LIÈGE université
Sciences Appliquées

Lecture 8 Laplace equation

Mathématiques appliquées (MATH0504-1)

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Learning objectives

Become familiar with two important properties of Laplace equation:

- the maximum principle
- the rotational invariance.

Be able to solve the equation in series form in rectangles, disks (incl. Poisson formula), and related shapes.

Become aware of key properties of the solutions, such as the *mean value* property.

Outline

1. Introduction to Laplace’s equation
2. Maximum and minimum principle
3. Invariance and fundamental solutions
4. Rectangles and cubes
5. Poisson’s formula
6. Exterior of a circle, wedges and annuli

1 – Introduction to Laplace’s equation

In this section, we introduce Laplace’s equation and show its practical relevance (Section 6.1 in Strauss, 2008).

For a stationary process, both the diffusion and the wave equations reduce to the Laplace equation

If a diffusion or wave process is stationary (independent of time), then $u_t \equiv 0$ and $u_{tt} \equiv 0$.

Therefore, both the diffusion and the wave equations reduce to the Laplace equation:

- in 1D: $u_{xx} = 0$
- in 2D: $\nabla \cdot \nabla u = \boxed{\Delta u} = u_{xx} + u_{yy} = 0$
- in 3D: $\nabla \cdot \nabla u = \boxed{\Delta u} = u_{xx} + u_{yy} + u_{zz} = 0$

A solution of the Laplace equation is called a *harmonic function*.

Two particular cases

In 1D,

- we have simply $u_{xx} = 0$;
- therefore, the only harmonic functions in 1D are $u(x) = A + B\,x$;
- this is so simple that it hardly gives a clue to what happens in higher dimensions.

The inhomogeneous version of Laplace's equation

$$\Delta u = f$$

with f a given function, is called *Poisson's equation*.

Laplace’s and Poisson’s equations are ubiquitous in Physics and Engineering applications

Example 1: steady fluid flow

- Assume that the flow is steady and irrotational (no eddies) so that $\mathrm{rot}\,\mathbf{v} = 0$, where $\mathbf{v} = \mathbf{v}(x, y, z)$ is the flow velocity.
- Hence, $\mathbf{v} = -\,\mathrm{grad}\,\phi$, with ϕ a scalar function (called *velocity potential*)
- Assume that the flow is incompressible, without sources nor sinks. Then $\mathrm{div}\,\mathbf{v} = 0$.
- Hence, the potential ϕ is governed by Laplace’s equation: $\Delta\phi = -\,\mathrm{div}\,\mathbf{v} = 0$.

Laplace's and Poisson's equations are ubiquitous in Physics and Engineering applications

Example 2: electrostatics

- We have $\text{rot}\,\mathbf{E} = 0$ and $\text{div}\,\mathbf{E} = 4\,\pi\,\rho$, where ρ is the charge density.
- $\text{rot}\,\mathbf{E} = 0$ implies that $\mathbf{E}$ can be written as: $\mathbf{E} = -\,\text{grad}\,\phi$, with ϕ a scalar function (called *electric potential*).
- Therefore,

$$\Delta\phi = \text{div}(\text{grad}\,\phi) = -\text{div}\,\mathbf{E} = -4\pi\rho$$

which is Poisson's equation (with $f = -\,4\,\pi\,\rho$).

Basic mathematical problem of interest here

Solve Laplace’s or Poisson’s equation in a given domain D with a condition on boundary bdy D:

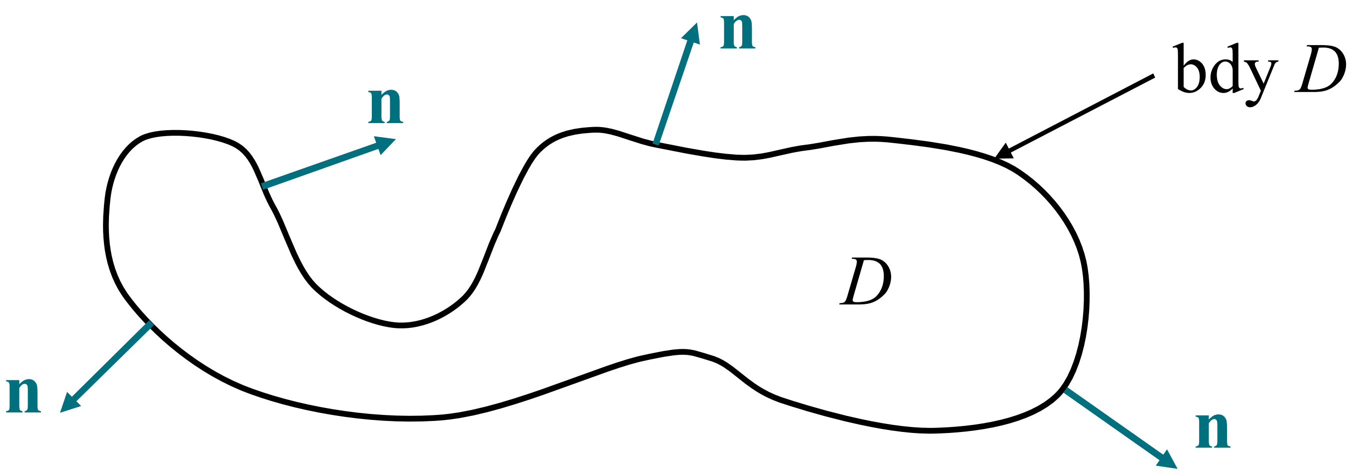
$$\Delta u = f \qquad \text{in} \qquad D$$

$$\text{with} \quad u = h \quad \text{or} \quad u_n = h$$

$$\text{or} \quad u_n + a\,u = h \quad \text{on bdy } D.$$

E.g.
- flux
- reaction force

E.g.
- temperature,
- displacement



2 – Maximum and minimum principle

In this section, we introduce the maximum and minimum principle and, as a consequence, demonstrate the unicity of the solution of the Dirichlet problem (Section 6.1 in Strauss, 2008).

Maximum principle: a harmonic function is its biggest and its smallest on the boundary

For Laplace's equation, the maximum principle is:

Open set
= a set w/o its boundary
= *domain*
= region

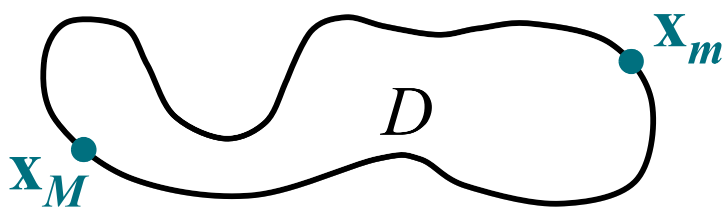
- Let D be a connected bounded open set (in 2D or 3D).
- Let either $u(x, y)$ or $u(x, y, z)$ be a harmonic function in D
- Let $u(x, y)$ or $u(x, y, z)$ be continuous on $D \cup (\text{bdy } D)$.
- Then the maximum and the minimum values of u are attained on bdy D and nowhere inside (unless $u \equiv$ constant).

