Lecture 7 Boundary Problems

Mathématiques appliquées (MATH0504-1)
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Reminder

We have derived explicit solutions for the wave equation and the diffusion equation on the whole line

For numerical approximations of the same equations we have seen how *separated solutions* can be used to get insights about stability

Learning objectives of this lecture

Understand and apply the method of separation of variables for problem with Dirichlet and Neumann boundary conditions

Learn how to solve the wave and the diffusion equation in bounded domains

Outline

Wave equation with homogeneous Dirichlet boundary conditions

Diffusion equation with homogeneous Dirichlet boundary conditions

Neumann boundary conditions

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1 – Wave equation with Dirichlet conditions

Wave equation with Dirichlet conditions

Consider the wave equation on a finite interval $0 < x < l$ with homogeneous Dirichlet boundary conditions:

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l$$

$$u(0, t) = 0 = u(l, t)$$

with initial conditions:

$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x).$$

We will build the general solution as a linear combination of special ones that are easy to find.

Wave equation with Dirichlet conditions

Let us look for *separated solutions* of the problem, i.e. solutions of the form

$$u(x,t) = X(x)T(t)$$

Plugging such solutions into the wave equation we get

$$X(x)T''(t) = c^2 X''(x)T(t)$$

Dividing* by $-c^2 XT$, we get

$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda$$

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda$$

Wave equation with Dirichlet conditions

The quantity λ must be constant, as

$$\partial\lambda/\partial x = 0 \quad \text{and} \quad \partial\lambda/\partial t = 0$$

Are there other constraints on λ?

1. Is $\lambda = 0$ allowed?

   Then $X'' = 0$, and thus $X(x) = C + Dx$.

   The boundary conditions $X(0) = X(l) = 0$ lead to $C = D = 0$, i.e. $X(x) \equiv 0$.

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda$$

Wave equation with Dirichlet conditions

The quantity λ must be constant, as

$$\partial\lambda/\partial x = 0 \quad \text{and} \quad \partial\lambda/\partial t = 0$$

Are there other constraints on λ?

2. Is $\lambda < 0$ allowed?

   Let $\lambda = -\gamma^2$, so that $X'' = \gamma^2 X$.

   We thus have $X(x) = C\cosh\gamma x + D\sinh\gamma x$.

   Then $0 = X(0) = C$ and $0 = X(l) = D\sinh\gamma l$.
   Hence $D = 0$ since $\sinh\gamma l \neq 0$, i.e. $X(x) \equiv 0$.

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda$$

Wave equation with Dirichlet conditions

Set let us assume that $\lambda > 0$, i.e. $\lambda = \beta^2$.
This leads to:

$$X'' + \beta^2 X = 0$$

whose solution is of the form

$$X(x) = C\cos\beta x + D\sin\beta x$$

with C and D two arbitrary constants.

The boundary conditions $X(0) = X(l) = 0$ imply that

$$0 = X(0) = C \quad \text{and} \quad 0 = X(l) = D\sin\beta l$$

$$\lambda = \beta^2$$
$$0 = X(l) = D \sin \beta l$$

Wave equation with Dirichlet conditions

We are not interested in $C = D = 0$, so we must have $\beta l = n\pi$, the roots of the sine function.

That is:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l} \quad (n = 1, 2, 3, \ldots)$$

are distinct solutions.

(Each sine function may be multiplied by an arbitrary constant.)

Wave equation with Dirichlet conditions

- The number $\lambda_n = (n\pi/l)^2$ are called *eigenvalues*
- The functions $X_n(x) = \sin(n\pi x/l)$ are called *eigenfunctions*

Why?

They satisfy an ODE with conditions at two points:

$$-\frac{d^2}{dx^2}X = \lambda X, \qquad X(0) = X(l) = 0.$$

If we denote by A the linear operator $-\frac{d^2}{dx^2}$, the ODE writes $AX = \lambda X$.

Wave equation with Dirichlet conditions

This is analogous to the more familiar case of an $N \times N$ matrix A. A vector X that satisfies $A X = \lambda X$ with $X \not\equiv 0$ is called an eigenvector and λ is called an eigenvalue.

For an $N \times N$ matrix there are at most N eigenvalues. But for our ODE there are an *infinite* number of eigenvalues: $\pi^2/l^2,\ 4\pi^2/l^2,\ 9\pi^2/l^2, \ldots$

In physics and engineering the eigenfunctions are sometimes called *normal modes* because they are the natural shapes of solutions that persist for all time.

