Finite Element Modeling of Electromagnetic Systems

Mathematical and numerical tools

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Introduction

- **Formulations of electromagnetic problems**
  - Maxwell equations, material relations
  - Electrostatics, electrokinetics, magnetostatics, magnetodynamics
  - Strong and weak formulations

- **Discretization of electromagnetic problems**
  - Finite elements, mesh, constraints
  - Very rich content of weak finite element formulations
Formulations of Electromagnetic Problems

Maxwell equations

Electrostatics
Electrokinetics
Magnetostatics
Magnetodynamics
Electromagnetic models

❖ Electrostatics
   ♦ Distribution of electric field due to static charges and/or levels of electric potential

❖ Electrokinetics
   ♦ Distribution of static electric current in conductors

❖ Electrodynamics
   ♦ Distribution of electric field and electric current in materials (insulating and conducting)

❖ Magnetostatics
   ♦ Distribution of static magnetic field due to magnets and continuous currents

❖ Magnetodynamics
   ♦ Distribution of magnetic field and eddy current due to moving magnets and time variable currents

❖ Wave propagation
   ♦ Propagation of electromagnetic fields

All phenomena are described by Maxwell equations
Maxwell equations

\[
\begin{align*}
\text{curl } \mathbf{h} &= \mathbf{j} + \partial_t \mathbf{d} \\
\text{curl } \mathbf{e} &= -\partial_t \mathbf{b} \\
\text{div } \mathbf{b} &= 0 \\
\text{div } \mathbf{d} &= \rho_v
\end{align*}
\]

Ampère equation

Faraday equation

Conservation equations

Principles of electromagnetism

Physical fields and sources

- \( \mathbf{h} \) magnetic field (A/m)
- \( \mathbf{b} \) magnetic flux density (T)
- \( \mathbf{j} \) current density (A/m²)
- \( \mathbf{e} \) electric field (V/m)
- \( \mathbf{d} \) electric flux density (C/m²)
- \( \rho_v \) charge density (C/m³)
Material constitutive relations

**Constitutive relations**

\[ \mathbf{b} = \mu \mathbf{H} \pm \mathbf{b}_s \]
\[ \mathbf{d} = \varepsilon \mathbf{E} \pm \mathbf{d}_s \]
\[ \mathbf{j} = \sigma \mathbf{E} \pm \mathbf{j}_s \]

**Magnetic relation**

**Dielectric relation**

**Ohm law**

**Characteristics of materials**

\( \mu \) magnetic permeability (H/m)
\( \varepsilon \) dielectric permittivity (F/m)
\( \sigma \) electric conductivity (\( \Omega^{-1} \text{m}^{-1} \))

**Possible sources**

\( b_s \) remnant induction, ...
\( d_s \) ...
\( j_s \) source current in stranded inductor, ...

**Constants (linear relations)**

**Functions of the fields**

**(nonlinear materials)**

**Tensors (anisotropic materials)**
**Electrostatics**

**Basis equations**
- \( \text{curl } \mathbf{e} = 0 \)
- \( \text{div } \mathbf{d} = \rho \)
- \( \mathbf{d} = \varepsilon \mathbf{e} \)

**& boundary conditions**
- \( \mathbf{n} \times \mathbf{e} \mid \Gamma_{0e} = 0 \)
- \( \mathbf{n} \cdot \mathbf{d} \mid \Gamma_{0d} = 0 \)

**Electric field (V/m)**
**Electric flux density (C/m\(^2\))**
**Electric charge density (C/m\(^3\))**
**Dielectric permittivity (F/m)**

**Electric scalar potential formulation**
- \( \text{div } \varepsilon \text{ grad } v = - \rho \)
- with \( \mathbf{e} = - \text{ grad } v \)

- Formulation for 
  - the exterior region \( \Omega_0 \)
  - the dielectric regions \( \Omega_{d,j} \)
- In each conducting region \( \Omega_{c,i} : v = v_i \rightarrow v = v_i \) on \( \Gamma_{c,i} \)

**Type of electrostatic structure**
- \( \Gamma_0 = \Gamma_{0e} \cup \Gamma_{0d} \)
- \( \Omega_0 \) Exterior region
- \( \Omega_{c,i} \) Conductors
- \( \Omega_{d,j} \) Dielectric

\( \mathbf{n} \times \mathbf{e} \mid \Gamma_{0e} = 0 \)
\( \mathbf{n} \cdot \mathbf{d} \mid \Gamma_{0d} = 0 \)
\( \text{curl } \mathbf{e} = 0 \)
\( \text{div } \mathbf{d} = \rho \)
\( \mathbf{d} = \varepsilon \mathbf{e} \)
**Electrokinetics**

