Discretization of Electromagnetic Problems

Nodal, edge, face and volume finite elements

Discrete mathematical structure

Continuous problem

Continuous function spaces & domain
Classical and weak formulations

Discretization ↓ **Approximation**

Discrete problem

Discrete function spaces piecewise defined
in a discrete domain (mesh)

Finite element method

Questions

Classical & weak formulations → ?
Properties of the fields → ?

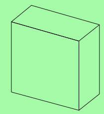
Objective

To build a discrete structure
as similar as possible
as the continuous structure

Discrete mathematical structure

Finite element

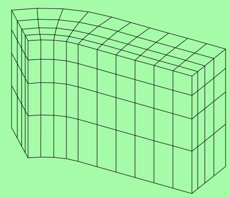
Interpolation in a geometric element of simple shape



$+ \ f$

Finite element space

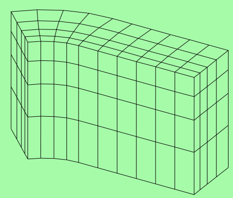
Function space
& Mesh



$+ \ \bigcup_i f_i$

Sequence of finite element spaces

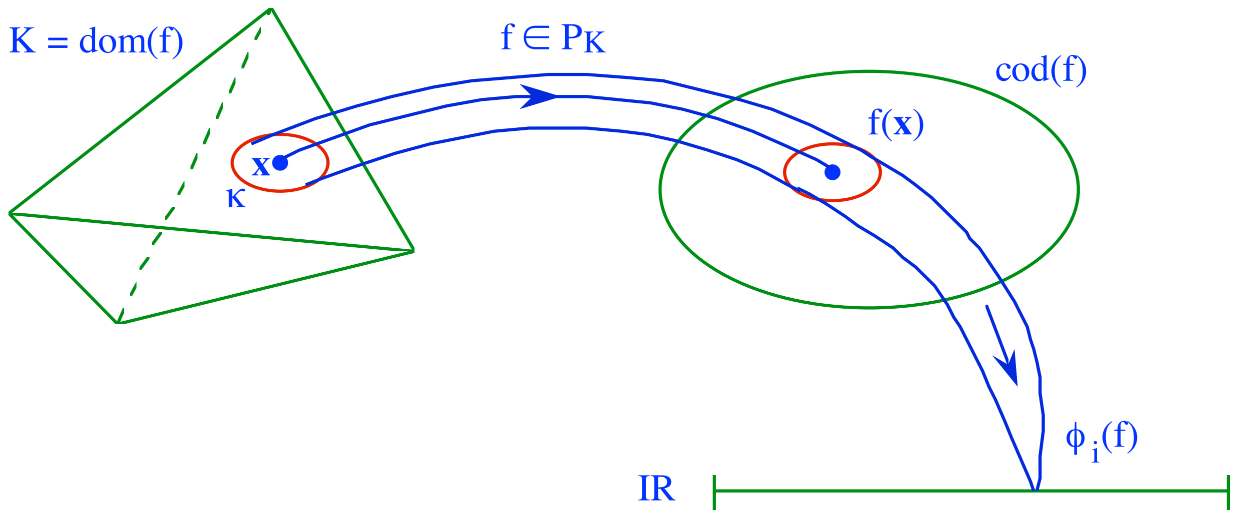
Sequence of function spaces
& Mesh



$+ \ \left\{ \bigcup_i f_i \right\}$

Finite elements

- Finite element (K, P_K, Σ_K)
  - K = domain of space (tetrahedron, hexahedron, prism)
  - P_K = function space of finite dimension n_K, defined in K
  - Σ_K = set of n_K degrees of freedom
    represented by n_K linear functionals ϕ_i, $1 \le i \le n_K$,
    defined in P_K and whose values belong to IR



Finite elements

- **Unisolvance**

  $\forall\, u \in P_K$, u is uniquely defined by the degrees of freedom

- **Interpolation**

$$u_K = \sum_{i=1}^{n_K} \phi_i(u)\, p_i$$

Degrees of freedom ($\phi_i(u)$)