Wave equation with Dirichlet conditions

What about the equation in T ?

With $\lambda_n = \beta_n^2$

$$-\frac{T_n''}{c^2 T_n} = -\frac{X_n''}{X_n} = \lambda_n$$

leads to

$$T_n'' + c^2 \beta_n^2 T_n = 0$$

Thus $T_n(t) = A_n \cos \beta_n ct + B_n \sin \beta_n ct$, with A_n and B_n two arbitrary constants.

Wave equation with Dirichlet conditions

There are thus an *infinite* (!) number of separated solutions of

$$u_{tt} = c^2 u_{xx} \quad \text{for } 0 < x < l$$
$$u(0, t) = 0 = u(l, t)$$

(one for each n). They are:

$$u_n(x, t) = \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

$$n = 1, 2, 3, \ldots$$

where A_n and B_n are arbitrary constants.

Wave equation with Dirichlet conditions

The sum of solutions is also a solution, so

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

is also a solution.

This solution also satisfies the initial conditions

$$u(x,0) = \phi(x) \quad \text{and} \quad u_t(x,0) = \psi(x)$$

if

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \quad \text{and} \quad \psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}.$$

Wave equation with Dirichlet conditions

$$\phi(x) = \sum_n A_n \sin\frac{n\pi x}{l} \quad \text{and} \quad \psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin\frac{n\pi x}{l}$$

are Fourier sine series!

*Provided that the series converge**, we know the coefficients:

$$A_m = \frac{2}{l}\int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx$$

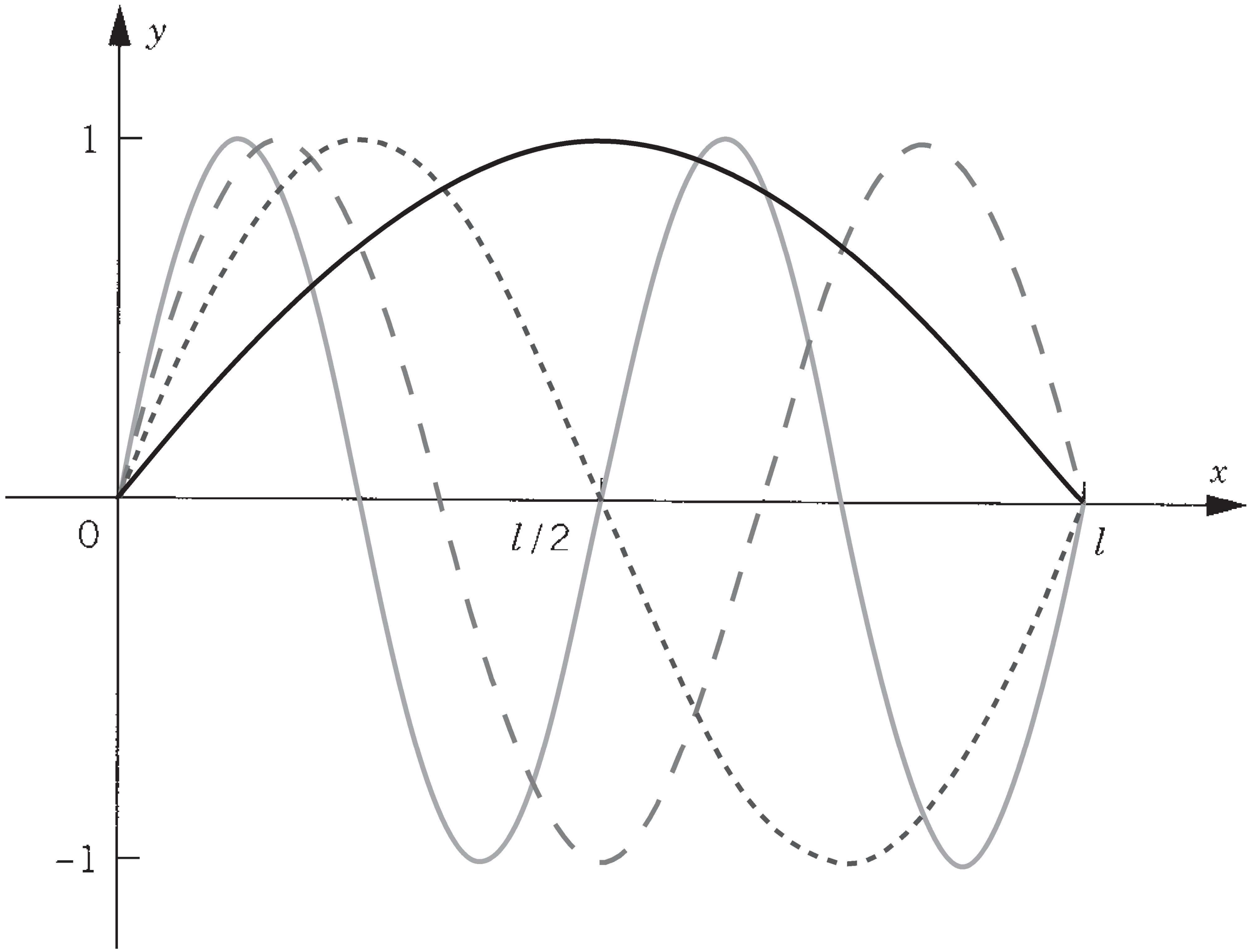
$$\frac{m\pi c}{l} B_m = \frac{2}{l}\int_0^l \psi(x)\sin\frac{m\pi x}{l}\,dx$$