Maximum principle: a harmonic function is its biggest and its smallest on the boundary

We use the following notations:

- $\mathbf{x} = (x, y)$ in 2D or $\mathbf{x} = (x, y, z)$ in 3D.
- $|\mathbf{x}| = (x^2 + y^2)^{1/2}$ or $|\mathbf{x}| = (x^2 + y^2 + z^2)^{1/2}$.

The maximum principle asserts that there are points $\mathbf{x}_M$ and $\mathbf{x}_m$ on bdy D such that $u(\mathbf{x}_m) \leq u(\mathbf{x}) \leq u(\mathbf{x}_M)$ for all $\mathbf{x} \in D$.



There could be several such points on the boundary.

In contrast, there are no points inside D with this property (unless $u \equiv$ constant).

Main idea underpinning the maximum principle

The overall idea behind the demonstration of the maximum principle is the following (in 2D).

- At a maximum point inside D, we would have $u_{xx} \leq 0$ and $u_{yy} \leq 0$. So $u_{xx} + u_{yy} \leq 0$.
- At most maximum points, $u_{xx} < 0$ and $u_{yy} < 0$, which would contradict Laplace's equation.

However,

- since it is possible that $u_{xx} = u_{yy} = 0$ at a maximum point …
- we have to work a little harder to get a proof!

3-step demonstration of the maximum principle

① Let

- $\epsilon > 0.$
- $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon\, |\mathbf{x}|^2.$

Then (in 2D)

$$\Delta v = \Delta u + \epsilon\, \Delta\, (x^2 + y^2) = 0 + 4\, \epsilon > 0 \quad \text{in} \;\; D.$$

If v has an interior maximum point, this would hold:

$$\Delta v = v_{xx} + v_{yy} \leq 0.$$

Since this result is in contradiction with the previous inequality, $v(\mathbf{x})$ has no interior maximum in D.

3-step demonstration of the maximum principle

② Function $v(\mathbf{x})$ being continuous, it must have a maximum *somewhere* in the closure $D = D \cup \text{bdy } D$.

Let us assume that the maximum of $v(\mathbf{x})$ is attained at $\mathbf{x}_0 \in \text{bdy } D$.

The closure is the union of the domain and its boundary

Then, for all $\mathbf{x} \in D$,

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{x}_0) = u(\mathbf{x}_0) + \epsilon\, |\mathbf{x}_0|^2$$

and

$$u(\mathbf{x}_0) + \epsilon\, |\mathbf{x}_0|^2 \leq \max_{\text{bdy } D} u + \epsilon\, l^2,$$

with l the greatest distance from bdy D to the origin.

3-step demonstration of the maximum principle

③ Since

$$u(\mathbf{x}) \le u(\mathbf{x}_0) + \epsilon\,|\mathbf{x}_0|^2 \le \max_{\text{bdy } D} u + \epsilon\, l^2,$$

is true for any $\epsilon > 0$, we have

The absence of a maximum inside D will be proved later

$$u(\mathbf{x}) \le \max_{\text{bdy } D} u \qquad \text{for all } \mathbf{x} \in D.$$

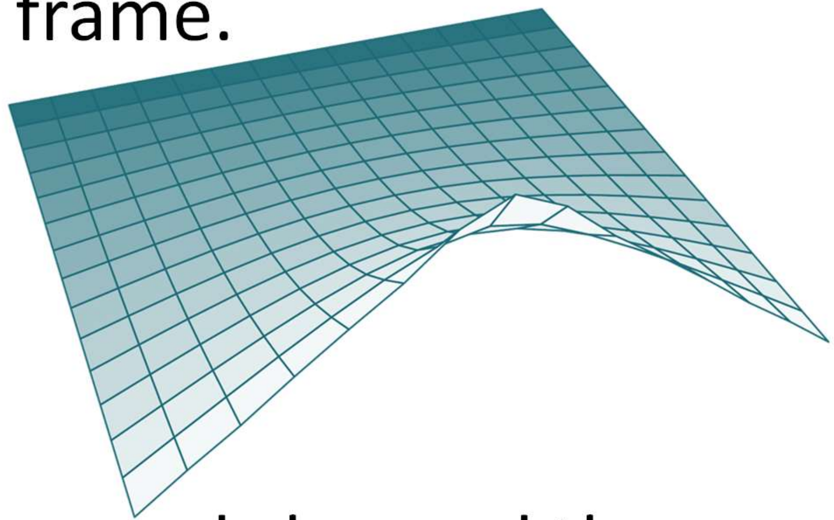
This maximum is attained at some point $\mathbf{x}_M \in \text{bdy } D$.

Consequently, $u(\mathbf{x}) \le u(\mathbf{x}_M)$ for all $\mathbf{x} \in D$, which is the desired conclusion!

A similar demonstration applies for a minimum ($\mathbf{x}_m$).

Intuitive visualization of the maximum-minimum principle

Consider the case of a membrane (or a soap film) extended over a rigid closed frame.



If we give the initially plane frame a small transverse deformation, we do not expect the membrane to bulge either upwards or downwards beyond the frame, unless external forces are applied.

Similarly, in the realm of thermal steady state, the temperature attains its maximum and minimum values at the boundaries of the region.

Uniqueness of the Dirichlet problem

To prove the uniqueness, consider two solutions u and v, so that

$$\begin{aligned} \Delta u &= f \quad \text{in } D \\ u &= h \quad \text{on bdy } D \end{aligned} \qquad\qquad \begin{aligned} \Delta v &= f \quad \text{in } D \\ v &= h \quad \text{on bdy } D \end{aligned}$$

Let us subtract the equations and let $w = u - v$.

By the maximum principle, since $w = 0$ on bdy D,

$$0 = w(\mathbf{x}_m) \leq w(\mathbf{x}) \leq w(\mathbf{x}_M) = 0 \quad \text{for all } \mathbf{x} \in D$$

Therefore, both the maximum and minimum of $w(x)$ are zero. This means that $w \equiv 0$ and $u \equiv v$.

Uniqueness (and existence) do not hold for all types of boundary conditions!

