LIÈGE université
Sciences Appliquées

Lecture 3 The wave equation

Mathématiques appliquées (MATH0504-1)

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Learning objectives of this lecture

Understand the fundamental properties of the wave equation

Write the general solution of the wave equation

Solve initial value problems with the wave equation

Understand the concepts of causality, domain of influence, and domain of dependence in relation with the wave equation

Become aware that the wave equation ensures conservation of energy

Outline

1. Reminder: physical significance and derivation of the wave equation, basic properties
2. General solution of the wave equation
3. Initial value problem
4. Causality
5. Energy
6. Generalized wave equation

1 - Reminders

Reminder

The 1D wave equation describes the small displacements of a flexible, elastic, homogenous string (e.g. guitar string or violin string), which undergoes transverse vibrations (in a plane).

The displacement from equilibrium position at time t and position x is noted $u(x, t)$.

Reminder

Using Newtown's law and these assumptions,

- the string is perfectly flexible, so that the tension (force) $\mathbf{T}(x, t)$ is directed tangentially along the string,
- the density ρ of the string (mass per unit length) is a constant because the string is homogeneous,
- purely transverse motion, no longitudinal motion,

leads to

T is independent of t a well as x,

$u_{tt} = c^2 u_{xx}$ where $c = \sqrt{\frac{T}{\rho}}$ is the *wave speed*.

Reminder

The 1D wave equation, or a variation of it, describes also other wavelike phenomena, such as

- vibrations of an elastic bar,
- sound waves in a pipe,
- long water waves in a straight channel,
- the electrical current in a transmission line …

The 2D and 3D versions of the equation describe:

- vibrations of a membrane / of an elastic solid,
- sound waves in air,
- electromagnetic waves (light, radar, etc.),
- seismic waves propagating through the earth …

For the sake of simplification, we consider here an infinite domain: $-\infty < x < +\infty$

Real physical situations are *often* on finite intervals.

However, we do not consider boundaries here, for two reasons:

- from a mathematical perspective, the absence of a boundary is a **big simplification**, which does not prevent shedding light on most of the fundamental properties of PDEs;
- from a physical perspective, far away from the boundary, it will take a certain time for the boundary to have a substantial effect on the process, and until that time the solutions derived here are valid.

Basic properties of the wave equation

The wave equation (WE) writes:

$$u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty.$$

where the following notation is used for the derivatives: $\partial u/\partial x = u_x$ ...

The WE has the following basic properties:

- it has two independent variables, x and t, and one dependent variable u (i.e. u is an unknown function of x and t);
- it is a *second*-order PDE, since the highest derivative in the equation is second order;
- it is a homogeneous linear PDE.

The wave equation is a hyperbolic PDE

Comparing the wave equation

$$u_{tt} = c^2 u_{xx}$$

to the general formulation

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0$$

reveals that

$$a_{12}^2 > a_{11}a_{22}$$

since $a_{12} = 0$, $a_{11} = -c^2$ and $a_{22} = 1$.

Hence, the wave equation is *hyperbolic*.

2 – Solution of the wave equation

In this section, we use two different approaches to derive the general solution of the wave equation (Section 2.1 in Strauss, 2008).

1^{st} approach
The operator in the wave equation factors

The wave equation

$$u_{tt} = c^2 u_{xx}$$

may be written as:

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\boxed{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u} = 0.$$

This is equivalent to two 1^{st} order PDEs:

$$u_t + cu_x = v$$

$$v_t - cv_x = 0$$

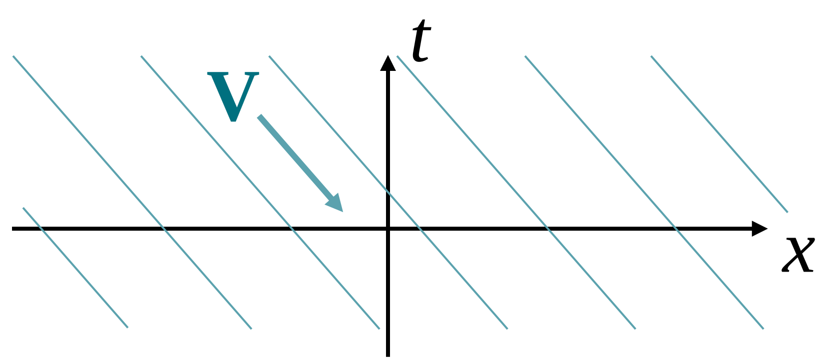
1^{st} approach
We solve each of the two 1^{st} order PDEs

As shown in Lecture 1 (Sect. 1.2), the general solution of $v_t - cv_x = 0$ is given by:

$$v(x,t) = h(x+ct)$$

where h is any function.

Indeed, $v_t - cv_x = 0$ expresses that the *directional derivative* of *v* along the direction $\mathbf{V} = (-c, 1)$ is zero.



1^{st} approach
We solve each of the two 1^{st} order PDEs

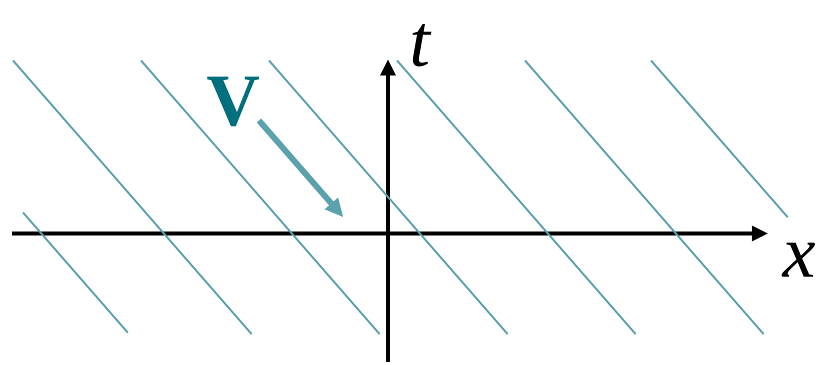
The lines parallel to $\mathbf{V} = (-c, 1)$ have the equations

$$x + c\,t = \text{constant.}$$

These lines are the *characteristic lines*.