**Basis equations**
- curl \( e = 0 \)
- div \( j = 0 \)
- \( j = \sigma e \)

\( e \) electric field (V/m)
\( j \) electric current density (C/m²)
\( \sigma \) electric conductivity (Ω⁻¹m⁻¹)

**Electric scalar potential formulation**
- div \( \sigma \) grad \( v \) = 0

with \( e = - \) grad \( v \)

**Type of electrokinetic structure**
- \( \Omega_c \) Conducting region
- \( \Gamma_{0e} \) to \( \Gamma_{0j} \)
- \( V = v^1 - v^0 \)

**Boundary conditions**
- \( n \times e \mid \Gamma_{0e} = 0 \)
- \( n \cdot j \mid \Gamma_{0j} = 0 \)

**Formulation for**
- the conducting region \( \Omega_c \)
- On each electrode \( \Gamma_{0e,i} \): \( v = v^i \rightarrow v = v^i \) on \( \Gamma_{0e,i} \)
Electrostatic problem

Basis equations

curl \( \mathbf{e} = 0 \)
\( \mathbf{d} = \varepsilon \mathbf{e} \)
\( \text{div} \mathbf{d} = \rho \)

\[ \varepsilon \mathbf{e} = \nabla v \]
\[ \mathbf{d} = \nabla \times \mathbf{u} \]
Electrokinetic problem

Basis equations

\[ \text{curl } \mathbf{e} = 0 \]
\[ \mathbf{j} = \sigma \mathbf{e} \]
\[ \text{div } \mathbf{j} = 0 \]

\[ \sigma \mathbf{e} = - \text{grad } \mathbf{v} \]
\[ \mathbf{j} = \text{curl } \mathbf{t} \]
Classical and weak formulations

Partial differential problem

**Classical formulation**

\[ \begin{align*}
L u &= f \quad \text{in} \ \Omega \\
B u &= g \quad \text{on} \ \Gamma = \partial \Omega \\
u &= \text{classical solution}
\end{align*} \]

**Weak formulation**

\[ \begin{align*}
(u, L^* v) - (f, v) + \int_{\Gamma} Q g (v) \, ds &= 0 \quad \forall \ v \in V(\Omega) \\
u &= \text{weak solution}
\end{align*} \]

**Notations**

\[ \begin{align*}
(u, v) &= \int_{\Omega} u(x) v(x) \, dx, \quad u, v \in L^2(\Omega) \\
(u, v) &= \int_{\Omega} u(x) \cdot v(x) \, dx, \quad u, v \in L^2(\Omega)
\end{align*} \]

**Notations**

\[ \begin{align*}
v &= \text{test function} \
\text{Continuous level} : \ \infty \times \infty \text{ system} \\
\text{Discrete level} : \ n \times n \text{ system} \
\Rightarrow \text{numerical solution}
\end{align*} \]
Classical and weak formulations

Application to the magnetostatic problem

\[ \text{curl } e = 0 \]
\[ \text{div } d = 0 \]
\[ d = \varepsilon e \]
\[ n \times e \big|_{\Gamma_e} = 0 \]
\[ n \cdot d \big|_{\Gamma_d} = 0 \]

Electrostatic classical formulation

\[ (d, \text{grad } v') = 0 \quad \forall \ v' \in V(\Omega) \quad \text{with } V(\Omega) = \{ v \in H^0(\Omega) ; v \big|_{\Gamma_e} = 0 \} \]

\[ \Rightarrow (\text{div } d, v') + <n \cdot d, v'>_{\Gamma} = 0 \quad \forall \ v' \in V(\Omega) \]

\[ \downarrow \quad \downarrow \]

\[ \text{div } d = 0 \quad n \cdot d \big|_{\Gamma_d} = 0 \]

\[ d = \varepsilon e \quad \& \quad e = - \text{grad } v \quad \Leftrightarrow \quad \text{curl } e = 0 \]

Weak formulation of \( \text{div } d = 0 \)
(+ boundary condition)

Electrostatic weak formulation with \( v \)

\[ (-\varepsilon \text{grad } v, \text{grad } v') = 0 \quad \forall \ v' \in V(\Omega) \]
Classical and weak formulations

Application to the magnetostatic problem

\( \nabla \times \mathbf{h} = \mathbf{j} \)
\( \nabla \cdot \mathbf{b} = 0 \)
\( \mathbf{b} = \mu \mathbf{h} \)
\( \mathbf{n} \times \mathbf{h} \mid_{\Gamma_h} = 0 \)
\( \mathbf{n} \cdot \mathbf{b} \mid_{\Gamma_e} = 0 \)

\( (\mathbf{b}, \nabla \phi') = 0, \quad \forall \phi' \in \Phi(\Omega) \quad \text{with} \quad \Phi(\Omega) = \{ \phi \in H^0(\Omega) ; \phi \mid_{\Gamma_h} = 0 \} \)