Consider the solution of the **Neumann problem**:

$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = g \quad \text{on bdy } D$$

1. Solutions do not exist, unless

$$\iiint_D f \, dx \, dy \, dz = \iint_{\text{bdy}(D)} g \, dS$$

Exercise
Prove this by integrating the equation

2. The solution is determined uniquely with an additive constant.

3 – Invariance and fundamental solutions

In this section, we introduce invariance properties of Laplace’s equation in 2D and 3D and derive particular solutions which have the same invariance properties (Section 6.1 in Strauss, 2008).

Invariance in 2D: Laplace equation is invariant under all rigid motions (translations, rotations)

A **translation** in the plane is a transformation

$$x' = x + a \qquad\qquad y' = y + b.$$

Invariance under translations means simply that

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'} \,.$$

A **rotation** by an angle α is given by

$$x' = x \cos \alpha + y \sin \alpha$$

$$y' = -x \sin \alpha + y \cos \alpha.$$

Let us use the chain rule …

Invariance in 2D: Laplace equation is invariant under all rigid motions (translations, rotations)

A **rotation** by an angle α is given by

$$x' = x\cos\alpha + y\sin\alpha$$

$$y' = -x\sin\alpha + y\cos\alpha.$$

Applying the chain rule to $u(x', y')$, we calculate

$$u_x = u_{x'}\cos\alpha - u_{y'}\sin\alpha$$

$$u_y = u_{x'}\sin\alpha + u_{y'}\cos\alpha$$

$$u_{xx} = (u_{x'}\cos\alpha - \underline{u_{y'}\sin\alpha})_{x'}\cos\alpha - (\underline{u_{x'}\cos\alpha} - u_{y'}\sin\alpha)_{y'}\sin\alpha$$

$$u_{yy} = (u_{x'}\sin\alpha + \underline{u_{y'}\cos\alpha})_{x'}\sin\alpha + (\underline{u_{x'}\sin\alpha} + u_{y'}\cos\alpha)_{y'}\cos\alpha.$$

By adding, we get:

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}\,.$$

Invariance in 2D: Laplace equation is invariant under all rigid motions (translations, rotations)

Interpretation: in engineering the laplacian Δ is a model for isotropic physical situations, in which there is no preferred direction.

The rotational invariance suggests that the 2D laplacian

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

should take a particularly simple form in polar coordinates.

Let us use the transformation

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Invariance in 2D: Laplace equation is invariant under all rigid motions (translations, rotations)

Applying the chain rule with

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

we get:

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta},$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

and we end up with:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Invariance in 2D: Laplace equation is invariant under all rigid motions (translations, rotations)

We investigate the existence of harmonic functions that themselves are rotationally invariant: $u(r, \cancel{\theta})$.

In 2D, this means that we use polar coordinates $(r,\ \theta)$ and look for solutions $u(r)$:

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}} \Rightarrow \quad 0 = u_{rr} + \frac{1}{r}u_r$$

This ODE is easy to solve:

$$(ru_r)_r = 0, \qquad ru_r = c_1, \qquad u = c_1 \ln r + c_2.$$

This function $\ln r$ will play a central role later.

The 3D laplacian is also invariant under rigid motion

A similar demonstration as in the 2D case can be elaborated using vector-matrix notation:

$$\mathbf{x}' = B\,\mathbf{x}$$

where B is an orthogonal matrix ($B^{\mathrm{T}}B = BB^{\mathrm{T}} = I$).

See details in the textbook.

For the 3D laplacian,

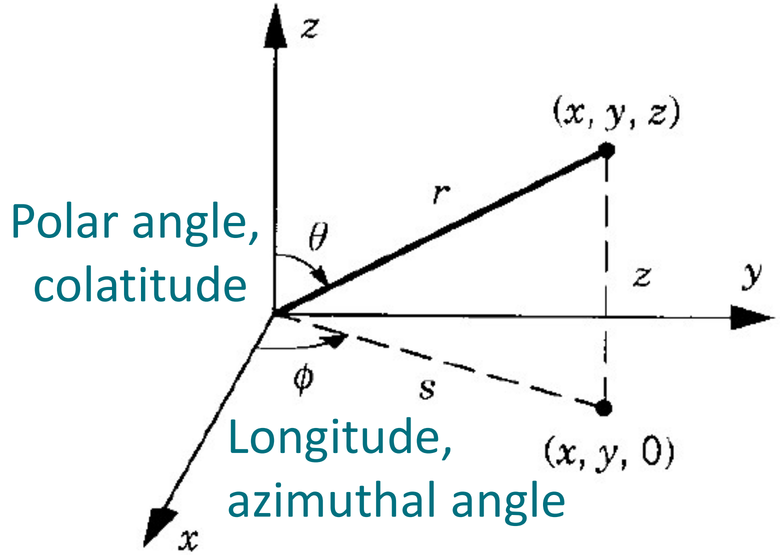
$$\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

it is also natural to use spherical coordinates (r, θ, ϕ).

The 3D laplacian is also invariant under rigid motion

The laplacian in spherical coordinates writes:

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}} + \cancel{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$



Let us look for harmonic functions in 3D which do not change under rotation ...

The 3D laplacian is also invariant under rigid motion

Harmonic functions which do not change under rotation, i.e. which depend only on r satisfy the ODE

$$0 = \Delta_3 u = u_{rr} + \frac{2}{r} u_r$$

So $(r^2 u_r)_r = 0$. It has the solutions $r^2 u_r = c_1$.
That is, $u = -c_1 r^{-1} + c_2$.

This important harmonic function

$$\frac{\mathbf{1}}{\mathbf{r}} = (x^2 + y^2 + z^2)^{-1/2}$$

is the analog of the 2D function $\ln(x^2 + y^2)^{1/2}$.

The 3D laplacian is also invariant under rigid motion

This result can also be easily obtained by assuming spherical symmetry from the very beginning, and this way, it becomes straightforward to extend it to n dimensions.

Indeed, consider you are looking for a solution in the form:

$$u(r) \quad \text{with} \quad r = (x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2)^{1/2}$$

in an n-dimensional space.

