, dl - \int_0^\infty \int_{-\infty}^{\xi(t)} \left[\cancel{u_t + A(u)_x}\right] \psi \, dx\, dt$$

$$- \int_{x=\xi(t)} u^+ \psi n_t + A\left(u^+\right)\psi n_x \, dl - \int_0^\infty \int_{\xi(t)}^{+\infty} \left[\cancel{u_t + A(u)_x}\right] \psi \, dx\, dt = 0$$

where (n_x, n_t) denotes the unit vector normal to the shock curve, which points to the right.

Since $u_t + A(u)_x = 0$ in the ordinary sense in both regions separately, the double integrals vanish.

We demonstrate in four steps that the solution at the shock can be computed using *jump relations*

④ Hence, it remains

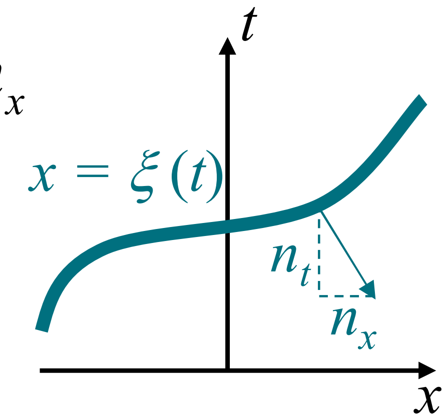
$$\int_{x=\xi(t)} \left[u^{+}\, n_t + A\left(u^{+}\right) n_x\right]\psi\, dl = \int_{x=\xi(t)} \left[u^{-}\, n_t + A\left(u^{-}\right) n_x\right]\psi\, dl$$

Because the test function ψ is arbitrary, the following relation must hold:

$$u^{+}\, n_t + A\left(u^{+}\right) n_x = u^{-}\, n_t + A\left(u^{-}\right) n_x$$

or

$$\frac{A\left(u^{+}\right) - A\left(u^{-}\right)}{u^{+} - u^{-}} = -\frac{n_t}{n_x} = s(t)$$

