Lecture 4  The Diffusion Equation

Mathématiques appliquées (MATH0504-1)
B. Dewals, C. Geuzaine
The Diffusion Equation

In this lecture we begin the study of the diffusion equation

$$u_t = ku_{xx}$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the PDEs.

Since solving the diffusion equation is harder than solving the wave equation, we start with a study of general properties of diffusions.
Learning objectives of this lecture

Learn general properties of the diffusion equation
Understand and prove the “maximum principle”
Prove the uniqueness and the stability of a diffusion problem
Outline

Maximum principle

Uniqueness for the Dirichlet problem for the diffusion equation

Stability of the Dirichlet problem
1 – Maximum Principle
Maximum Principle

If $u(x, t)$ satisfies the diffusion equation $u_t = ku_{xx}$ in a rectangle $R \ (0 \leq x \leq l, \ 0 \leq t \leq T)$ in space-time, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$).

![Diagram of a rectangle in space-time with axes labeled $x$, $t$, and $t = 0$, $x = 0$, $x = l$.]
Maximum Principle: interpretation

- If you have a rod with no internal heat source, the hottest spot and the coldest spot can occur only initially or at one of the two ends of the rod
  - Thus a hot spot at time zero will cool off (unless heat is fed into the rod at an end)
  - You can burn one of its ends but the maximum temperature will always be at the hot end
- Similarly, if you have a substance diffusing along a tube, its highest concentration can occur only initially or at one of the ends of the tube
Maximum Principle: interpretation

If we draw a “movie” of the solution, the maximum drops down while the minimum comes up.
So the diffusion equation tends to *smooth* the solution out.
This is very different from the behavior of the wave equation!
Maximum Principle: idea for a proof

If the maximum is at an interior point, we know that at this point we have $u_t = 0$, and $u_{xx} \leq 0$.

First idea:

- If we could assume that $u_{xx} \neq 0$ at the maximum, then we would have $u_t \neq ku_{xx}$, which would contradict the diffusion equation; which would mean that the maximum must be on the boundary.

- Unfortunately we can have $u_{xx} = 0$ at the maximum... So we need a slightly more elaborate strategy to prove the maximum principle.
Maximum Principle: proof

Let $M$ denote the maximum value of $u(x, t)$ on the three sides $t = 0$, $x = 0$, and $x = l$.

We must show that $u(x, t) \leq M$ throughout the rectangle $R$.

Let $\epsilon$ be a positive constant ($\epsilon > 0$) and let

$$v(x, t) = u(x, t) + \epsilon x^2$$

Our goal is to show that $v(x, t) \leq M + \epsilon l^2$ throughout $R$.

(Indeed, then $u(x, t) \leq M + \epsilon (l^2 - x^2)$ for any positive $\epsilon$, which proves $u(x, t) \leq M$ throughout $R$)
Maximum Principle: proof

It is clear that \( v(x, t) \leq M + \epsilon l^2 \) on \( t = 0, x = 0 \) and \( x = l \).

What about in the interior and on the top side?

The function \( v(x, t) = u(x, t) + \epsilon x^2 \) satisfies the “diffusion inequality”

\[
v_t - kv_{xx} = u_t - k(u + \epsilon x^2)_{xx}
\]

\[
= u_t - ku_{xx} - 2\epsilon k
\]

\[
= -2\epsilon k
\]

\[
< 0
\]
Maximum Principle: proof

Let’s suppose that \( v(x, t) \) attains a maximum at an interior point \((x_0, t_0)\). Then \( v_t = 0 \) and \( v_{xx} \leq 0 \) at the point \((x_0, t_0)\): this violates the diffusion inequality.

Let’s suppose that \( v(x, t) \) attains a maximum on the top edge. Then

\[
v_t(x_0, t_0) = \lim_{\delta \to 0} \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \geq 0 \]

and \( v_{xx} \leq 0 \): this again violates the diffusion inequality.

Hence the maximum has to be on the bottom or the sides.
Minimum Principle

The minimum value has the same property: it too can be attained only on the bottom or the lateral sides.

To prove the minimum principle, just apply the maximum principle to $-u(x, t)$!
2 – Uniqueness
Uniqueness

The maximum principle can be used to give a proof of *uniqueness for the Dirichlet problem for the diffusion equation.*

That is, there is at most one solution of

\[ u_t - ku_{xx} = f(x, t) \quad \text{for } 0 < x < l \text{ and } t > 0 \]
\[ u(x, 0) = \phi(x) \]
\[ u(0, t) = g(t) \quad u(l, t) = h(t) \]

for the given functions \( f, \phi, g, \) and \( h \)
Uniqueness: proof

Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of the Dirichlet problem.