Using the chain rule, you get ($x_i = x_1 \ldots x_n$):

$$\frac{\partial u}{\partial x_i} = \frac{du}{dr}\frac{\partial r}{\partial x_i} = u_r \frac{1}{2}\frac{2x_i}{r} = u_r \frac{x_i}{r}$$

The 3D laplacian is also invariant under rigid motion

Similarly,

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(u_r \frac{x_i}{r}\right) = \frac{u_r}{r} + u_r x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) + \frac{\partial u_r}{\partial x_i}\frac{x_i}{r}$$

$$= \frac{u_r}{r} - u_r x_i \frac{1}{r^2}\frac{1}{2}\frac{2x_i}{r} + u_{rr}\frac{\partial r}{\partial x_i}\frac{x_i}{r} = \frac{u_r}{r} - \frac{x_i^2}{r^2}\frac{u_r}{r} + \frac{x_i^2}{r^2}u_{rr}$$

Summing up over all independent variables x_i, you get:

$$\Delta u = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = n\frac{u_r}{r} - \frac{\sum_{i=1}^{n} x_i^2}{r^2}\frac{u_r}{r} + \frac{\sum_{i=1}^{n} x_i^2}{r^2}u_{rr} = (n-1)\frac{u_r}{r} + u_{rr}$$

The 3D laplacian is also invariant under rigid motion

It may be concluded that, **assuming spherical symmetry**, Laplace equation writes

- in n dimensions: $$\Delta_n u = (n-1)\frac{u_r}{r} + u_{rr} = 0$$

$$\Rightarrow u = c_1 \frac{1}{r^{n-2}} + c_2 \qquad n > 2$$

Check this on your own!

- in 3D: $$\Delta_3 u = 2\frac{u_r}{r} + u_{rr} = 0 \qquad \Rightarrow u = c_1 \boxed{\frac{1}{r}} + c_2$$

- in 2D: $$\Delta_2 u = \frac{u_r}{r} + u_{rr} = 0 \qquad \Rightarrow u = c_1 \boxed{\ln r} + c_2$$

4 – Rectangles and cubes

In this section, we solve Laplace equation in a rectangle by separating variables and we provide an overview of the solution of the Dirichlet problem in a cube (Section 6.2 in Strauss, 2008).

Laplace equation can be solved in particular geometries by separating the variables

The general procedure is the same as in Lecture 7.

1. Look for separated solutions of the PDE.
2. Put in the **homogeneous** boundary conditions to get the eigenvalues. This is the step which is dependent on the considered geometry.
3. Sum the series.
4. Put in the **inhomogeneous** (initial or) boundary conditions.

Laplace equation can be solved in particular geometries by separating the variables

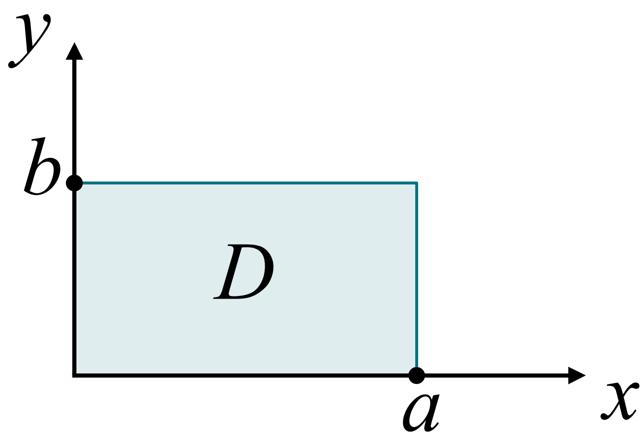
Let us consider

$$\Delta_2 u = u_{xx} + u_{yy} = 0 \qquad \text{in } D$$

where D is the rectangle $\{0 < x < a,\, 0 < y < b\}$.

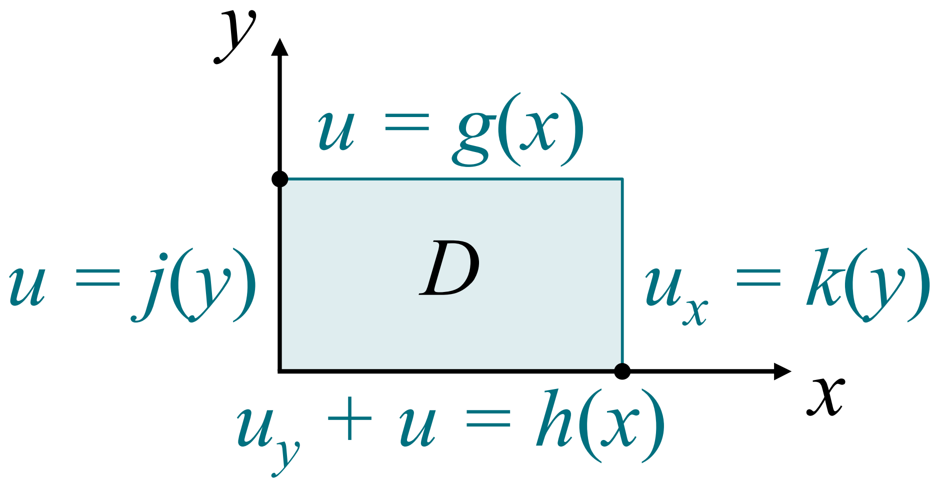
On each side of the rectangle, one of the standard boundary conditions is prescribed:

- Dirichlet
- Neumann
- or Robin.



Example 1

Let us consider the following BCs:



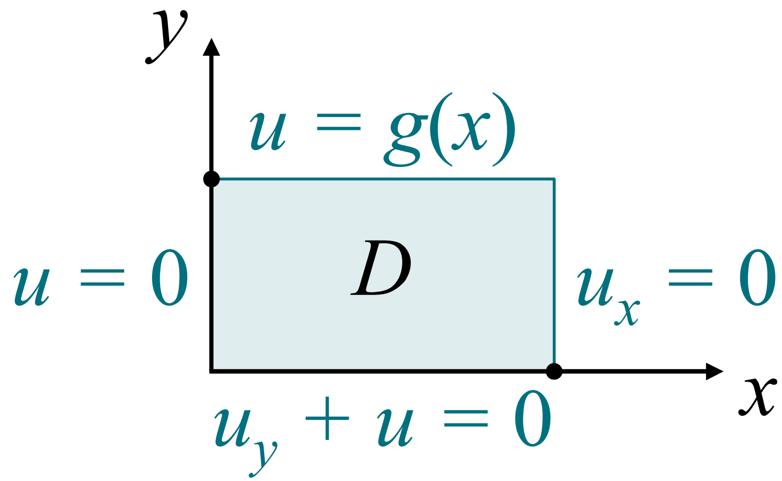
If we call the solution u with data (g, h, j, k), then $u = u_1 + u_2 + u_3 + u_4$ where

- u_1 has data $(g, 0, 0, 0)$,
- u_2 has data $(0, h, 0, 0)$, and so on ...

Example 1

Step 1: Look for separated solutions of the PDE

For simplicity, let's assume that $h = 0, j = 0, k = 0$



① We separate the variables: $u(x, y) = X(x)\ Y(y)$.