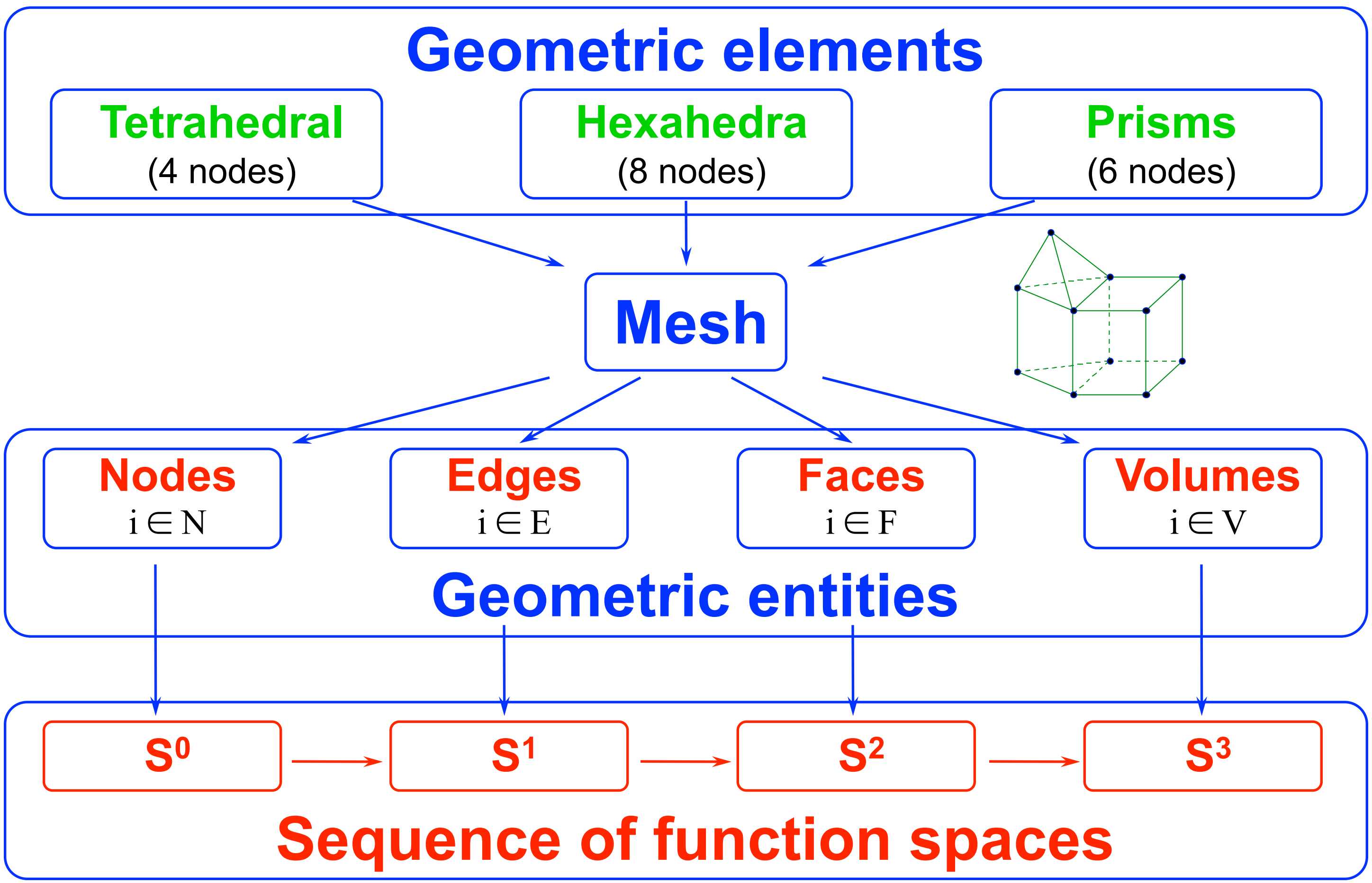
Basis functions (p_i)

- **Finite element space**

  Union of finite elements (K_j, P_{Kj}, Σ_{Kj}) such as :

  - the union of the K_j fill the studied domain (≡ mesh)
  - some continuity conditions are satisfied across the element interfaces