Since the function v must remain constant on each such line, v depends only on $x + c\,t$:

$$v(x,t) = h(x + ct)$$



1^{st} approach
We solve each of the two 1^{st} order PDEs

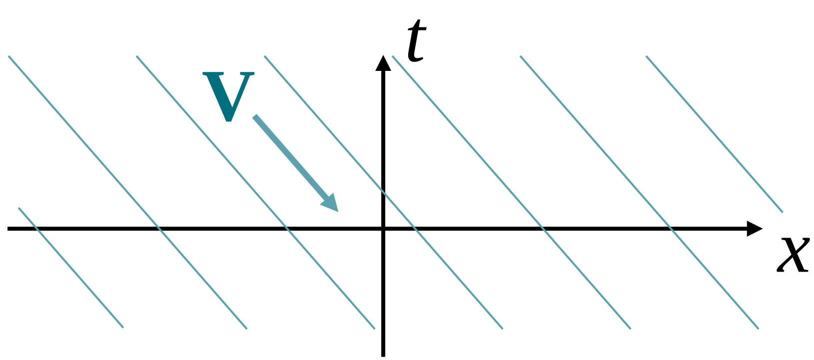
Now, the second 1^{st} order equation takes the form

$$u_t + cu_x = h(x + ct)$$

It is easy to check directly by differentiation that <u>one</u> solution is:

$$u(x, t) = f(x + ct)\text{, where } f'(s) = h(s)/2c$$

[A prime ($'$) denotes the derivative of a function of one variable]



1^{st} approach
We solve each of the two 1^{st} order PDEs

To the particular solution $f(x + c\,t)$ of equation

$$u_t + cu_x = h(x + ct)$$

we can add the solution $g\,(x - c\,t)$ of the homogeneous equation

$$u_t + cu_x = 0$$

to get another solution (since the equation is linear).

Therefore, the most general solution is expressed as a particular solution plus any solution of the homogeneous equation:

$$u(x, t) = f\,(x + c\,t) + g(x - c\,t).$$

2^{nd} approach
Introduce the characteristic coordinates

Consider the following change of coordinates:

$$\xi = x + ct \qquad \eta = x - ct$$

By the chain rule, one obtains:

$$\partial_x = \partial_\xi + \partial_\eta \text{ and } \partial_t = c\partial_\xi - c\partial_\eta$$

Therefore,

$$\partial_t - c\partial_x = -2c\partial_\eta \text{ and } \partial_t + c\partial_x = 2c\partial_\xi$$

So, the wave equation takes the form:

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (-2c\partial_\eta)(2c\partial_\xi)u = 0$$

2nd approach
Introduce the characteristic coordinates

Since $c \neq 0$,

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (-2c\partial_\eta)(2c\partial_\xi)u = 0$$

is equivalent to:

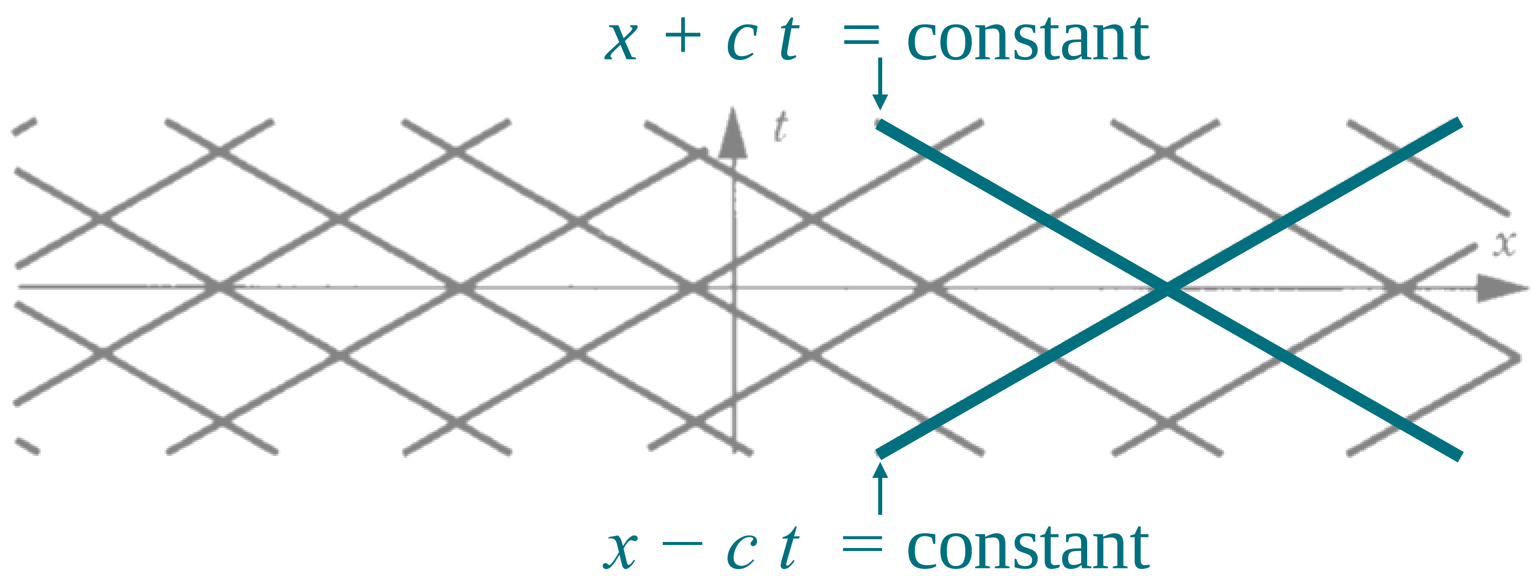
$$u_{\xi\eta} = 0$$

The solution of this transformed equation is

$$u = f(\xi) + g(\eta)$$

which agrees exactly with the result obtained from the 1st approach.