We get:

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Example 1

Step 2: Use the homogeneous BCs → eigenvalues

Since $X''(x) \,/\, X = -\,Y''(y) \,/\, Y(y)$, each side of this equation must be a constant (say $-\lambda$):

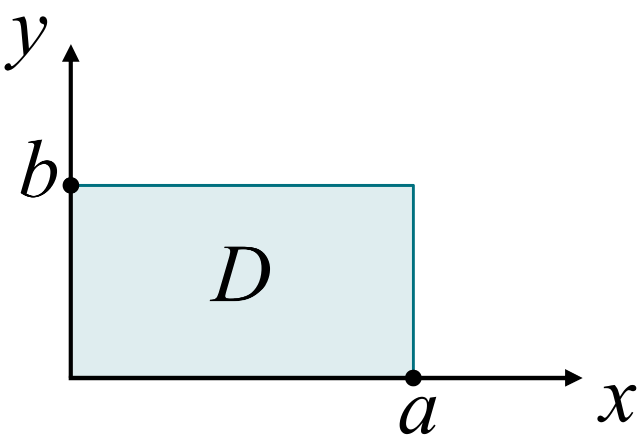
- $X''(x) + \lambda X = 0$ for $0 \le x \le a$

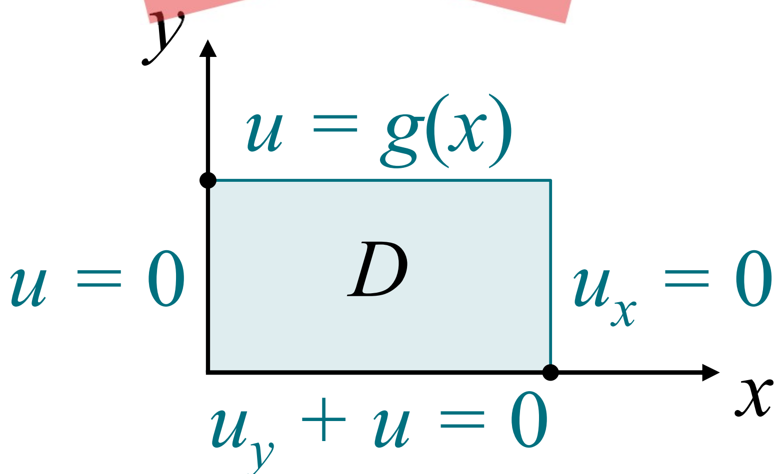
  with $X(0) = 0$ and $X'(a) = 0$

- $Y''(y) - \lambda Y = 0$ for $0 \le y \le b$

  with $Y'(0) + Y(0) = 0$ ~~and $Y(b) = \dots$~~

2





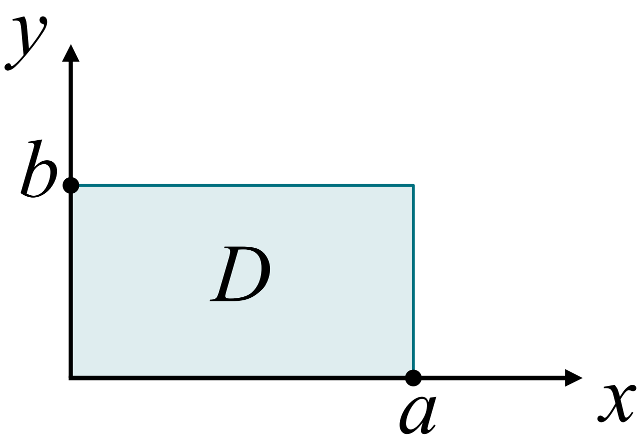
Example 1

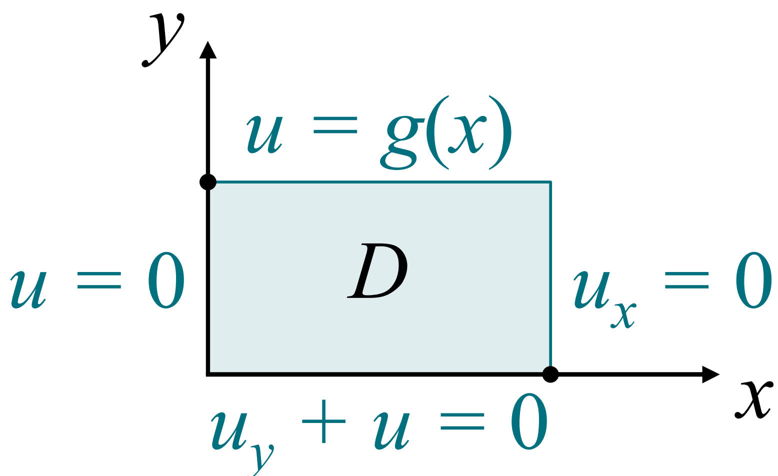
Step 2: Use the homogeneous BCs → eigenvalues

The solution for $X(x)$ verifying $X''(x) + \lambda X = 0$
for $0 \le x \le a$, with $X(0) = 0$ and $X'(a) = 0$, writes:

$$X_n(x) = \sin \frac{(n + \frac{1}{2})\pi x}{a}$$

and $\lambda_n = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} \quad (n = 0,\ 1,\ 2,\ 3, \ldots)$





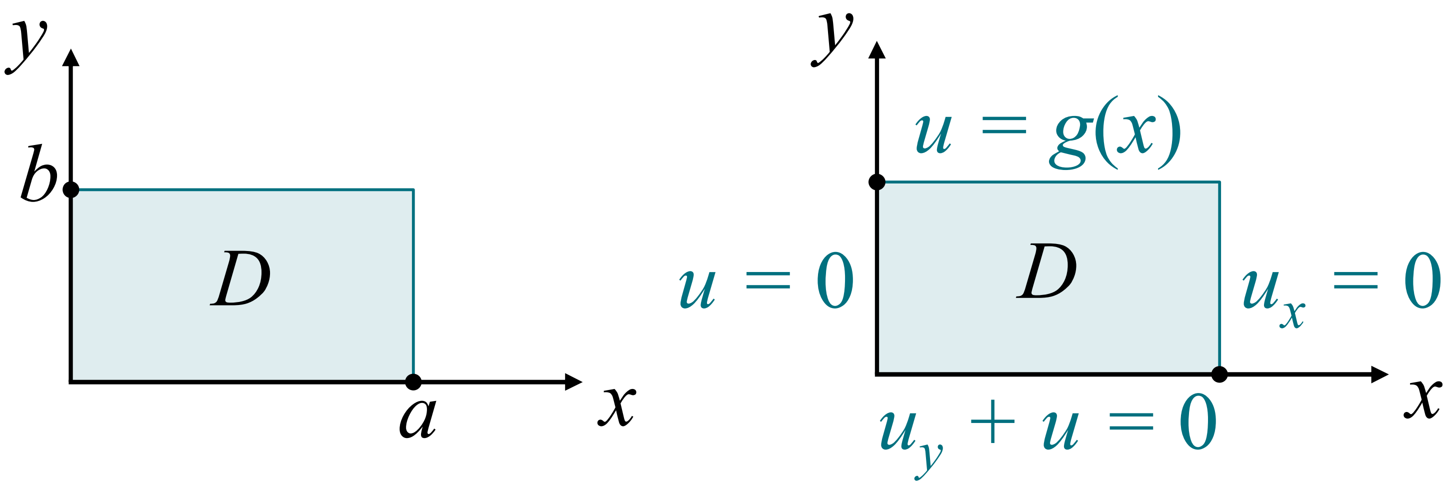
Example 1
Step 2: Use the homogeneous BCs → eigenvalues