And we have found an explicit solution to the wave equation with homogeneous Dirichlet conditions!

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

Wave equation with Dirichlet conditions

Sketch of first few functions $\sin(\pi\, x/l)$, $\sin(2\pi\, x/l)$, ...



$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

Wave equation with Dirichlet conditions

The functions $\cos(n\pi\, ct/l)$ and $\sin(n\pi\, ct/l)$ which describe the behavior in *time* have a similar form.

The coefficients of t inside the sines and cosines, namely $n\pi\, c/l$, are called the (angular) *frequencies*.

For the vibrating string, remember that $c = \sqrt{T/\rho}$: the frequencies are thus

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad \text{for } n = 1, 2, 3, \ldots$$

The "fundamental" note of the string is the smallest of these, $\pi\sqrt{T}/(l\sqrt{\rho})$. The "overtones" are *exactly* the double, the triple, and so on, of the fundamental!

*Reminder on the convergence of series

1. An infinite series $\sum_{n=1}^{\infty} f_n(x)$ *converges* to $f(x)$ *pointwise* in (a, b) if it converges to $f(x)$ for *each* $a < x < b$. That is, for each $a < x < b$ we have

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as } N \to \infty$$

2. The series *converges in the mean-square (or L2) sense* to $f(x)$ in (a, b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \quad \text{as } N \to \infty$$

*Reminder on the convergence of classical Fourier series (sine, cosine or full)

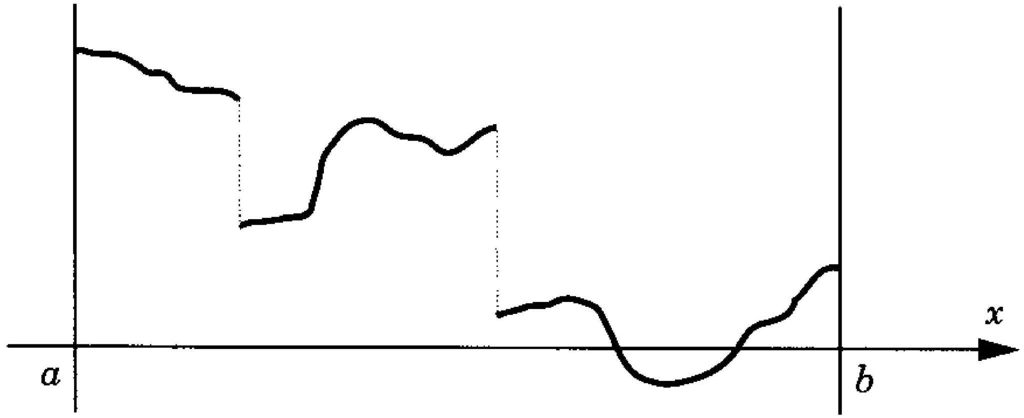
1. The Fourier series converges to $f(x)$ in the mean-square *(or L2)* sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2 \, dx \text{ is finite}$$

2. The Fourier series converges to $f(x)$ pointwise on (a, b) provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

*Reminder on the convergence of classical Fourier series (sine, cosine or full)

3. More generally, if $f(x)$ itself is only piecewise continuous on $a \leq x \leq b$ and $f'(x)$ is also piecewise continuous on $a \leq x \leq b$, then the Fourier series converges at every point x. At a jump discontinuity the series converges to the *average* of the limits from the right and from the left.