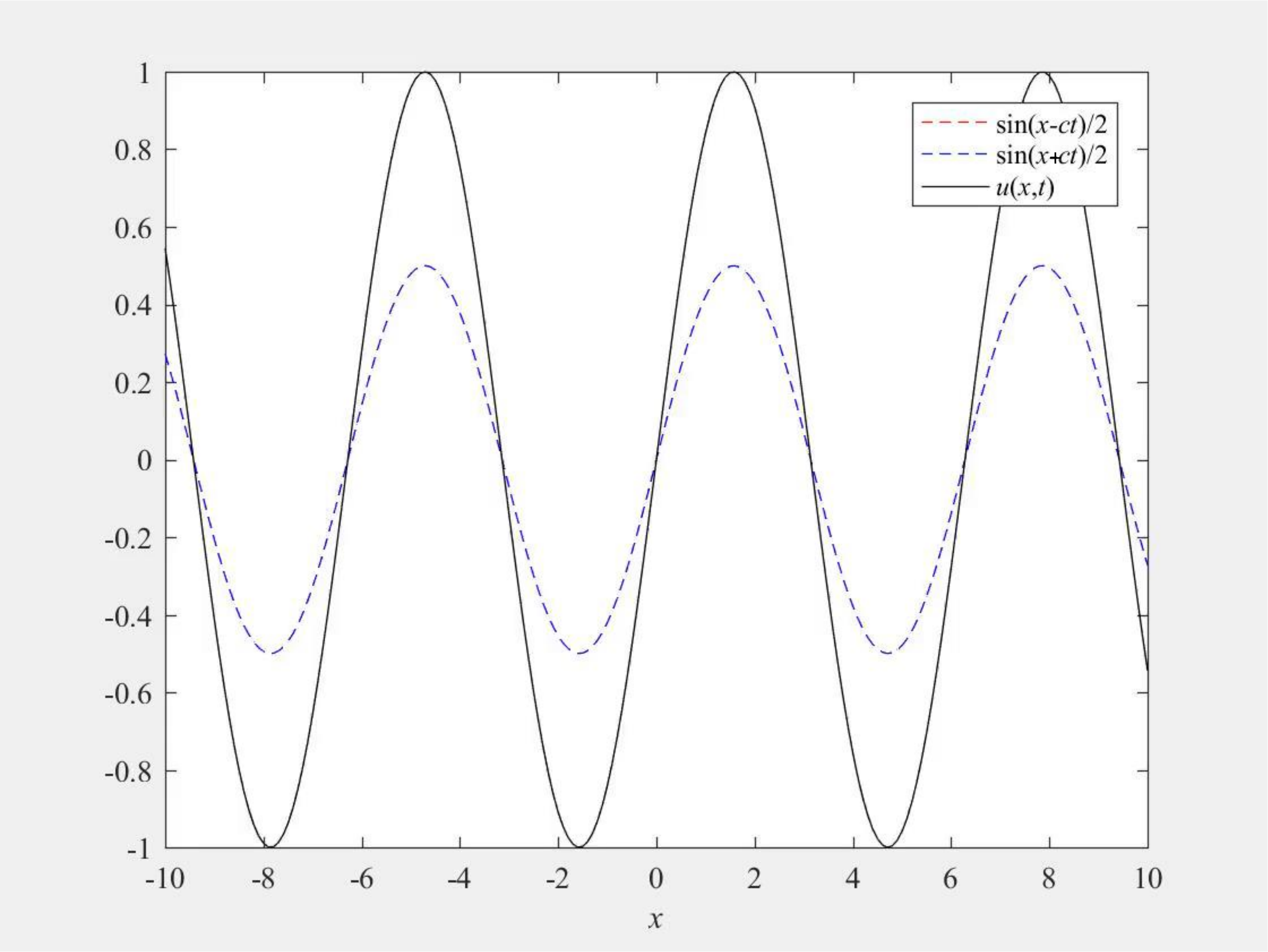
The wave equation has two families of characteristic lines: $x \pm c\,t = \text{constant}$



The most general solution of the *wave equation* is the sum of two functions, i.e. two waves of arbitrary shape each:

- $g\,(x - c\,t)$, traveling to the right at speed c;
- $f\,(x + c\,t)$, traveling to the left at speed c.

Here, we anticipate the result of a numeric example detailed later on ...



This is how propagation of information at a finite speed looks like in two dimensions …

3 – Initial value problem

In this section, we solve the initial value problem and present a few worked out examples (Section 2.1 in Strauss, 2008)

The initial-value problem, i.e. the wave equation and its initial conditions, has one and only one solution

The initial-value problem (IVP) consists in solving the wave equation

$$u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty$$

with the initial conditions (IC):

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x),$$

where ϕ and ψ are arbitrary functions of x.

This problem has one, and only one, solution, as we show hereafter.

The solution of the IVP is easily found from the general formula $u(x, t) = f(x + ct) + g(x - ct)$

Setting $t = 0$ in $u(x, t) = f(x + ct) + g(x - ct)$,

we get:

$$\phi(x) = f(x) + g(x)$$

Using the chain rule, we differentiate

$u\,(x, t) = f\,(x + ct) + g\,(x - ct)$ with respect to t

and set $t = 0$:

$$\psi(x) = cf'(x) - cg'(x)$$

The solution of the IVP is easily found from the general formula $u(x, t) = f(x + ct) + g(x - ct)$

By differentiating $\phi(x) = f(x) + g(x)$,
one obtains:

$$\phi' = f' + g'$$

Combining with

$$\frac{1}{c}\psi = f' - g'$$

gives us:

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \qquad \text{and} \qquad g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right)$$

The solution of the IVP is easily found from the general formula $u(x, t) = f(x + ct) + g(x - ct)$

Integrating, we get:

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi \, ds + A$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi \, ds + B$$

where A and B are constants.