The solution for $Y(y)$ verifying $Y''(y) - \lambda\, Y = 0$ for $0 \leq y \leq b$ writes (with $\beta_n = \lambda_n^{1/2}$):

$$Y(y) = A \cosh \beta_n y + B \sinh \beta_n y$$

The BC $Y'(0) + Y(0) = 0$ implies $B\,\beta_n + A = 0.$

Since the modes may be multiplied by any arbitrary constant, we may choose $B = -1$, so that $A = \beta_n$.



Example 1

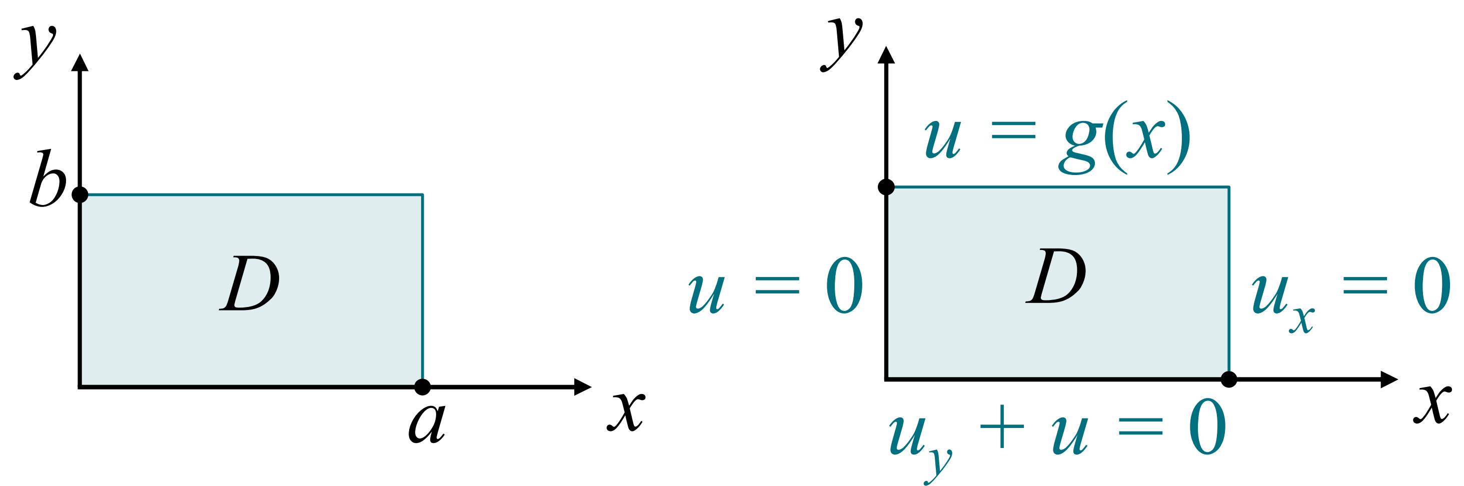
Step 3: Sum the series

③ Therefore, the sum

$$u(x, y) = \sum_{n=0}^{\infty} A_n \sin \beta_n x \ (\beta_n \cosh \beta_n y - \sinh\beta_n y)$$

is a harmonic function in D that satisfies all three homogeneous BCs.

In the rectangle, this function is also bounded.



Example 1
Step 4: Put in the inhomogeneous BCs

④ The remaining BC $u(x, b) = g(x)$ requires that

$$g(x) = \sum_{n=0}^{\infty} A_n(\beta_n \cosh \beta_n b - \sinh \beta_n b) \cdot \sin \beta_n x$$

for $0 < x < a$.

This is simply a Fourier series in the eigenfunctions $\sin \beta_n x$.

From Fourier series theory, the coefficients A_n are given by the formula:

$$A_n = \frac{2}{a}(\beta_n \cosh \beta_n b - \sinh \beta_n b)^{-1} \int_0^a g(x) \sin \beta_n x \, dx .$$

Example 2: Dirichlet problem in a 3D "box" $\{0 < x < a,\ 0 < y < b,\ 0 < z < c\}$

Consider the particular case of a cube:

$$\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in } D$$

$$D = \{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}$$

$$u(\pi, y, z) = g(y, z)$$

$$u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0.$$

To solve,

① • separate variables: $u = X(x)Y(y)Z(z)$

② • use the five homogeneous BCs

$$X(0) = Y(0) = Z(0) = Y(\pi) = Z(\pi) = 0.$$

Example 2: Dirichlet problem in a 3D "box" $\{0 < x < a,\ 0 < y < b,\ 0 < z < c\}$

③ Evaluating the eigenfunctions and eigenvalues gives:

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2}\, x) \sin my \sin nz$$

④ Plugging the inhomogeneous BC at $x = \pi$ leads to a double Fourier sine series in the variables y and z:

$$A_{mn} = \frac{4}{\pi^2 \sinh(\sqrt{m^2 + n^2}\, \pi)} \int_0^{\pi} \int_0^{\pi} g(y, z) \sin my \sin nz \, dy \, dz$$

Hence the solution is expressed as
a doubly infinite series!

5 – Poisson’s formula

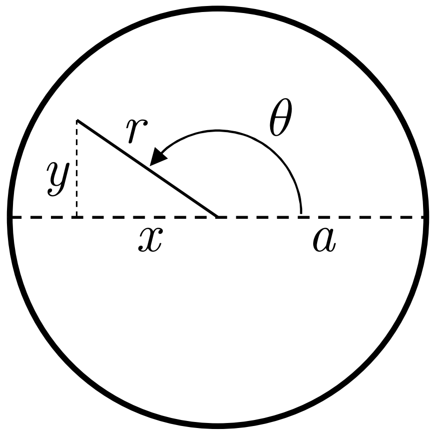
In this section, we consider the Dirichlet problem in a disk and we find a closed form of the solution, namely the *Poisson formula*. We show that this result has several important consequences, including the *mean value* property of harmonic functions (Section 6.3 in Strauss, 2008).