\( \Rightarrow (\nabla \cdot \mathbf{b}, \phi') + \langle \mathbf{n} \cdot \mathbf{b}, \phi' \rangle_{\Gamma} = 0, \quad \forall \phi' \in \Phi(\Omega) \)

\( \nabla \cdot \mathbf{b} = 0 \quad \mathbf{n} \cdot \mathbf{b} \mid_{\Gamma_e} = 0 \)

\( \mathbf{b} = \mu \mathbf{h} \quad \& \quad \mathbf{h} = \mathbf{h}_s - \nabla \phi \quad \text{(with} \quad \nabla \times \mathbf{h}_s = \mathbf{j}) \quad \Leftrightarrow \quad \nabla \times \mathbf{h} = \mathbf{j} \)

(\mu (\mathbf{h}_s - \nabla \phi), \nabla \phi') = 0, \quad \forall \phi' \in \Phi(\Omega)
Quasi-stationary approximation

\[ \text{curl } \mathbf{h} = \mathbf{j} + \partial_t \mathbf{d} \]

Dimensions \(<\) wavelength

Conduction current density

Displacement current density

Electrotechnic apparatus (motors, transformers, ...)
Frequencies from Hz to a few 100 kHz
Magnetostatics

Equations

\[ \text{curl } \mathbf{h} = \mathbf{j} \]  
Ampère equation

\[ \text{div } \mathbf{b} = 0 \]  
Magnetic conservation equation

Constitutive relations

\[ \mathbf{b} = \mu \mathbf{h} + \mathbf{b}_s \]  
Magnetic relation

\[ \mathbf{j} = \mathbf{j}_s \]  
Ohm law & source current

Type of studied configuration

\( \Omega \)  
Studied domain

\( \Omega_m \)  
Magnetic domain

\( \Omega_s \)  
Inductor
Magnetodynamics

Equations

- $\text{curl } \mathbf{h} = \mathbf{j}$ (Ampère equation)
- $\text{curl } \mathbf{e} = -\frac{\partial}{\partial t} \mathbf{b}$ (Faraday equation)
- $\text{div } \mathbf{b} = 0$ (Magnetic conservation equation)

Constitutive relations

- $\mathbf{b} = \mu \mathbf{h} + \mathbf{b}_s$ (Magnetic relation)
- $\mathbf{j} = \sigma \mathbf{e} + \mathbf{j}_s$ (Ohm law & source current)

Type of studied configuration

- $\Omega$: Studied domain
  - $\Omega_p$: Passive conductor and/or magnetic domain
  - $\Omega_a$: Active conductor
  - $\Omega_s$: Inductor

- $\Omega_p$: Passive conductor and/or magnetic domain
- $\Omega_a$: Active conductor
- $\Omega_s$: Inductor

- $I_a$: Source current
- $V_a$: Voltage
Magnetic constitutive relation

\[ b = \mu \cdot h \]
\[ \mu = \mu_r \cdot \mu_0 \]
\[ \mu_r \text{ relative magnetic permeability} \]

- **Diamagnetic and paramagnetic materials**
  - Linear material \( \mu_r \approx 1 \) (silver, copper, aluminium)

- **Ferromagnetic materials**
  - Nonlinear material \( \mu_r \gg 1 \), \( \mu_r = \mu_r(h) \) (steel, iron)

**b-h characteristic of steel**

**\( \mu_r \)-h characteristic of steel**
Magnetostatic formulations

Maxwell equations
(magnetic - static)

\[ \text{curl } h = j \]
\[ \text{div } b = 0 \]
\[ b = \mu \cdot h \]

ϕ Formulation

"h" side

a Formulation

"b" side
Magnetostatic formulations

**Basis equations**

- \( \text{curl } h = j \) (h)
- \( b = \mu h \) (m)
- \( \text{div } b = 0 \) (b)

**\( \phi \) Formulation**

- Magnetic scalar potential \( \phi \)
  - \( h = h_s - \text{grad } \phi \) \( \Rightarrow \) (h) OK
  - \( h_s \) given such as \( \text{curl } h_s = j \) (non-unique)
  - \( \text{div } (\mu (h_s - \text{grad } \phi)) = 0 \) \( \Leftarrow \) (b) & (m)

**a Formulation**

- Magnetic vector potential \( a \)
  - (b) OK \( \iff \) \( b = \text{curl } a \)
  - (h) & (m) \( \rightarrow \) \( \text{curl } (\mu^{-1} \text{curl } a) = j \)