Sequence of finite element spaces



Sequence of finite element spaces

	Functions	Properties	Functionals	Degrees of freedom		
S^0	$\{s_i\ ,\ i \in N\}$	$s_i(x_j) = \delta_{ij}\quad \forall\, i,j \in N$	Point evaluation	Nodal value	→	Nodal element
S^1	$\{\mathbf{s}_i\ ,\ i \in E\}$	$\int_j \mathbf{s}_i \cdot d\mathbf{l} = \delta_{ij}\quad \forall\, i,j \in E$	Curve integral	Circulation along edge	→	Edge element
S^2	$\{\mathbf{s}_i\ ,\ i \in F\}$	$\int_j \mathbf{s}_i \cdot \mathbf{n}\, ds = \delta_{ij}\quad \forall\, i,j \in F$	Surface integral	Flux across face	→	Face element
S^3	$\{s_i\ ,\ i \in V\}$	$\int_j s_i\, dv = \delta_{ij}\quad \forall\, i,j \in V$	Volume integral	Volume integral	→	Volume element

↓ Bases (from Functions)

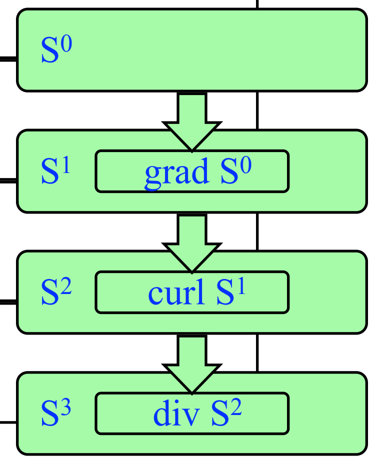
↓ (from Functionals)

$$u_K = \sum_i \phi_i(u)\, s_i$$

↑ Finite elements (to the element types)

Sequence of finite element spaces

	Base functions	Continuity across element interfaces	Codomains of the operators
S^0	$\{s_i\ ,\ i \in N\}$	value	
S^1	$\{\mathbf{s}_i\ ,\ i \in E\}$	tangential component	$\text{grad}\ S^0 \subset S^1$
S^2	$\{\mathbf{s}_i\ ,\ i \in F\}$	normal component	$\text{curl}\ S^1 \subset S^2$
S^3	$\{s_i\ ,\ i \in V\}$	discontinuity	$\text{div}\ S^2 \subset S^3$



Conformity

Sequence

$$S^0 \xrightarrow{\text{grad}} S^1 \xrightarrow{\text{curl}} S^2 \xrightarrow{\text{div}} S^3$$

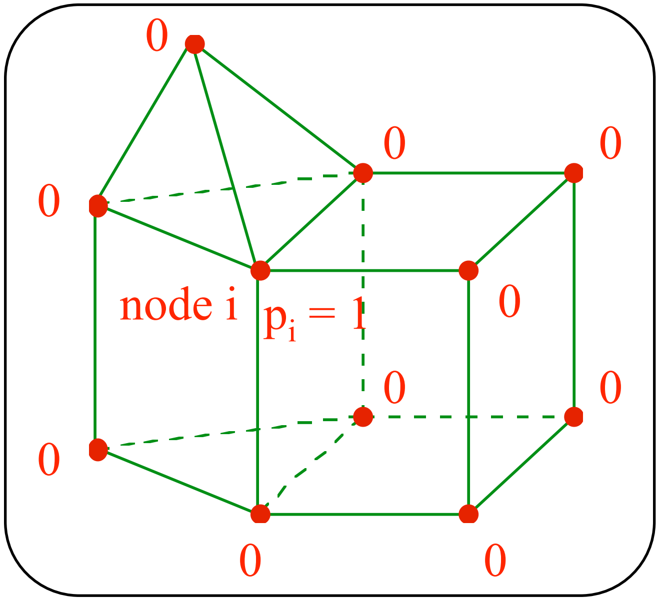
Function spaces S^0 et S^3

For each node $i \in N$ → scalar field

$$s_i(\mathbf{x}) = p_i(\mathbf{x}) \quad \in S^0$$

$$p_i = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at all other nodes} \end{cases}$$