2 – Diffusion equation with Dirichlet conditions

Diffusion equation with Dirichlet conditions

The analogous problem for diffusion is

$$u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty)$$
$$u(0, t) = u(l, t) = 0$$
$$u(x, 0) = \phi(x).$$

We separate the variables $u = T(t)\, X(x)$ as before, which leads to:

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{constant}$$

Diffusion equation with Dirichlet conditions

For $X(x)$ we get

$$-X'' = \lambda X \quad \text{in } 0 < x < l \quad \text{with} \quad X(0) = X(l) = 0$$

which is exactly the same as for the wave equation.

For $T(t)$ we get

$$T' = -\lambda k T$$

whose solution is $T(t) = Ae^{-\lambda k t}$.

Diffusion equation with Dirichlet conditions

We thus obtain the solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l}$$

which is solution of the stated diffusion problem provided that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

The solution is thus expressible again for each t as a Fourier sine series in x provided that the initial data are.

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3 – Neumann conditions

Diffusion equation with Neumann conditions

Consider

$$u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty)$$
$$u_x(0, t) = u_x(l, t) = 0$$
$$u(x, 0) = \phi(x).$$

The eigenfunctions are the solution $X(x)$ of

$$-X'' = \lambda X, \qquad X'(0) = X'(l) = 0,$$

other than the trivial solution $X(x) \equiv 0$.

Diffusion equation with Neumann conditions

1. Let's first search for the positive eigenvalues

$$\lambda = \beta^2 > 0$$

As before, $X(x) = C\cos\beta x + D\sin\beta x$, so that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$$

Neumann B.C. on the left: $0 = X'(0) = D\beta$

$\Rightarrow D = 0$

Neumann B.C. on the right: $0 = X'(l) = -C\beta\sin\beta l$

$\Rightarrow \sin\beta l = 0$ since we don't want $C = 0$

Diffusion equation with Neumann conditions

Thus $\beta = \pi/l,\ 2\pi/l,\ 3\pi/l, \ldots$ and we have

- Eigenvalues: $\left(\frac{\pi}{l}\right)^2, \left(\frac{2\pi}{l}\right)^2, \ldots$
- Eigenfunctions: $X_n(x) = \cos\frac{n\pi x}{l} \quad (n = 1, 2, \ldots)$

2. Let's now check if zero is an eigenvalue

We get $X'' = 0$, so that $X(x) = C + Dx$, $X'(x) \equiv D$.

The Neumann B.C.s are satisfied if $D = 0$: C can be any number.

Therefore, $\lambda = 0$ *is an eigenvalue*, and any constant function is its eigenfunction!

Diffusion equation with Neumann conditions

3. If $\lambda < 0$ you can verify that, as in the Dirichlet case, there is no eigenfunction

So the solution of the diffusion with homogeneous Neumann B.C.s is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l}$$

where the term corresponding to the zero eigenvalue is written separately, in the form $\frac{1}{2}A_0$ for convenience.

The solution requires the initial data to have the Fourier cosine expansion

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

Wave equation with Neumann conditions

We could solve the wave equation with Neumann conditions in exactly the same way.

The zero eigenvalue leads to:

- $X(x)$ being constant
- and $T''(t) = \lambda c^2 T(t) = 0$

which has the solution $T(t) = A + Bt$.

Therefore, the complete solution writes

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}$$

Wave equation with Neumann conditions

The initial data must satisfy:

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}$$

Check the details as an exercise!

Take-home messages

The method of separation of variables allowed us to find explicit solutions of the wave equation and the diffusion equation in bounded domains, with homogeneous Dirichlet or Neumann boundary conditions.

The solution involves Fourier series, in terms of (an infinite number of) eigenfunctions (or "normal modes").

Robin boundary conditions are a bit more demanding: you will examine them in exercises.

Next lecture: Laplace equation

Important properties

- maximum principle
- rotational invariance
- mean value

Solution in series form for simple geometrical domains (rectangle, disk)