**Multivalued potential**

- Cuts

**Non-unique potential**

- Gauge condition
Kernel of the curl (in a domain $\Omega$)
\[
\text{ker (curl)} = \{ \mathbf{v} : \text{curl} \mathbf{v} = 0 \}
\]

\[
\text{cod (grad)} \subset \text{ker (curl)}
\]
Multivalued scalar potential - Cut

\[ \text{curl } h = 0 \quad \text{in } \Omega \]

\[ h = - \text{grad } \phi \quad \text{in } \Omega \]

Circulation of \( h \) along path \( \gamma_{AB} \) in \( \Omega \)

\[ \int_{\gamma_{AB}} h \cdot dl = \int_{\gamma_{AB}} - \text{grad } \phi \cdot dl = \phi_A - \phi_B \]

Closed path \( \gamma_{AB} \) (A\( \equiv \)B) surrounding a conductor (with current I)

\[ \Rightarrow \phi_A - \phi_B = 0 \neq I \]

\( \phi \) must be discontinuous ... through a cut

\[ \Delta \phi = I \]
Vector potential - gauge condition

\[ \text{div } b = 0 \quad \text{in } \Omega \]

\[ b = \text{curl } a \quad \text{in } \Omega \]

**Non-uniqueness of vector potential** \( a \)

\[ b = \text{curl } a = \text{curl } (a + \text{grad } \eta) \]

**Gauge condition**

**Coulomb gauge** \( \text{div } a = 0 \)

**Gauge** \( a \cdot \omega = 0 \)

\( \omega \) vector field with non-closed lines linking any 2 points in \( \Omega \)

**Example:** \( w(r) = r \)

ex.: \( w(r) = r \)
Magnetodynamic formulations

Maxwell equations (quasi-stationary)

\[
\begin{align*}
\text{curl } \mathbf{h} &= \mathbf{j} \\
\text{curl } \mathbf{e} &= -\partial_t \mathbf{b} \\
\text{div } \mathbf{b} &= 0 \\
\mathbf{b} &= \mu \mathbf{h} \\
\mathbf{j} &= \sigma \mathbf{e}
\end{align*}
\]

"h" side \quad | \quad "b" side
Magnetodynamic formulations

**H-φ Formulation**

**Magnetic field h**

**Magnetic scalar potential φ**

\[
\begin{align*}
\text{curl } h &= j \\
\mathbf{b} &= \mu \mathbf{h} \\
j &= \sigma \mathbf{e} \\
\text{div } \mathbf{b} &= 0
\end{align*}
\]

**Basis equations**

\[
\begin{align*}
\text{curl } h &= j \\
b &= \mu h \\
j &= \sigma e \\
curl e &= - \partial_t b \\
\text{div } b &= 0
\end{align*}
\]

\[
\begin{align*}
h &= \mathbf{h}_s - \text{grad } \phi \\
h = \mathbf{h}_s - \text{grad } \phi \quad \text{ds } \Omega_c
\end{align*}
\]

\[
\begin{align*}
\text{curl } \mathbf{h}_s &= j_s \\
\Rightarrow (\text{h}) \text{ OK}
\end{align*}
\]

\[
\begin{align*}
\text{curl } (\sigma^{-1} \text{curl } h) + \partial_t (\mu h) &= 0 \\
\text{div } (\mu (h_s - \text{grad } \phi)) &= 0
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \text{ (h) OK} \\
\text{in } \Omega_c \rightarrow \\
\text{in } \Omega_c^C \rightarrow
\end{align*}
\]

**T-ω Formulation**

**Electric vector potential t**

**Magnetic scalar potential ω**

\[
\begin{align*}
j &= \text{curl } t \\
h &= t - \text{grad } \omega
\end{align*}
\]

\[
\begin{align*}
curl (\sigma^{-1} \text{curl } t) + \partial_t (\mu (t - \text{grad } \omega)) &= 0 \\
\text{div } (\mu (t - \text{grad } \omega)) &= 0
\end{align*}
\]

\[
\begin{align*}
\Rightarrow (\text{h}) \text{ OK} \\
\text{in } \Omega_c \leftarrow \\
\text{in } \Omega_c^C \leftarrow
\end{align*}
\]

+ Gauge
Magnetodynamic formulations

<table>
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<tr>
<th>Basis equations</th>
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<tr>
<td>( \text{curl } h = j ) (h)</td>
</tr>
<tr>
<td>( b = \mu h )</td>
</tr>
<tr>
<td>( j = \sigma e )</td>
</tr>
<tr>
<td>( \text{curl } e = -\partial_t b ) (b)</td>
</tr>
<tr>
<td>( \text{div } b = 0 )</td>
</tr>
</tbody>
</table>