p_i continuous in Ω



For each Volume $v \in V$ → scalar field

$$s_v = 1 / \mathrm{vol}(v) \quad \in S^3$$

Edge function space S^1

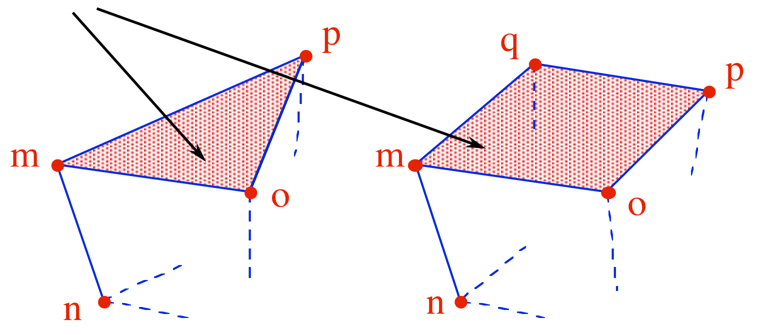
For each edge $e_{ij} = \{i, j\} \in E \rightarrow$ vector field

$$\mathbf{s}_{e_{ij}} = p_j \, \mathrm{grad} \sum_{r \in N_{F,j\bar{i}}} p_r - p_i \, \mathrm{grad} \sum_{r \in N_{F,i\bar{j}}} p_r$$

$\mathbf{s}_e \in S^1$

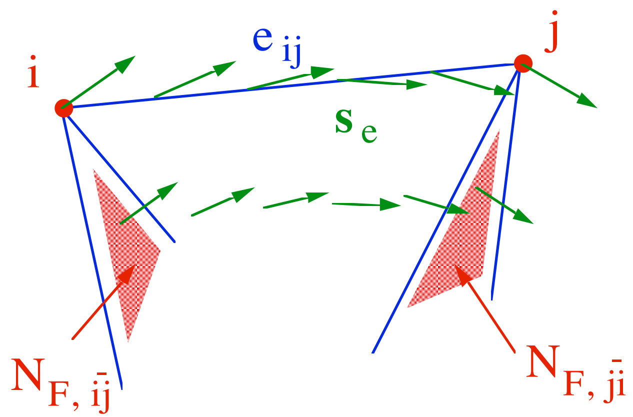
Definition of the set of nodes $N_{F,m\bar{n}}$