Since $\phi(x) = f(x) + g(x)$, we have $A + B = 0$.

The solution of the IVP is easily found from the general formula $u(x, t) = f(x + ct) + g(x - ct)$

Substituting

- $s = x + c\,t$ into the formula for f
- and $s = x - c\,t$ into that of g,

we get:

$$u(x,t)=\frac{1}{2}\phi(x+ct)+\frac{1}{2c}\int_0^{x+ct}\psi\,ds+\frac{1}{2}\phi(x-ct)-\frac{1}{2c}\int_0^{x-ct}\psi\,ds$$

This simplifies to:

$$u(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds.$$

A first worked out example

Considering $\phi(x) = \sin x$ and $\psi(x) = 0$,
one obtains from the general solution:

$$u(x, t) = [\ \sin(x + c\,t) + \sin(x - c\,t)\ ] / 2$$

Hence,

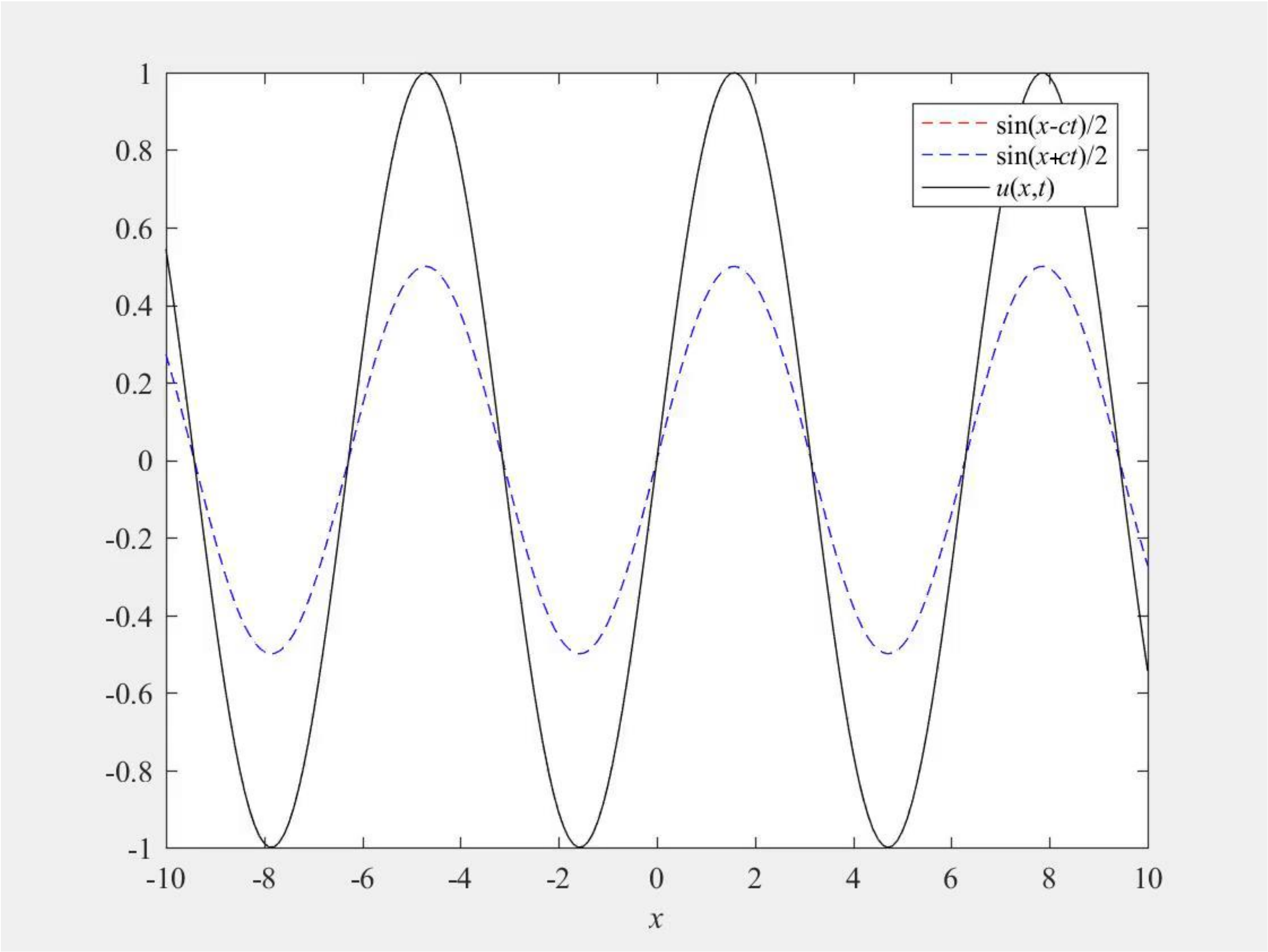
$$u(x, t) = \sin x \cos(c\,t).$$

This can be checked easily by substituting the expression found for $u(x, t)$ into the wave equation.

$$\boxed{\begin{aligned}\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b)\\ \sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b)\end{aligned}}$$

A first worked out example

$$u(x, t) = [\ \sin(x + c\,t) + \sin(x - c\,t)\ ] / 2$$



Another example

Let us consider now $\phi(x) = 0$ and $\psi(x) = \cos x$.