**a* Formulation**

**Magnetic vector potential a***

\[
\begin{align*}
 b &= \text{curl } a^* \\
 e &= -\partial_t a^* \\
 \text{curl } (\mu^{-1} \text{curl } a^*) + \sigma \partial_t a^* &= j_s \\
 + \text{Gauge in } \Omega_c^C
\end{align*}
\]

**a-v Formulation**

**Magnetic vector potential a**

**Electric scalar potential v**

\[
\begin{align*}
 b &= \text{curl } a \\
 e &= -\partial_t a - \text{grad } v \\
 \text{curl } (\mu^{-1} \text{curl } a) + \sigma (\partial_t a + \text{grad } v)) &= j_s \\
 + \text{Gauge in } \Omega
\end{align*}
\]
Magnetostatic problem

Basis equations

curl \mathbf{h} = \mathbf{j} \quad \mathbf{b} = \mu \mathbf{h} \quad \text{div} \mathbf{b} = 0

\begin{align*}
\mathbf{h} &= -\text{grad} \phi \\
\mathbf{b} &= \text{curl} \mathbf{a}
\end{align*}

"h" side

"b" side
Magnetodynamic problem

Basis equations

\[ \text{curl } \mathbf{h} = \mathbf{j} \]
\[ \mathbf{b} = \mu \mathbf{h} \]
\[ \mathbf{j} = \sigma \mathbf{e} \]
\[ \text{curl } \mathbf{e} = - \partial_t \mathbf{b} \]
\[ \text{div } \mathbf{b} = 0 \]

\[ \mathbf{h} = \mathbf{t} - \text{grad } \phi \]
\[ \mathbf{b} = \text{curl } \mathbf{a} \]
\[ \mathbf{e} = - \partial_t \mathbf{a} - \text{grad } \mathbf{v} \]
Continuous mathematical structure

Domain \( \Omega \), Boundary \( \partial \Omega = \Gamma_h \cup \Gamma_e \)

**Basis structure**

Function spaces \( F^0_h \subset L^2, F^1_h \subset L^2, F^2_h \subset L^2, F^3_h \subset L^2 \)

- \( \text{dom} (\text{grad}_h) = F^0_h = \{ \phi \in L^2(\Omega) ; \text{grad} \phi \in L^2(\Omega) , \phi |_{\Gamma_h} = 0 \} \)
- \( \text{dom} (\text{curl}_h) = F^1_h = \{ h \in L^2(\Omega) ; \text{curl} \ h \in L^2(\Omega) , \mathbf{n} \wedge h |_{\Gamma_h} = 0 \} \)
- \( \text{dom} (\text{div}_h) = F^2_h = \{ j \in L^2(\Omega) ; \text{div} j \in L^2(\Omega) , \mathbf{n} \cdot j |_{\Gamma_h} = 0 \} \)

\( \text{grad}_h F^0_h \subset F^1_h, \text{curl}_h F^1_h \subset F^2_h, \text{div}_h F^2_h \subset F^3_h \)

**Boundary conditions on \( \Gamma_h \)**

Sequence

\[ F^0_h \xrightarrow{\text{grad}_h} F^1_h \xrightarrow{\text{curl}_h} F^2_h \xrightarrow{\text{div}_h} F^3_h \]

**Basis structure**

Function spaces \( F^0_e \subset L^2, F^1_e \subset L^2, F^2_e \subset L^2, F^3_e \subset L^2 \)

- \( \text{dom} (\text{grad}_e) = F^0_e = \{ v \in L^2(\Omega) ; \text{grad} v \in L^2(\Omega) , v |_{\Gamma_e} = 0 \} \)
- \( \text{dom} (\text{curl}_e) = F^1_e = \{ a \in L^2(\Omega) ; \text{curl} a \in L^2(\Omega) , \mathbf{n} \wedge a |_{\Gamma_e} = 0 \} \)
- \( \text{dom} (\text{div}_e) = F^2_e = \{ b \in L^2(\Omega) ; \text{div} b \in L^2(\Omega) , \mathbf{n} \cdot b |_{\Gamma_e} = 0 \} \)

\( \text{grad}_e F^0_e \subset F^1_e, \text{curl}_e F^1_e \subset F^2_e, \text{div}_e F^2_e \subset F^3_e \)

**Boundary conditions on \( \Gamma_e \)**

Sequence

\[ F^3_e \xleftarrow{\text{div}_e} F^2_e \xleftarrow{\text{curl}_e} F^1_e \xleftarrow{\text{grad}_e} F^0_e \]
Discretization of Electromagnetic Problems

Nodal, edge, face and volume finite elements
Discrete mathematical structure

Continuous problem
Continuous function spaces & domain
Classical and weak formulations

Discretization
Approximation

Discrete problem
Discrete function spaces piecewise defined in a discrete domain (mesh)

Finite element method

Questions
Classical & weak formulations $\rightarrow$ ?
Properties of the fields $\rightarrow$ ?