$$N_{F,m\bar{n}} = \{ i \in N \, ; \, i \in f_{mop(q)} \, , \, o,p,q \neq n \}$$

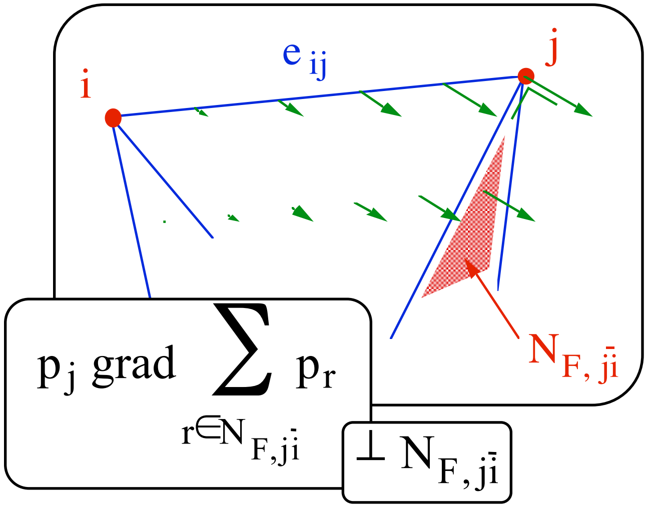


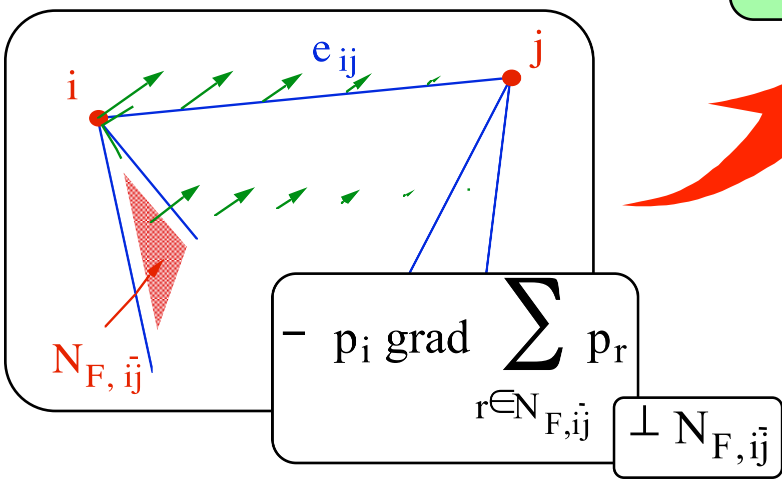
N.B.: In an element : 3 edges/node

Illustration of the vector field $\mathbf{s}_e$



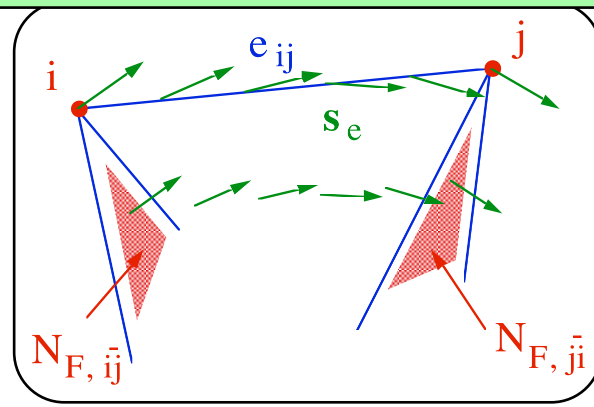
Edge function space S^1





Geometric interpretation of the vector field s_e

$$s_{e_{ij}} = p_j \,\mathrm{grad} \sum_{r \in N_{F,j\bar{i}}} p_r - p_i \,\mathrm{grad} \sum_{r \in N_{F,i\bar{j}}} p_r$$



Function space S^2

For each facet $f \in F$ $\rightarrow$ vector field

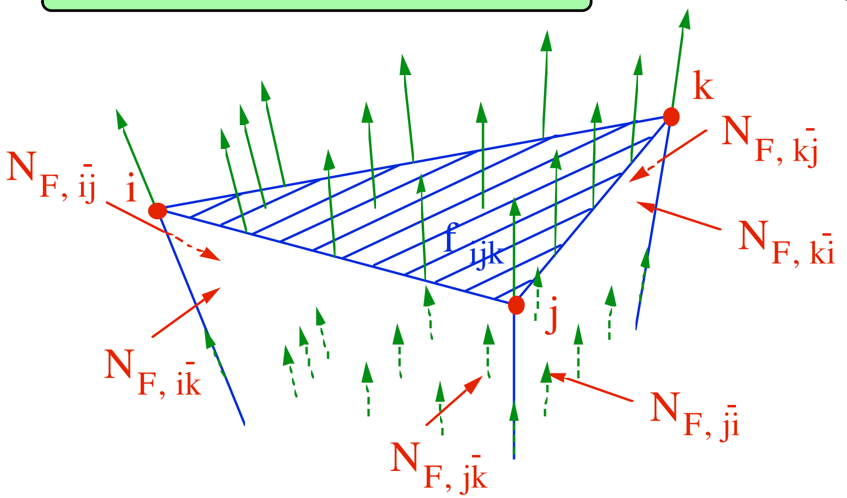
$f = f_{ijk(l)} = \{i, j, k\ (, l)\} = \{q_1, q_2, q_3\ (, q_4)\}$

$$\mathbf{s}_f = a_f \sum_{c=1}^{\#N_f} p_{q_c}\ \mathrm{grad}\left(\sum_{r \in N_{F,q_c \bar{q}_{c+1}}} p_r\right) \wedge \mathrm{grad}\left(\sum_{r \in N_{F,q_c \bar{q}_{c-1}}} p_r\right)$$