The solution writes:

$$u(x, t) = [\ \sin(x + c\,t) - \sin(x - c\,t)\ ] / (2\,c)$$

Hence,

$$u(x, t) = \cos x \sin(c\,t) / c\ .$$

Again, this can be checked easily by substituting the result into the wave equation and the IC.

$$\boxed{\begin{aligned} \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\ \sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b) \end{aligned}}$$

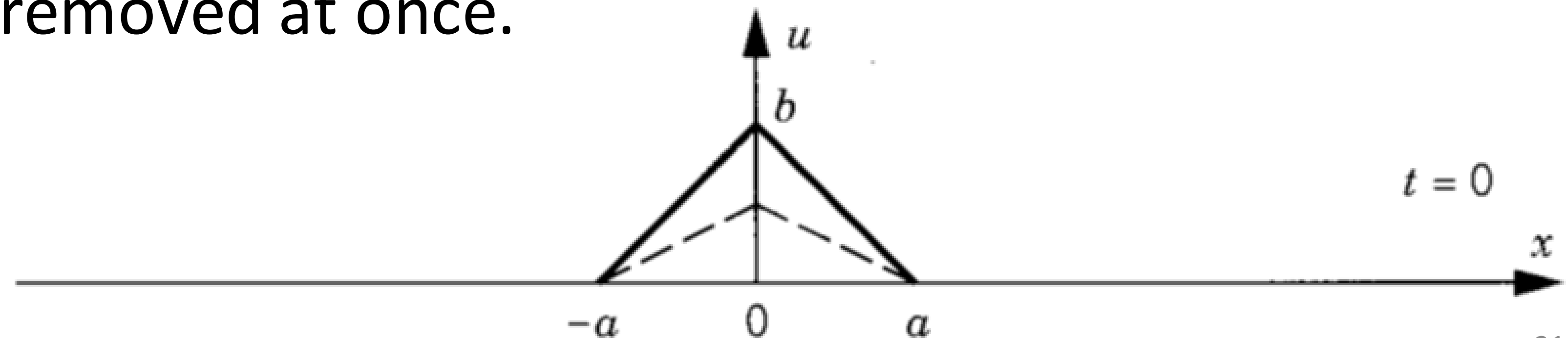
The plucked string

Consider an infinitely long string with initial position:

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

and initial velocity $\psi(x) = 0$ for all x.

This is a “three-finger” pluck, with all three fingers removed at once.