Objective
To build a discrete structure as similar as possible as the continuous structure
Discrete mathematical structure

Finite element
Interpolation in a geometric element of simple shape

Finite element space
Function space & Mesh

Sequence of finite element spaces
Sequence of function spaces & Mesh
Finite elements

- Finite element \((K, P_K, \Sigma_K)\)
  - \(K\) = domain of space (tetrahedron, hexahedron, prism)
  - \(P_K\) = function space of finite dimension \(n_K\), defined in \(K\)
  - \(\Sigma_K\) = set of \(n_K\) degrees of freedom represented by \(n_K\) linear functionals \(\phi_i\), \(1 \leq i \leq n_K\), defined in \(P_K\) and whose values belong to \(\mathbb{IR}\)
Finite elements

❖ **Unisolvance**

∀ u ∈ P_K, u is uniquely defined by the degrees of freedom

❖ **Interpolation**

\[ u_K = \sum_{i=1}^{n_K} \phi_i(u) p_i \]

 Degrees of freedom

 Basis functions

❖ **Finite element space**

Union of finite elements (K_j, P_{Kj}, \Sigma_{Kj}) such as:

- the union of the K_j fill the studied domain (≡ mesh)
- some continuity conditions are satisfied across the element interfaces
Sequence of finite element spaces

Geometric elements
- Tetrahedral (4 nodes)
- Hexahedra (8 nodes)
- Prisms (6 nodes)

Mesh

Geometric entities
- Nodes \( i \in N \)
- Edges \( i \in E \)
- Faces \( i \in F \)
- Volumes \( i \in V \)

Sequence of function spaces
- \( S^0 \)
- \( S^1 \)
- \( S^2 \)
- \( S^3 \)
Sequence of finite element spaces

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<td>${s_i, i \in N}$</td>
<td>$s_i(x_j) = \delta_{ij}$</td>
<td>Point evaluation</td>
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<td>$S^1$</td>
<td>${s_i, i \in E}$</td>
<td>$\int_j s_i \cdot dl = \delta_{ij}$</td>
<td>Curve integral</td>
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<td>$S^2$</td>
<td>${s_i, i \in F}$</td>
<td>$\int_j s_i \cdot n , ds = \delta_{ij}$</td>
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<td>$S^3$</td>
<td>${s_i, i \in V}$</td>
<td>$\int_j s_i , dv = \delta_{ij}$</td>
<td>Volume integral</td>
</tr>
</tbody>
</table>

\[ u_K = \sum_i \phi_i(u) s_i \]

Bases

Finite elements
## Sequence of finite element spaces

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<td>(S^0) ({s_i, \ i \in N})</td>
<td>value</td>
<td>(\text{grad } S^0 \subset S^1)</td>
</tr>
<tr>
<td>(S^1) ({s_i, \ i \in E})</td>
<td>tangential component</td>
<td>(\text{curl } S^1 \subset S^2)</td>
</tr>
<tr>
<td>(S^2) ({s_i, \ i \in F})</td>
<td>normal component</td>
<td>(\text{div } S^2 \subset S^3)</td>
</tr>
<tr>
<td>(S^3) ({s_i, \ i \in V})</td>
<td>discontinuity</td>
<td></td>
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</tbody>
</table>

**Conformity**

**Sequence**

\[S^0 \xrightarrow{\text{grad}} S^1 \xrightarrow{\text{curl}} S^2 \xrightarrow{\text{div}} S^3\]
For each node $i \in N \rightarrow$ scalar field

$$s_i(x) = p_i(x) \in S^0$$

$$p_i = \begin{cases} 
1 & \text{at node } i \\
0 & \text{at all other nodes}
\end{cases}$$

$p_i$ continuous in $\Omega$

For each Volume $v \in V \rightarrow$ scalar field

$$s_v = \frac{1}{\text{vol}(v)} \in S^3$$
Edge function space $S^1$

For each edge $e_{ij} = \{i, j\} \in E \rightarrow$ vector field

$$s_{e_{ij}} = p_j \text{ grad } \sum_{r \in N_{F,ij}} p_r - p_i \text{ grad } \sum_{r \in N_{F,ij}} p_r$$

$s_e \in S^1$

Definition of the set of nodes $N_{F,mn}$

$N_{F,mn} = \{i \in N; \ i \in f_{mop}(q), \ o,p,q \neq n\}$

Illustration of the vector field $s_e$

N.B.: In an element : 3 edges/node
Edge function space $S^1$

Geometric interpretation of the vector field $s_e$

$$s_{e,ij} = p_j \text{ grad } \sum_{r \in N_{F,j\hat{i}}} p_r - p_i \text{ grad } \sum_{r \in N_{F,i\hat{j}}} p_r$$
Function space $S^2$