Dirichlet problem for a disk of radius a

Let us consider the Dirichlet problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad \text{for } x^2 + y^2 < a^2 \\ u &= h(\theta) \qquad \text{for } x^2 + y^2 = a^2 \end{aligned}$$

We solve again by separating the variables in polar coordinates: $u(r, \theta) = R(r)\,\Theta(\theta)$:



$$0 = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''.$$

Dirichlet problem for a disk of radius a

Dividing by $R\,\Theta$ and multiplying by r^2, we find that

$$\Theta'' + \lambda\Theta = 0$$

$$r^2R'' + rR' - \lambda R = 0$$

For $\Theta(\theta)$, periodic BCs are required:

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad \text{for} \quad -\infty \le \theta \le +\infty$$

Thus (with $\lambda = n^2$):

$$\Theta(\theta) = A\cos n\theta + B\sin n\theta \qquad (n = 1, 2, \ldots)$$

or $\lambda = 0$ with $\Theta(\theta) = A$.

Dirichlet problem for a disk of radius a

The equation for R (*Euler type*):

$$r^2 R'' + r R' - \lambda R = 0$$

has solutions of the form $R(r) = r^\alpha$.

Since $\lambda = n^2$, it reduces to

$$\alpha (\alpha - 1) r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0$$

Hence, $\alpha = \pm n$. Thus $R(r) = C r^n + D r^{-n}$
and we have the separated solutions:

$$u = \left(C r^n + \frac{D}{r^n} \right) (A \cos n\theta + B \sin n\theta)$$

for $n = 1, 2, 3, \ldots$

Dirichlet problem for a disk of radius a

In case $n = 0$, we also have a second linearly independent solution (besides $R = \text{constant}$):

$$R(r) = \ln r \quad \text{(obtained from simple calculus)}$$

So we also have the solutions: $u = C + D \ln r$.

Similarly to prescribing a BC at $r = 0$, we require that the considered harmonic functions are bounded.

By rejecting the obtained harmonic functions which are infinite at the origin (r^{-n} and $\ln r$), we get:

$$u = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

Dirichlet problem for a disk of radius a

Finally, we prescribe the inhomogeneous BC at $r = a$:

$$h(\theta) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

This is precisely the full Fourier series for $h(\theta)$, so that the full solution of our problem is

$$u = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

with
$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi \, d\phi$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi \, d\phi.$$

Dirichlet problem for a disk of radius a

Amazingly, this series can be summed explicitly!

Indeed, using geometric series of complex numbers, it is possible to show that the solution

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

writes in the form of *Poisson's formula*:

$$u(r,\theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$$

It expresses any harmonic function inside a circle in terms of its boundary values.

Mathematical statement of Poisson's formula

Let $h(\phi) = u(\mathbf{x}')$ be any continuous function on the circle $C = \text{bdy } D$.

Then the Poisson formula

$$u(r, \theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$$

provides the only harmonic function in D for which

$$\lim_{\mathbf{x} \to \mathbf{x}_0} u(\mathbf{x}) = h(\mathbf{x}_0) \qquad \text{for all } \mathbf{x}_0 \in C$$

Hence, $u(\mathbf{x})$ is a continuous function on $D = D \cup C$.

It is also differentiable to all orders inside D.

Poisson formula has several key consequences

Let's take a closer look at the solution at the origin (i.e., for $r = 0$):

$$u = \frac{1}{2}A_0 + \cancel{\sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)}$$

All terms corresponding to $n \geq 1$ vanish!

So, the solution at the origin writes:

$$u(0,\theta) = \frac{1}{2}A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\, d\phi = \frac{1}{2\pi a}\int_0^{2\pi} a\, h(\phi)\, d\phi$$

This is nothing but the average of h over the circle!

Mean value property

Let u be a harmonic function in a disk D, continuous in its closure (circumference).

Then the value of u at the centre of D equals the average of u on its circumference.

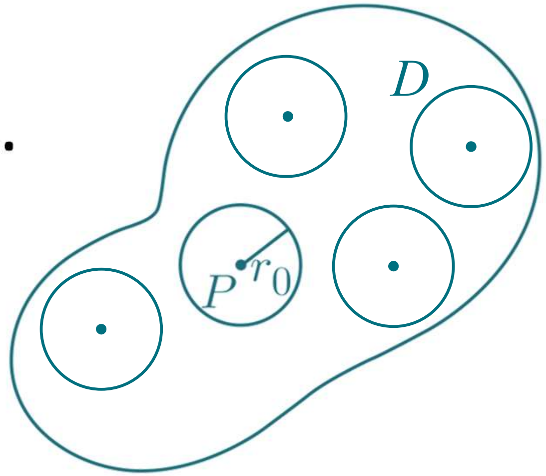
Alternate proof, based on Poisson formula

- Consider the origin $\mathbf{0}$ at the centre of the circle.
- Put $r = 0$ in Poisson's formula:

$$u(\mathbf{0}) = \frac{a^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{a^2}\, ds'$$

This is the average of u on the circumference $|\mathbf{x}'| = a$.

The mean value property holds also in a more general setting than the specific case of a disk



Suppose that we wish to solve Laplace's equation in any domain D.

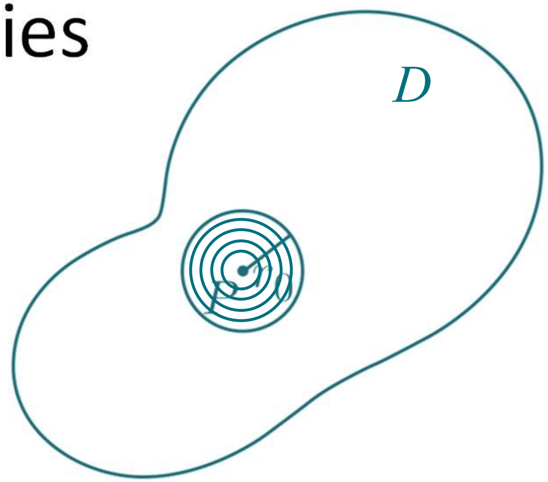
Consider any point P inside D and a circle of any radius r_0 (such that the circle remains inside D).

Let the value of u on the circle be $f(\theta)$, using polar coordinates centred at P.

Our previous analysis still holds, and thus the value of a harmonic function at any point is the average of its values along any circle centred at that point, and lying inside the domain D.