$\mathbf{s}_f \in S^2$

$$\#N_f = \begin{cases} \blacktriangle\ 3 \rightarrow a_f = 2 \\ \blacksquare\ 4 \rightarrow a_f = 1 \end{cases}$$

Illustration of the vector field $\mathbf{s}_f$



Particular subspaces of S^1

Kernel of the curl operator

Applications

$\mathbf{h} \in S^1(\Omega)$; curl $\mathbf{h} = 0$ in $\Omega_c^C \subset \Omega$ → $\mathbf{h} \equiv$?

→ Ampere equation in a domain Ω_c^C without current ($\leftrightarrow \Omega_c$)

Gauged subspace

$\mathbf{a} \in S^1(\Omega)$; $\mathbf{b} = \text{curl}\ \mathbf{a} \in S^2(\Omega)$ → $\mathbf{a} \equiv$?

Gauge $\mathbf{a} \cdot \omega = 0$
to fix $\mathbf{a}$

→ Gauge condition on a vector potential

→ Definition of a generalized source field $\mathbf{h}_s$ such that curl $\mathbf{h}_s = \mathbf{j}_s$

Kernel of the curl operator

Case of simply connected domains

$$H = \{ \mathbf{h} \in S^1(\Omega) \,;\, \mathrm{curl}\,\mathbf{h} = 0 \text{ in } \Omega_c^C \}$$

$\mathbf{h} = -\,\mathrm{grad}\,\phi$ in Ω_c^C

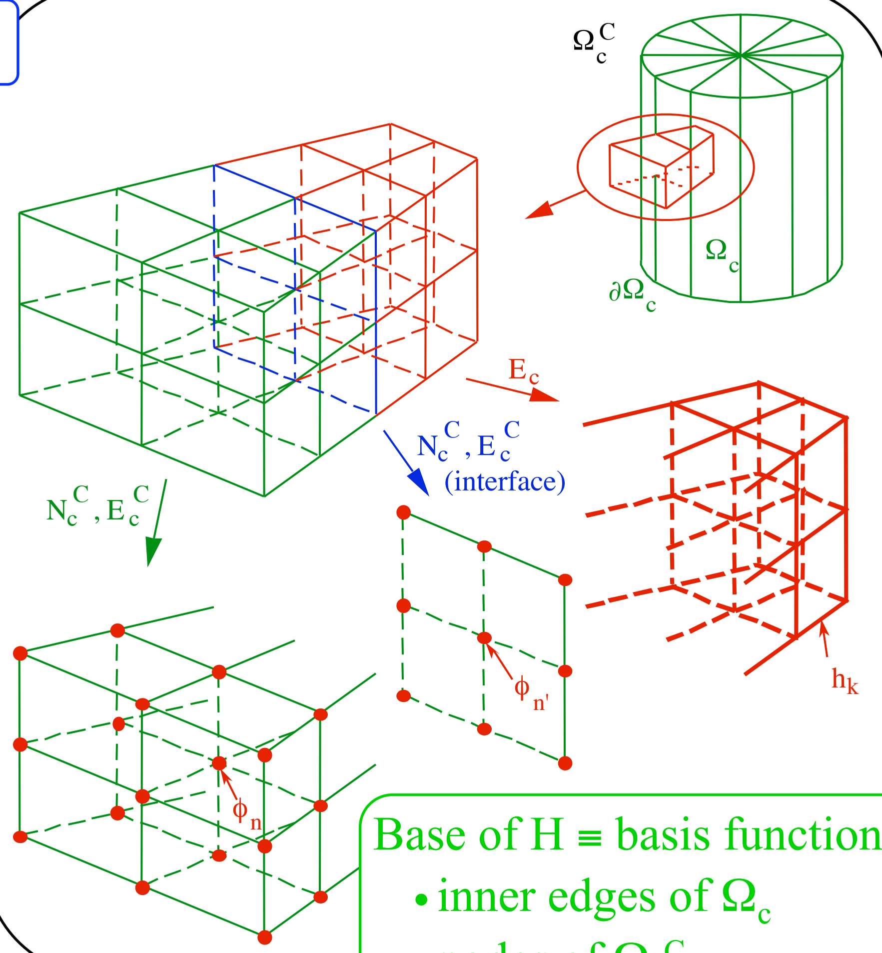
$$\mathbf{h} = \sum_{e \in E} h_a\, \mathbf{s}_a = \sum_{k \in E_c} h_k\, \mathbf{s}_k + \sum_{l \in E_c^C} h_l\, \mathbf{s}_l$$