Let $w = u_1 - u_2$ be their difference.

Then $w_t - kw_{xx} = 0$, $w(x, 0) = w(0, t) = w(l, t) = 0$.

By the maximum principle, $w(x, t)$ has its maximum for the rectangle on its bottom or sides—exactly where it vanishes. So $w(x, t) \leq 0$.

The same type of argument for the minimum shows that $w(x, t) \geq 0$. Therefore, $w(x, t) \equiv 0$, so that $u_1(x, t) \equiv u_2(x, t)$ for all $t \geq 0$. 
Uniqueness: alternative proof ("energy method")

Multiplying the equation for \( w = u_1 - u_2 \) by \( w \) itself, we can write:

\[
0 = 0 \cdot w = (w_t - kw_{xx})(w) = \left( \frac{1}{2}w^2 \right)_t + (-kw_x w)_x + kw_x^2
\]

Integrating over \( 0 < x < l \), we get

\[
0 = \int_0^l \left( \frac{1}{2}w^2 \right)_t \, dx - kw_x w \bigg|_{x=0}^{x=l} + k \int_0^l w_x^2 \, dx
\]

Since \( w = 0 \) at \( x = 0 \) and at \( x = l \), we get

\[
\frac{d}{dt} \int_0^l \frac{1}{2} [w(x, t)]^2 \, dx = -k \int_0^l [w_x(x, t)]^2 \, dx \leq 0
\]
Uniqueness: alternative proof ("energy method")

This means that \( \int w^2 \, dx \) is decreasing in time, and thus:

\[
\int_0^l [w(x, t)]^2 \, dx \leq \int_0^l [w(x, 0)]^2 \, dx \quad \text{for } t \geq 0.
\]

The right-hand side vanishes since the initial conditions on \( u \) and \( v \) are identical, and thus \( w(x, 0) = u(x, 0) - v(x, 0) = 0 \).

Hence \( \int [w(x, t)]^2 \, dx = 0 \) for \( t > 0 \).

Thus \( w(x, t) \equiv 0 \), so that \( u_1(x, t) \equiv u_2(x, t) \) for all \( t \geq 0 \).
3 – Stability
Consider two solutions $u_1$ and $u_2$ of the Dirichlet diffusion problem with initial data $\phi_1$ and $\phi_2$.

We have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle $R$ and $w = \phi_1 - \phi_2$ on the bottom.

The maximum principle asserts that throughout the rectangle

$$u_1(x, t) - u_2(x, t) \leq \max |\phi_1 - \phi_2|$$

The “minimum” principle says that

$$u_1(x, t) - u_2(x, t) \geq -\max |\phi_1 - \phi_2|$$
Stability: using the maximum principle

Therefore:

\[
\max_{0 \leq x \leq l} |u_1(x, t) - u_2(x, t)| \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|
\]

valid for all \( t > 0 \).

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at \( t = 0 \)), we stay nearby.

This is a measure of stability; called stability in the “uniform” sense.
Stability: using the energy method

Consider again two solutions $u_1$ and $u_2$ of the Dirichlet problem with initial data $\phi_1$ and $\phi_2$, and with $f = g = h = 0$.

Then $w \equiv u_1 - u_2$ is the solution with initial data $\phi_1 - \phi_2$ and the inequality from slide 18

$$\int_0^l [w(x, t)]^2 \, dx \leq \int_0^l [w(x, 0)]^2 \, dx$$

becomes

$$\int_0^l [u_1(x, t) - u_2(x, t)]^2 \, dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2 \, dx$$
Stability: using the energy method

This again measures on the right the nearness of the initial data for two solutions, and on the left the nearness of the solutions at any later time.

This is another measure of stability; called stability in the “square integral” (or energy) sense.
Take-home messages

• The maximum (and minimum) principle provide deep insights into the qualitative behavior of the solution of the diffusion equation
  • Diffusion tends to *smooth* the solution out

• We can prove uniqueness and stability using the this maximum principle

• We can also prove uniqueness and stability using an “energy” method
What’s next?

After the break we will learn how to discretize the diffusion equation in a stable manner.

Next we will derive a formula for the solution of the diffusion equation on the whole line. This will introduce the concept of fundamental solutions (or Green’s functions).