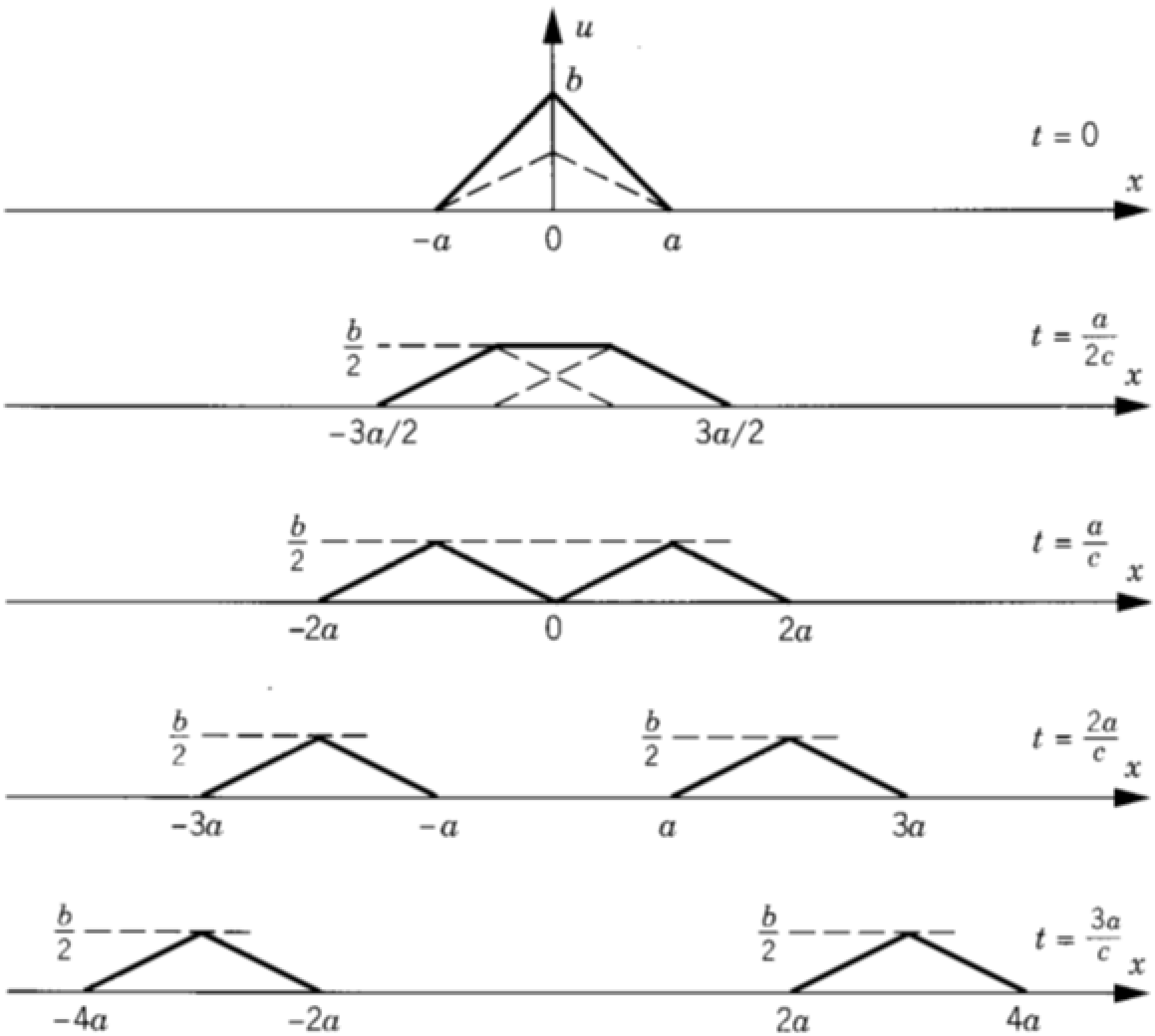
The plucked string

The solution

$$u(x, t) = f(x + c\,t) + g(x - c\,t)$$

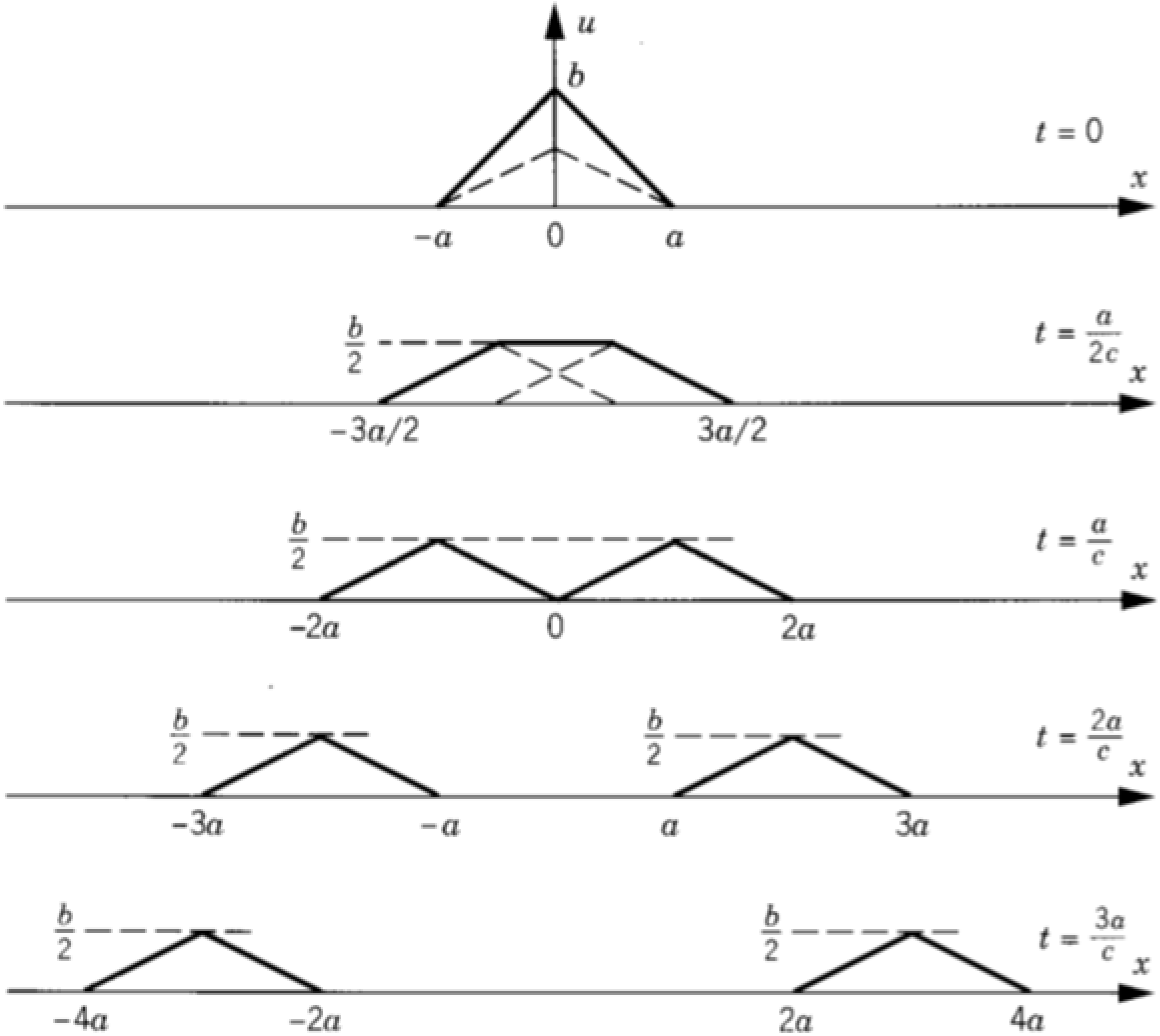
is the sum of two triangle functions,

- one moving to the right
- and one to the left.



The plucked string

This solution is not twice differentiable; but it is a "weak" solution, as discussed later (Section 12.1).

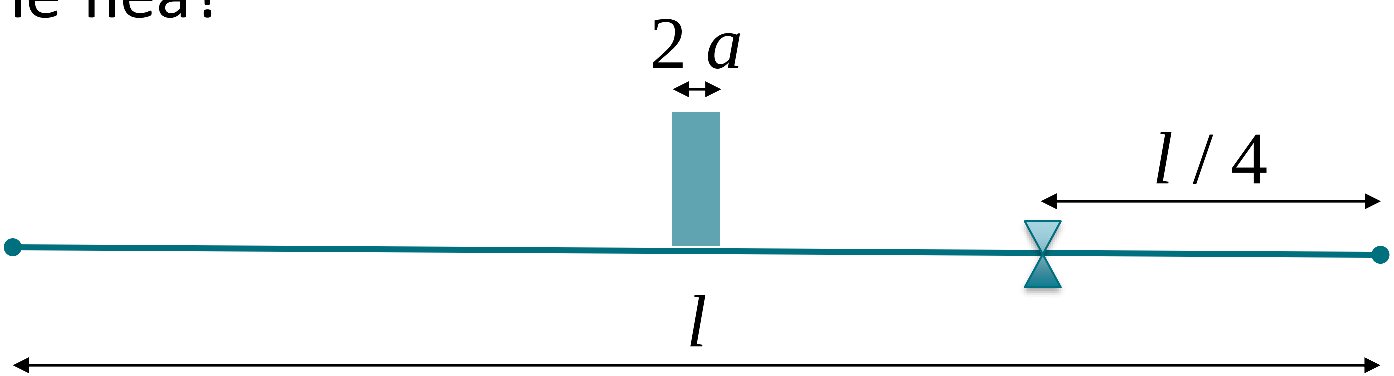


Poor flea ...

The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2\,a$.

A flea is sitting at a distance $l\,/\,4$ from one end. (Assume that $a < l\,/\,4$; otherwise, poor flea!)

How long does it take for the disturbance to reach the flea?