$$h_l = \int_{l_{ab}} \mathbf{h} \,.\, \mathbf{dl} = \int_{l_{ab}} -\mathrm{grad}\,\phi \,.\, \mathbf{dl} = \phi_{a_l} - \phi_{b_l}$$

$$\mathbf{h} = \sum_{k \in E_c} h_k\, \mathbf{s}_k + \sum_{l \in E_c^C} \left(\phi_{a_l} - \phi_{b_l}\right) \mathbf{s}_l$$

$$\mathbf{h} = \sum_{k \in E_c} h_k\, \mathbf{s}_k + \sum_{n \in N_c^C} \phi_n\, \mathbf{v}_n$$

with $\mathbf{v}_n = \sum_{nj \in E_c^C} \mathbf{s}_{nj}$



Base of H ≡ basis functions of
- inner edges of Ω_c
- nodes of Ω_c^C, with those of $\partial\Omega_c$

Kernel of the curl operator

Case of multiply connected domains

$$H = \{ \mathbf{h} \in S^1(\Omega) \,;\, \mathrm{curl}\, \mathbf{h} = 0 \text{ in } \Omega_c^C \}$$

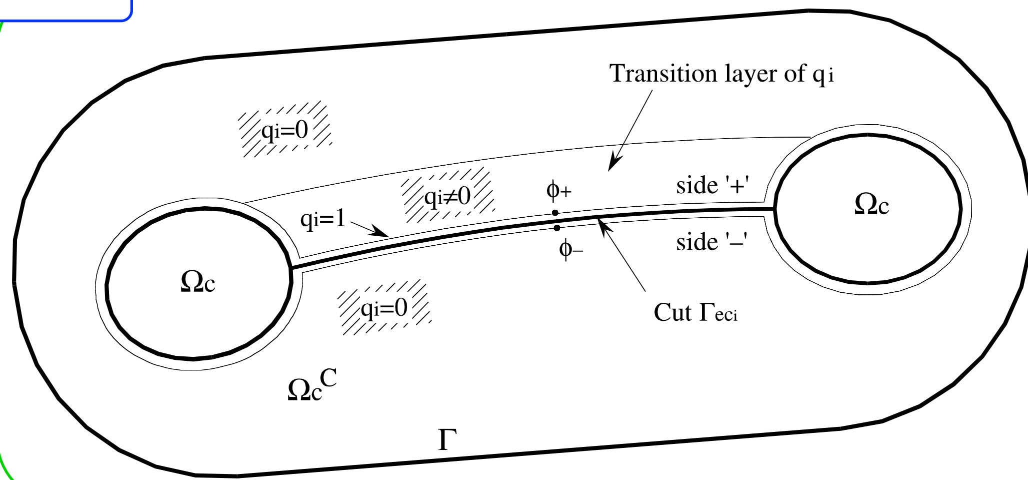
$\mathbf{h} = -\,\mathrm{grad}\, \phi$ in Ω_c^C (cuts)

$$\phi = \phi^{cont} + \phi^{disc}$$

$$\phi^+ - \phi^- = \phi\,|_{\Gamma^+_{eci}} - \phi\,|_{\Gamma^-_{eci}} = I_i$$

$$\phi^{disc} = \sum_{i \in C} I_i \, q_i$$

discontinuity of ϕ^{disc}



q_i
- defined in Ω_c^C
- unit discontinuity across Γ_{eci}
- continuous in a transition layer
- zero out of this layer

$$\mathbf{h} = \sum_{k \in A_c} h_k \, \mathbf{s}_k + \sum_{n \in N_c^C} \phi^{cont}{}_n \, \mathbf{v}_n + \sum_{i \in C} I_i \, \mathbf{c}_i$$

with $\mathbf{c}_i = \sum \mathbf{s}_{nj}$

edges of Ω_c^C ← $nj \in A_c^C$

starting from a node of the cut ← $n \in N_{ec_i}$

and located on side '+' ← $j \in N_c^{C+}$

but not on the cut ← $j \notin N_{ec_i}$

Basis of H ≡ basis functions of
- inner edges of Ω_c
- nodes of Ω_c^C
- cuts of C