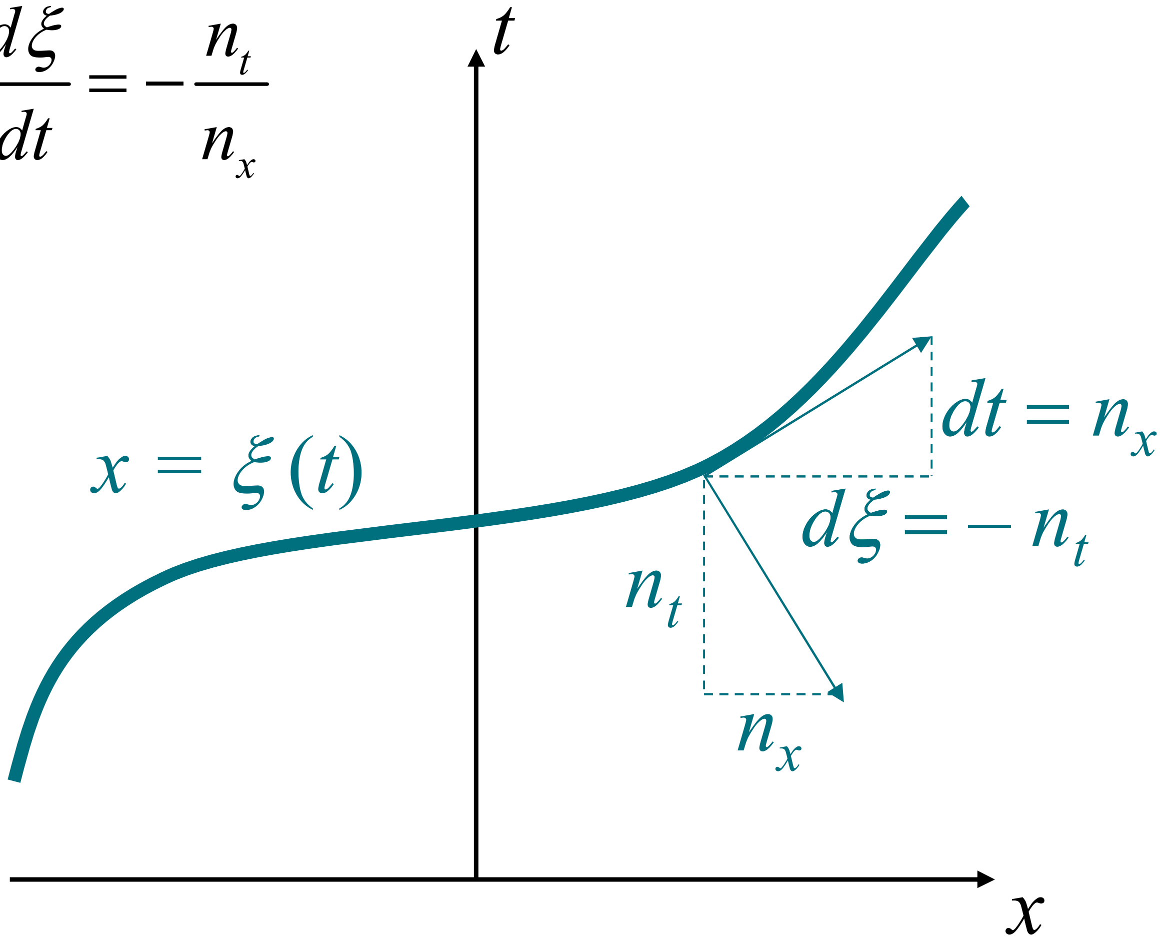


This is the *Rankine–Hugoniot formula* for the speed of the shock wave.

We demonstrate in four steps that the solution at the shock can be computed using *jump relations*

④

$$s(t) = \frac{d\xi}{dt} = -\frac{n_t}{n_x}$$



4 – Solitons

In this section, we briefly depict the concept of *soliton*, which is a remarkably stable wave solution of a nonlinear PDE (Section 14.2 in Strauss, 2008).

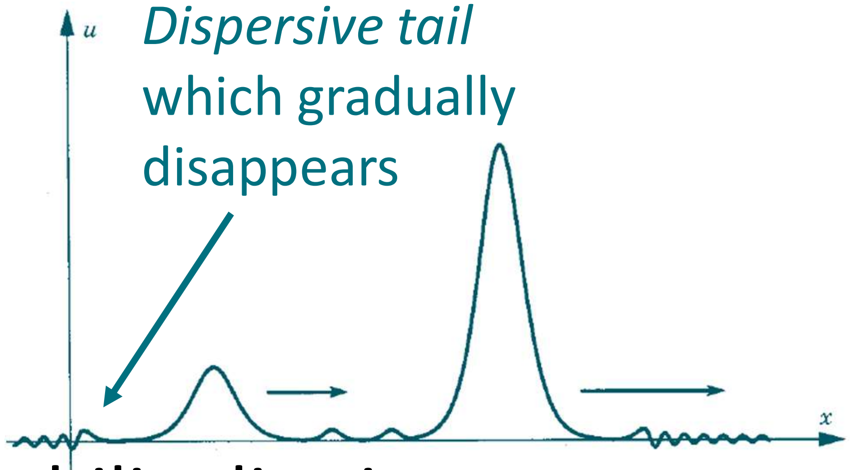
A soliton is a localized traveling wave solution of a *nonlinear* PDE that is remarkably stable.