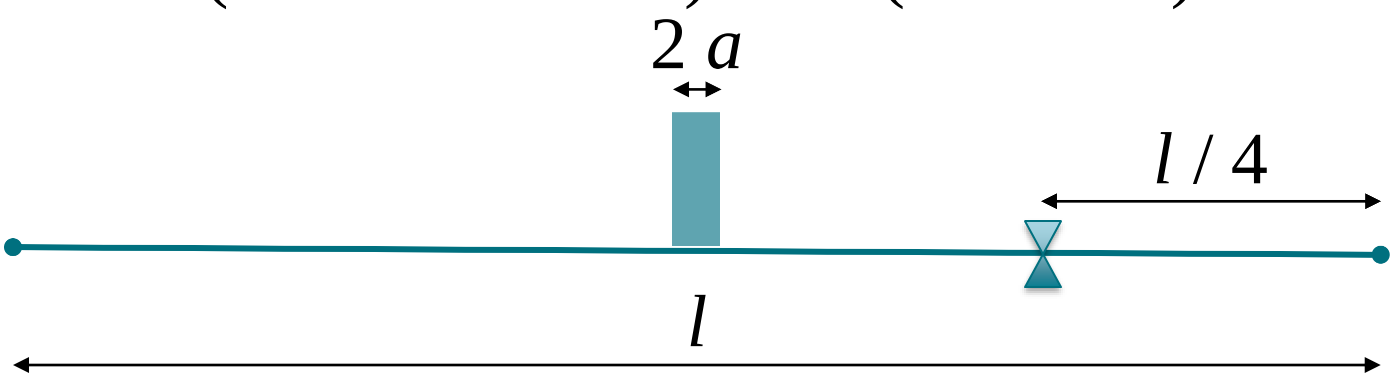
Poor flea …

The wave celerity c is given by:

$$c = \sqrt{\frac{T}{\rho}}$$

Hence, the travelling time from the edge of the hammer to the flea is:

$$(l / 2 - a - l / 4) / c = (l / 4 - a) / c$$



4 – Causality in the wave equation

In this section, we introduce the concepts of zones of *influence* and of *dependence* (Section 2.2 in Strauss, 2008)

Principle of causality:
no part of the waves goes faster than speed c

We have just learned that

- the effect of an initial position $\phi(x)$ is a pair of waves traveling in either direction at speed c and at half the original amplitude;
- the effect of an initial velocity $\psi(x)$ is a wave spreading out at speed $\leq c$ in both directions.

So, part of the wave may lag behind (if there is an initial velocity), but

no part goes faster than speed c.

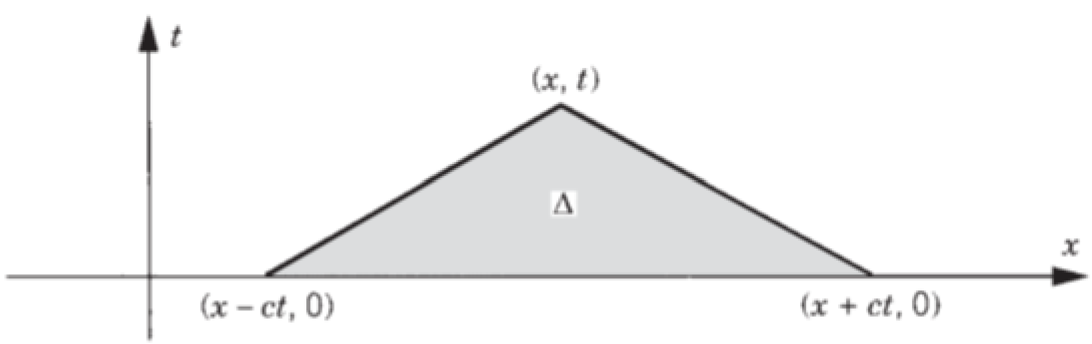
This is the principle of causality.

$u(x, t)$ depends only on the IC within the interval $(x - ct, x + ct)$, called *domain of dependence*

The value of $u(x, t)$ at any point (x, t) depends only on the values of

- ϕ at the two points $x \pm ct$,
- and ψ within the interval $[x - ct, x + ct]$.

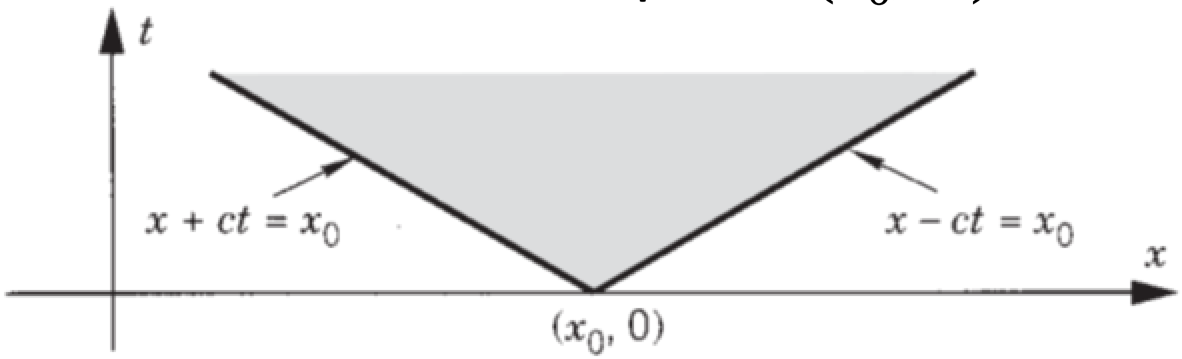
This interval is called the domain of dependence of the point (x, t) on $t = 0$. It is bounded by the pair of characteristic lines that pass through (x, t).



Vice versa, an IC at a given point affects the solution only in the domain of influence of the point

Here is an "inverse" way to express causality.

An initial condition (position or velocity or both) at the point $(x_0, 0)$ can affect the solution for $t > 0$ only in the shaded sector, which is called the domain of influence of the point $(x_0, 0)$.



Similarly, if ϕ and ψ vanish for $|x| > R$, then $u(x, t) = 0$ for $|x| > R + ct$: the domain of influence of an interval ($|x| \leq R$) is a sector ($|x| \leq R + ct$).