Gauged subspace of S^1

Gauged space in Ω

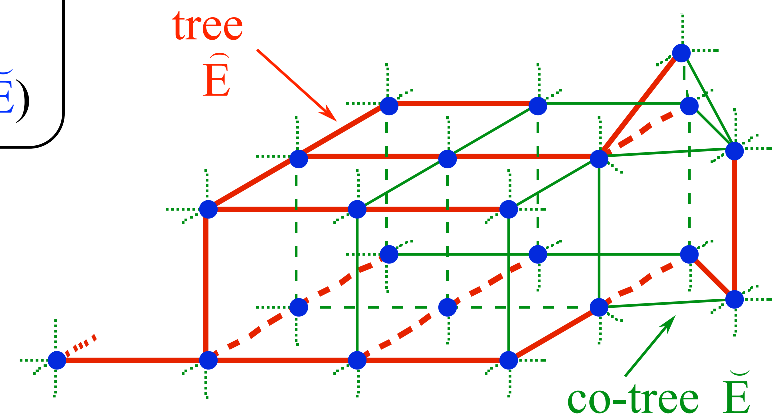
$\mathbf{b} = \text{curl}\, \mathbf{a}$ with $\mathbf{a} = \sum_{e \in E} a_e\, \mathbf{s}_e \in S^1(\Omega)$, $\mathbf{b} = \sum_{f \in F} b_f\, \mathbf{s}_f \in S^2(\Omega)$

$b_f = \sum_{e \in E} i(e,f)\, a_e$, $f \in F$ ➡ matrix form: $[b_f] = [\mathbf{C}_{FE}]\,[a_e]$

Face-edge incidence matrix

Tree ≡ set of edges connecting
(in Ω) all the nodes of Ω without forming any loop ($\hat{E}$)

Co-tree ≡ complementary set of the tree ($\breve{E}$)



Gauged space of $S^1(\Omega)$

$\breve{S}^1(\Omega) = \{\mathbf{a} \in S^1(\Omega) \,;\, a_j = 0 \,,\, \forall\, j \in \breve{E}\}$

$\mathbf{a} = \sum_{i \in \breve{E}} a_i\, \mathbf{s}_i \in \breve{S}^1(\Omega)$

Basis of $\breve{S}^1(\Omega)$ ≡ co-tree edge basis functions (explicit gauge definition)

Mesh of electromagnetic devices

- **Electromagnetic fields extend to infinity (unbounded domain)**
  - **Approximate boundary conditions:**
    - **zero fields at finite distance**
  - **Rigorous boundary conditions:**
    - **"infinite" finite elements (geometrical transformations)**
    - **boundary elements (FEM-BEM coupling)**
- **Electromagnetic fields are confined (bounded domain)**
  - **Rigorous boundary conditions**

Mesh of electromagnetic devices

- **Electromagnetic fields enter the materials up to a distance depending of physical characteristics and constraints**

  - **Skin depth δ (δ<< if ω, σ, μ >>)**

$$\delta = \sqrt{\frac{2}{\omega \, \sigma \, \mu}}$$

  - **mesh fine enough near surfaces (material boundaries)**
  - **use of surface elements when $\delta \rightarrow 0$**

Mesh of electromagnetic devices

- **Types of elements**
  - 2D : triangles, quadrangles
  - 3D : tetrahedra, hexahedra, prisms, pyramids
  - Coupling of volume and surface elements
    - boundary conditions
    - thin plates
    - interfaces between regions
    - cuts (for making domains simply connected)
  - Special elements (air gaps between moving pieces, ...)