One PDE that has solitons as solutions is the Korteweg–deVries (KdV) equation

$$u_t + 6\, u\, u_x + u_{xxx} = 0 \qquad (-\infty < x < \infty).$$

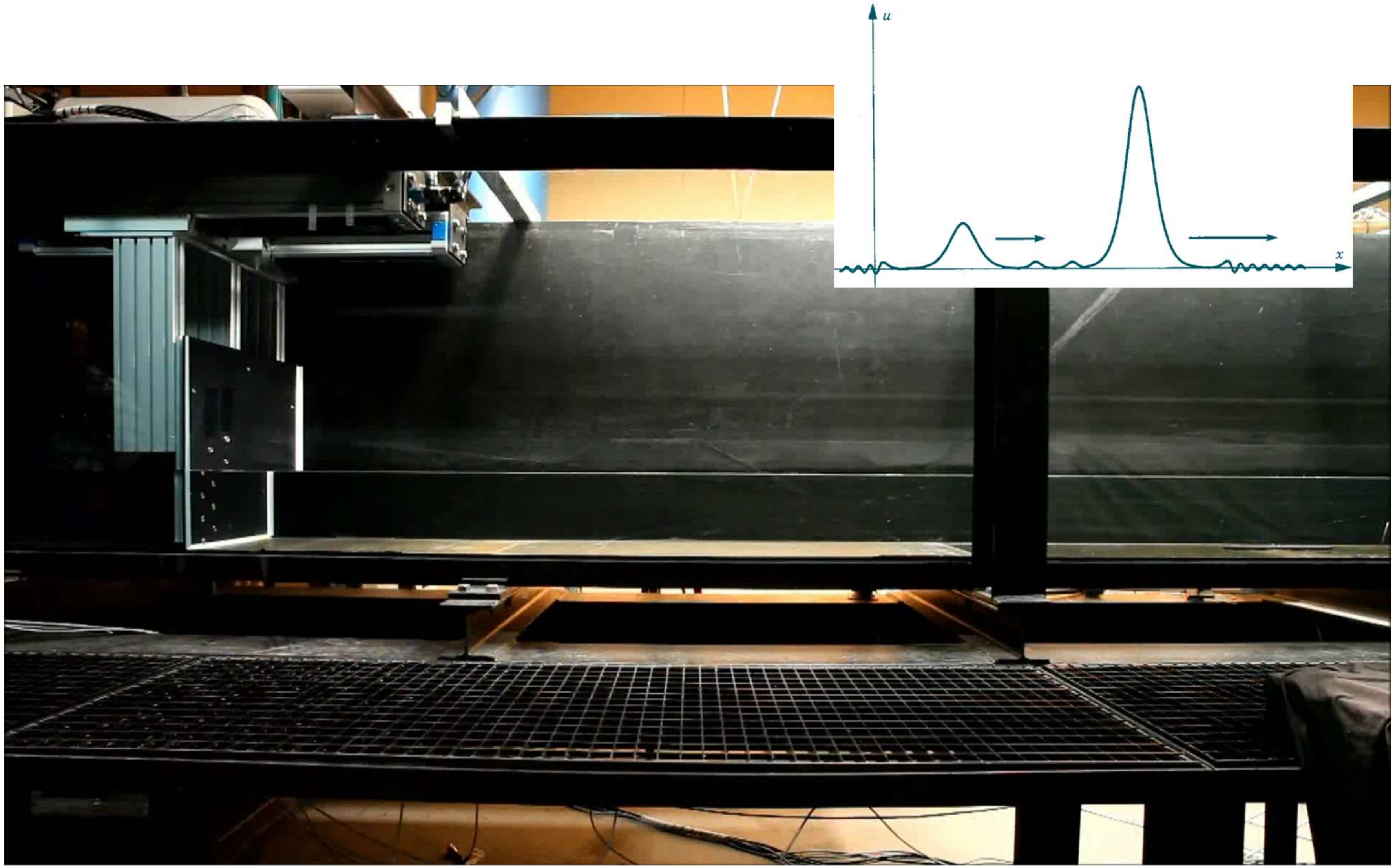
It describes water waves in a channel.

It also appears in several other branches of physics (e.g. as a mathematical model of a *stable* elementary particle).