The mean value property has a straightforward corollary

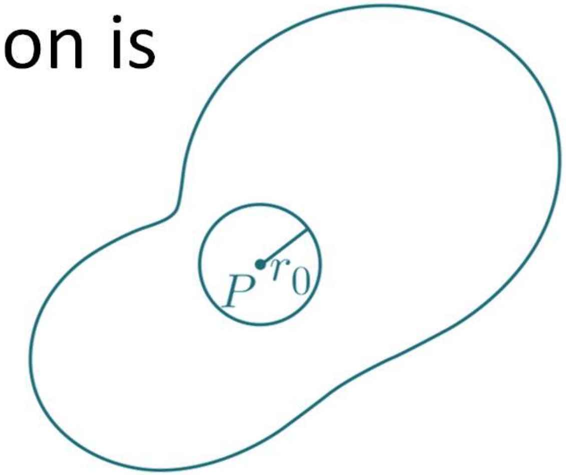
Since the mean value property applies for *all* circles centred at the considered point and lying inside the domain, it also applies to a disk centred at the considered point.



Hence,

the value of a harmonic function at a point is also equal to the **average** over any disk (or ball) with centre at that point.

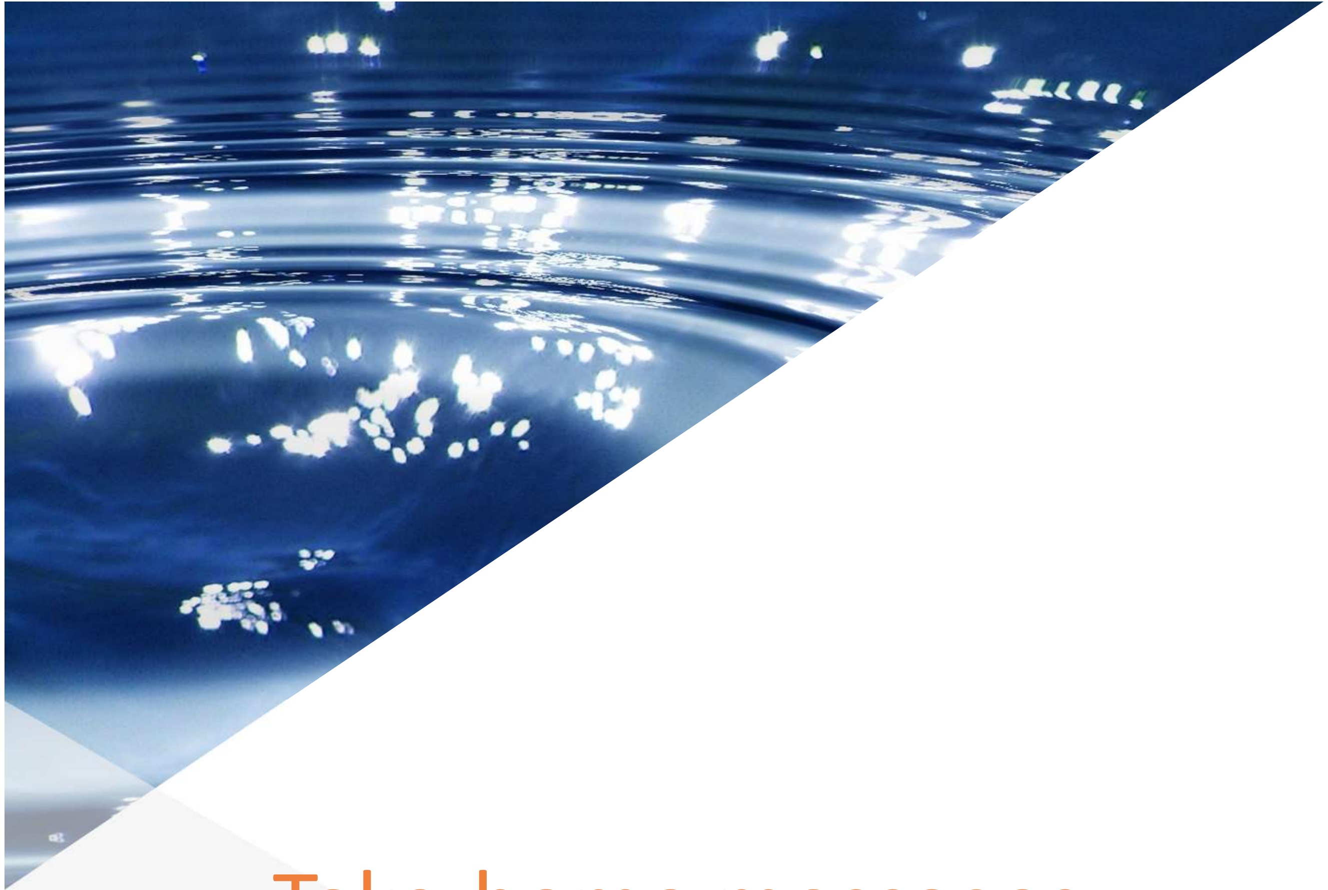
The mean value property enables deriving a proof of the strong form of the maximum principle



The maximum of a harmonic function is not in the domain; but only on the boundary, unless it is a constant.

The **proof** is by contradiction.

- Suppose that the maximum was at point P.
- However, this should be the average of all points on any circle centred on P.
- It is impossible for the function at P to be larger.
- This contradicts the original assumption, which thus cannot hold.



Take-home messages

Take-home messages

A solution of the Laplace equation is called a *harmonic* function.

The inhomogeneous version of Laplace's equation is called *Poisson's equation*.

Laplace's and Poisson's equations are of broad interest in physics and in engineering.

The maximum and the minimum values of a harmonic function u are attained on the boundary of the considered domain (unless $u \equiv \text{constant}$).

Take-home messages

We have shown the uniqueness of the solution of the Dirichlet problem (**not for Neumann problem**).

Laplace equation is invariant under all rigid motions (translations, rotations).

In engineering the Laplacian is a model used for isotropic physical situations (no preferred direction).

We have found these rotationnally invariant harmonic functions:

$\ln(x^2+y^2)^{1/2}$ (2D) and $\dfrac{1}{\mathbf{r}} = (x^2+y^2+z^2)^{-1/2}$ (3D)

Take-home messages

By separating variables, we get the solution of Laplace problems in various geometries, in the form

- of Fourier series in a rectangle (2D)
- of double Fourier series in a box (3D)

The solution of the Dirichlet problem in a circle takes a closed form, called Poisson formula.

Poisson formula has several important consequences on the properties of harmonic functions, including their “mean value property”.