For each facet $f \in F \rightarrow$ vector field $f = f_{ijk(l)} = \{i, j, k, (l)\} = \{q_1, q_2, q_3, (l)\}$

$\mathbf{s}_f = a_f \sum_{c=1}^{#N_f} p_{q_c} \text{grad} \left( \sum_{r \in F, q_c \tilde{q}_{c+1}} p_r \right) \wedge \text{grad} \left( \sum_{r \in F, q_c \tilde{q}_{c-1}} p_r \right)$

$\mathbf{s}_f \in S^2$

Illustration of the vector field $\mathbf{s}_f$

$\begin{align*}
#N_f &= 3 \rightarrow a_f = 2 \\
#N_f &= 4 \rightarrow a_f = 1
\end{align*}$
Particular subspaces of $S^1$

**Kernel of the curl operator**

$$h \in S^1(\Omega) \ ; \ \text{curl } h = 0 \ \text{in } \Omega_c^c \subset \Omega \rightarrow h \equiv ?$$

**Gauged subspace**

$$a \in S^1(\Omega) \ ; \ b = \text{curl } a \in S^2(\Omega) \rightarrow a \equiv ?$$

Gauge condition $a \cdot \omega = 0$

to fix $a$

**Applications**

Ampere equation in a domain $\Omega_c^c$
without current ($\leftrightarrow \Omega_c$)

Gauge condition on a vector potential

Definition of a generalized source field $h_s$
such that $\text{curl } h_s = j_s$
Kernel of the curl operator

Case of simply connected domains

\[ H = \{ h \in \mathbb{S}^1(\Omega); \text{curl} \ h = 0 \text{ in } \Omega_c^C \} \]

\[ h = \sum_{k \in E_c} h_k \ s_k + \sum_{l \in E_c} h_l \ s_l \]

\[ h_1 = \int_{l_{ab}} h \cdot dl = \int_{l_{ab}} -\text{grad} \phi \cdot dl = \phi_{a_1} - \phi_{b_1} \]

\[ h = \sum_{k \in E_c} h_k \ s_k + \sum_{l \in E_c} \left( \phi_{a_1} - \phi_{b_1} \right) s_l \]

\[ h = \sum_{k \in E_c} h_k \ s_k + \sum_{n \in N_c^C} \phi_n \ v_n \]

with \[ v_n = \sum_{nj \in E_c^C} s_{nj} \]

Base of \( H \equiv \) basis functions of
- inner edges of \( \Omega_c \)
- nodes of \( \Omega_c^C \),
with those of \( \partial \Omega_c \)
Kernel of the curl operator

Case of multiply connected domains

\[ H = \{ h \in S^1(\Omega) ; \text{curl } h = 0 \in \Omega_c^C \} \]

\[ \phi = \phi^\text{cont} + \phi^\text{disc} \]

\[ h = - \text{grad } \phi \text{ in } \Omega_c^C \text{ (cuts)} \]

\[ \phi^+ - \phi^- = \phi |_{\Gamma_{eci}^+} - \phi |_{\Gamma_{eci}^-} = I_i \]

\[ \phi^\text{disc} = \sum_{i \in C} I_i \cdot q_i \]

\[ h = \sum_{k \in A_c} h_k \cdot s_k + \sum_{n \in N_c^C} \phi^\text{cont}_n \cdot v_n + \sum_{i \in C} I_i \cdot c_i \]

with

\[ c_i = \sum_{j \in N_c^C} s_{nj} \]

edges of \( \Omega_c^C \)

starting from a node of the cut

and located on side '+'

but not on the cut

\[ \sum_{nj \in A_c \cap A_{eci}} s_{nj} \]

\[ \sum_{n \in N_{eci}} \phi^\text{disc}_n \]

\[ \sum_{i \in C} q_i = 1 \]

\[ \sum_{q_i = 0} \phi_i = 0 \]

\[ \phi^+ \]

\[ \phi^- \]

Transition layer of \( q_i \)

\( q_i \) • defined in \( \Omega_c^C \)

• unit discontinuity across \( \Gamma_{eci} \)

• continuous in a transition layer

• zero out of this layer

Basis of \( H \equiv \) basis functions of

• inner edges of \( \Omega_c \)

• nodes of \( \Omega_c^C \)

• cuts of \( C \)
Gauged subspace of $S^1$

**Gauged space in $\Omega$**

$b = \text{curl } a$ with $a = \sum_{e \in E} a_e \, s_e \in S^1(\Omega)$, $b = \sum_{f \in F} b_f \, s_f \in S^2(\Omega)$