5 – Energy in the wave equation

In this section, we demonstrate that the wave equation ensures conservation of energy (Section 2.2 in Strauss, 2008)

The wave equation ensures conservation of energy

Consider an infinite string with constants ρ and T.

The transverse displacement $u(x,t)$ is governed by:

$$\rho\, u_{tt} = T\, u_{xx} \quad \text{for} \quad -\infty < x < +\infty.$$

The kinetic energy K is given by:

$$K = \frac{1}{2}\rho \int_{-\infty}^{+\infty} u_t^2 \mathrm{d}x$$

To ensure integral convergence, we assume that $\phi(x)$ and $\psi(x)$ vanish outside an interval $\{|x| \leq R\}$.

Consequently, as mentioned above, $u(x, t)$ [and therefore $u_t(x, t)$] vanish for $|x| > R + ct$.

The wave equation ensures conservation of energy

Differentiating the kinetic energy, we can pass the derivative under the integral

$$\frac{dK}{dt} = \frac{1}{2}\rho\frac{d}{dt}\left(\int_{-\infty}^{+\infty} u_t^2 \mathrm{d}x\right) = \rho\int_{-\infty}^{+\infty} u_t u_{tt} \mathrm{d}x$$

Next, we substitute the PDE $\rho\, u_{tt} = T\, u_{xx}$
and integrate by parts to get

$$\frac{dK}{dt} = T\int_{-\infty}^{+\infty} u_t u_{xx} \mathrm{d}x = T\left[u_t u_x\right]_{-\infty}^{+\infty} - T\int_{-\infty}^{+\infty} u_{tx} u_x \mathrm{d}x$$

Term evaluated at $x = \pm\infty$ and so it vanishes.

The wave equation ensures conservation of energy

The final term is a pure derivative since:

$$\frac{dK}{dt} = -T\int_{-\infty}^{+\infty} u_{tx}u_x \mathrm{d}x = -T\int_{-\infty}^{+\infty} \frac{\partial}{\partial t}\left(\frac{1}{2}u_x^2\right)\mathrm{d}x = -\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{1}{2}Tu_x^2\ \mathrm{d}x$$

Let us define the potential energy P as:

$$P = \int_{-\infty}^{+\infty}\frac{1}{2}Tu_x^2\ \mathrm{d}x$$

Consequently, the total energy

$$E = K + P = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho u_t^2 + Tu_x^2\right)dx$$

remains constant since $\dfrac{dE}{dt} = \dfrac{dK}{dt} + \dfrac{dP}{dt} = 0$

4 – Generalization

Through one example, we show here that a range of more general equations can be solved in a similar way as the wave equation discussed so far.

Exercise: solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,$

$$u(x,0) = x^2, \quad u_t(x,0) = e^x$$

The PDE factors as follows:

$$\left(-4\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u = 0$$

or

$$-4\left(\frac{\partial}{\partial t} - \frac{1}{4}\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)u = 0$$

which is equivalent to two 1st order PDEs:

$$v_t - \frac{1}{4}v_x = 0$$

$$u_t + u_x = v$$

Exercise: solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,$

$$u(x,0) = x^2, \quad u_t(x,0) = e^x$$

As shown in Lecture 1, the general solution of

$$au_x + bu_y = 0$$

writes:

$$u(x,y) = f(bx - ay)$$

Hence, $v_t - \frac{1}{4}v_x = 0$ leads to:

$$v = h\left(x + \frac{1}{4}t\right)$$

where h is an arbitrary function.

Exercise: solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,$

$u(x, 0) = x^2, \quad u_t(x, 0) = e^x$

Now, the second 1st order equation takes the form:

$$u_t + u_x = h\left(x + \frac{1}{4}t\right)$$

By adding (since the PDE is linear) one particular solution f and the general solution g of the homogeneous PDE, we obtain:

$$u = f\left(x + \frac{1}{4}t\right) + g(x - t)$$

with $f'(s) = \dfrac{4}{5}h(s)$.

Exercise: solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,$
$u(x,0) = x^2, \quad u_t(x,0) = e^x$

An alternate solution strategy consists in using a change of variable.

Consider

$$\xi = x + \frac{1}{4}t$$

$$\eta = x - t$$

The PDE becomes:

$$u_{\xi\eta} = 0$$

and the general solution writes:

$$u = f\left(x + \frac{1}{4}t\right) + g(x - t)$$

Exercise: solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,$

$$u(x, 0) = x^2, \quad u_t(x, 0) = e^x$$

By applying the same procedure as followed earlier to solve the IVP, we get:

$$f(x) + g(x) = x^2 \quad \text{and} \quad \frac{1}{4} f'(x) - g'(x) = e^x$$

Hence,

$$f' + g' = 2s \quad \text{and} \quad f' - 4g' = 4e^s$$

Leading, in the end, to:

$$u = \frac{4}{5}\left[\exp\left(x + \frac{1}{4}t\right) - \exp(x - t)\right] + \frac{4}{5}\left(x + \frac{1}{4}t\right)^2 + \frac{1}{5}(x - t)^2$$

Take-home messages

The basic properties of the wave equation include:

- the IVP has one, and only one, solution,
- information gets transported in both directions (along the characteristic lines) at a finite speed,
- consequently, an initial condition at a given point affects the solution only in a finite interval, called the domain of influence,
- vice-versa for the domain of dependence, the solution in not smoothed over time, which is reflected in the energy conservation property.

What will be next?

The one-dimensional diffusion equation (DE) writes:

$$u_t = k\, u_{xx}$$

Although it differs from the wave equation (WE)

$$u_{tt} = c^2\, u_{xx}$$

"just" by one unit in the order of the time derivative,

- this equation has mathematical properties strongly contrasting with those of the WE
- it also reflects a physical process which is totally different from waves...

The DE equation is harder to solve than the WE ... 😐