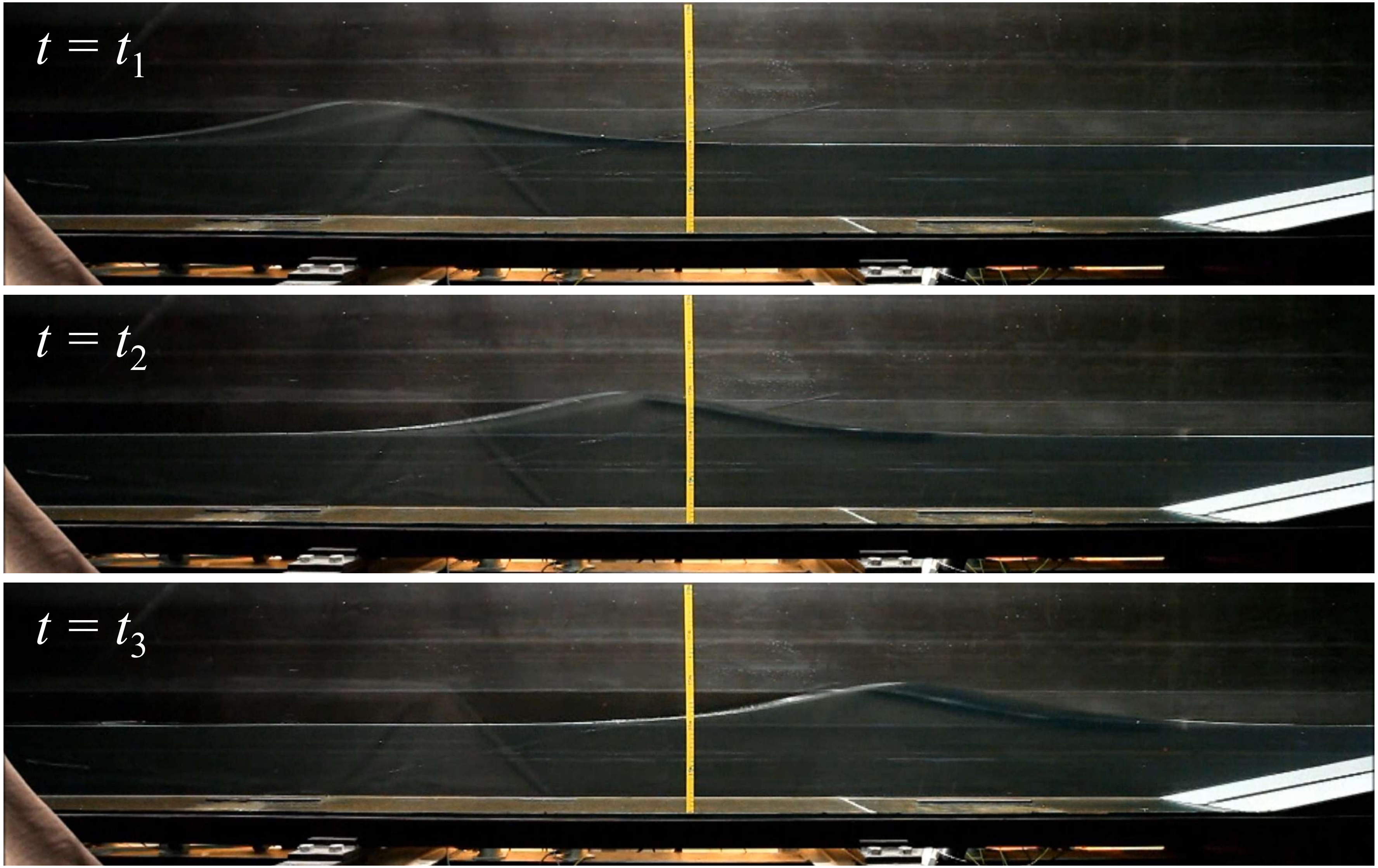
The explanation of soliton stability lies in a complicated relationship with a *linear* equation!

Example: solitary wave in a laboratory flume



Lab experiment of solitary impulse wave (ETHZ, Courtesy of Prof. W. Hager)

Example: solitary wave in a laboratory flume



Lab experiment of solitary impulse wave (ETHZ, Courtesy of Prof. W. Hager)

Take-home messages

For nonlinear PDE, the principle of superposition ceases to hold.

For nonlinear 1st order PDEs,

- the method of characteristics works,
  as it does for linear transport equations,
  as long as the characteristics do not intersect;
- each characteristic curve is a straight line;
- the solution $u(x, t)$ remains constant
  on each characteristic line.

Take-home messages

In nonlinear PDEs, the wave speed depends on the solution itself, so that characteristics may intersect.

Therefore, *shock waves* may occur when a faster part of the wave tends to overtake a slower part.

This leads to a *jump discontinuity* in the solution.

To accommodate *discontinuous* solutions, the PDE is regarded in the sense of *distributions*.

A solution of this type is called a *weak solution*.

It verifies Rankine–Hugoniot formula at the jump.

LIÈGE université
Sciences Appliquées

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Mathématiques appliquées (MATH0504-1)

B. Dewals, Ch. Geuzaine