$b_f = \sum_{e \in E} i(e,f) \, a_e$, $f \in F$ → matrix form: $b_f = [C_{FE}] \, a_e$

- **Tree** (in $\Omega$) = set of edges connecting all the nodes of $\Omega$ without forming any loop ($\tilde{E}$)
- **Co-tree** = complementary set of the tree ($\tilde{E}$)

**Gauged space of $S^1(\Omega)$**

$\tilde{S}^1(\Omega) = \{a \in S^1(\Omega) \mid a_j = 0, \forall j \in \tilde{E}\}$

$a = \sum_{i \in E} a_i \, s_i \in \tilde{S}^1(\Omega)$

Basis of $\tilde{S}^1(\Omega) \equiv$ co-tree edge basis functions (explicit gauge definition)
Mesh of electromagnetic devices

- Electromagnetic fields extend to infinity (unbounded domain)
  - Approximate boundary conditions:
    - zero fields at finite distance
  - Rigorous boundary conditions:
    - "infinite" finite elements (geometrical transformations)
    - boundary elements (FEM-BEM coupling)

- Electromagnetic fields are confined (bounded domain)
  - Rigorous boundary conditions
Mesh of electromagnetic devices

- Electromagnetic fields enter the materials up to a distance depending on physical characteristics and constraints
  - Skin depth $\delta$ ($\delta \ll \omega, \sigma, \mu$)
    \[ \delta = \sqrt{\frac{2}{\omega \sigma \mu}} \]
  - Mesh fine enough near surfaces (material boundaries)
  - Use of surface elements when $\delta \to 0$
Mesh of electromagnetic devices

❖ Types of elements

♦ 2D : triangles, quadrangles

♦ 3D : tetrahedra, hexahedra, prisms, pyramids

♦ Coupling of volume and surface elements
  ○ boundary conditions
  ○ thin plates
  ○ interfaces between regions
  ○ cuts (for making domains simply connected)

♦ Special elements (air gaps between moving pieces, ...)

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Constraints in partial differential problems

- **Local constraints (on local fields)**
  - Boundary conditions
    - i.e., conditions on local fields on the boundary of the studied domain
  - Interface conditions
    - e.g., coupling of fields between sub-domains

- **Global constraints (functional on fields)**
  - Flux or circulations of fields to be fixed
    - e.g., current, voltage, m.m.f., charge, etc.
  - Flux or circulations of fields to be connected
    - e.g., circuit coupling

**Weak formulations for finite element models**

**Essential and natural constraints, i.e., strongly and weakly satisfied**
Constraints in electromagnetic systems

- Coupling of scalar potentials with vector fields
  - e.g., in h-φ and a-v formulations

- Gauge condition on vector potentials
  - e.g., magnetic vector potential a, source magnetic field h_s

- Coupling between source and reaction fields
  - e.g., source magnetic field h_s in the h-φ formulation, source electric scalar potential v_s in the a-v formulation

- Coupling of local and global quantities
  - e.g., currents and voltages in h-φ and a-v formulations (massive, stranded and foil inductors)

- Interface conditions on thin regions
  - i.e., discontinuities of either tangential or normal components

Interest for a “correct” discrete form of these constraints

Sequence of finite element spaces
Complementary 3D formulations

**Magnetodynamic h-formulation**

$$\partial_t (\mu \, h, h')_\Omega + (\sigma^{-1} \, \text{curl} \, h, \text{curl} \, h')_\Omega + <n \times e_s, h'>_{\Gamma_e} = 0 \quad \forall h' \in F_h^1(\Omega)$$

**Magnetodynamic a-formulation**

$$(\mu^{-1} \, \text{curl} \, a, \text{curl} \, a')_\Omega + (\sigma \, \partial_t a, a')_\Omega + <n \times h_s, a'>_{\Gamma_h} = 0 \quad \forall a' \in F_e^1(\Omega)$$

How to enforce global fluxes?
h-φ formulation

- h-φ magnetodynamic finite element formulations with massive and stranded inductors
- Use of edge and nodal finite elements for h and φ
  - Natural coupling between h and φ
  - Definition of current in a strong sense with basis functions either for massive or stranded inductors
  - Definition of voltage in a weak sense
  - Natural coupling between fields, currents and voltages
  - etc.
a-v formulation

- $a-v_s$ Magnetodynamic finite element formulation with massive and stranded inductors

- Use of edge and nodal finite elements for $a$ and $v_s$
  - Definition of a source electric scalar potential $v_s$ in massive inductors in an efficient way (limited support)
  - Natural coupling between $a$ and $v_s$ for massive inductors
  - Adaptation for stranded inductors: several methods
  - Natural coupling between local and global quantities, i.e. fields and currents and voltages
  - etc.